A proof-theoretic approach to ignorance

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1 Introduction

Several recent works in epistemic logic focus on finding a way to model the notion of ignorance (see, e.g., [10], [8], [1], [3]). One of the difficulties in achieving this task is that there is no agreement on which notion of ignorance to model. Indeed, van der Hoek & Lomuscio [10] take ignorance to be 'not knowing whether'; Steinsvold [8] considers ignorance as 'unknown truth'; finally, Kubyshkina and Petrolo [3] introduce a primitive ignorance operator relying on the factive nature of ignorance. We argue that these three different approaches should not be considered as exclusive alternatives, but as representing different aspects of the polysemic notion of ignorance. From this perspective, these three types of ignorance should coexist in the same formal framework. On the basis of this pluralist view, our main objective is to provide a unified framework expressing all the aforementioned types of ignorance, in order to analyse their behaviour and interactions.

We introduce a class of Kripke models, *ignorance models*, which interpret the three types of ignorance. We then define a labelled sequent calculus called *lab*WUDI, and prove its soundness and completeness with respect to ignorance models. Completeness is proved by constructing a countermodel from a failed and finite proof search tree. In future work we plan to define a Hilbert-style axiomatization for ignorance models, to prove admissibility of cut for *lab*WUDI, and to investigate alternative non-labelled calculi to treat ignorance. Furthermore, to study the interactions between ignorance and knowledge modalities, we intend to strengthen our models by imposing (combinations of) reflexivity, transitivity and symmetry on the accessibility relation, and to define sequent calculi formalising these frameworks.

2 Ignorance models

Given a countable set of propositional variables $Atm = \{p, q, \dots\}$, formulas of our language are constructed using the following grammar: $\phi ::= p \mid \bot \mid \phi \rightarrow \phi \mid \Box\phi \mid I^w\phi \mid I^u\phi \mid I^d\phi$. Negation is set to be $\neg\phi := \phi \rightarrow \bot$, and the other propositional connectives can be standardly defined. Operator I^w, for *ignorance whether*, was introduced by van der Hoek & Lomuscio [10]; I^u, for *unknown truth*, by Steinsvold [8, 9], and I^d by Kubyshkina & Petrolo [3]. Differently from [3], we intuitively interpret I^d as representing a specific type of ignorance, namely, *disbelieving ignorance*, which is characterized by Peels [7] as follows: "[a subject] S is disbelievingly ignorant that p iff (i) it is true that p, and (ii) S disbelieves that p."

For each ignorance operator there exists a complete Hilbert-style system. However, no unified framework for all the three ignorance operators is present in the literature.

Definition 2.1. An *ignorance model* is a triple $\mathcal{M} = \langle W, R, v \rangle$, where W is a set of possible worlds, $R \subseteq W \times W$ and $v : Atm \to 2^W$ is a valuation of propositional variables. We assume R to satisfy the *two-worlds property*, that is: for all $x \in W$, there is a $y \in W$ such that xRy and $x \neq y$. The satisfiability relation of formulas in a world x of a model $\mathcal{M}$ is defined as:

$$
\begin{array}{lll}
\mathcal{M}, x \models p & \textit{iff} & x \in v(p) \quad \text{and} \quad \mathcal{M}, x \not\models \bot; \\
\mathcal{M}, x \models \phi \to \psi & \textit{iff} & \mathcal{M}, x \not\models \phi \text{ or } \mathcal{M}, x \models \psi; \\
\mathcal{M}, x \models \Box\phi & \textit{iff} & \text{for all } y \in W, \text{ if } xRy \text{ then } \mathcal{M}, y \models \phi; \\
\mathcal{M}, x \models I^w\phi & \textit{iff} & \text{there are } y, z \in W \text{ s.t. } xRy,\ xRz,\ \mathcal{M}, y \not\models \phi \text{ and } \mathcal{M}, z \models \phi; \\
\mathcal{M}, x \models I^u\phi & \textit{iff} & \mathcal{M}, x \models \phi \text{ and there is } y \in W \text{ s.t. } xRy \text{ and } \mathcal{M}, y \not\models \phi; \\
\mathcal{M}, x \models I^d\phi & \textit{iff} & \mathcal{M}, x \models \phi \text{ and for all } y \in W, \text{ if } xRy \text{ and } y \neq x \text{ then } \mathcal{M}, y \not\models \phi.
\end{array}
$$

We say that ϕ *is valid in* $\mathcal{M}$ and write $\mathcal{M} \models \phi$ if $\mathcal{M}, x \models \phi$ for all x in W. If for all $\mathcal{M}$ we have $\mathcal{M} \models \phi$, we say that ϕ *is valid*, and write $\models \phi$.

Ignorance whether and unknown truth can be defined in terms of the $\Box$ operator as follows: $I^w\phi := \neg\Box\phi \wedge \neg\Box\neg\phi$ and $I^u\phi = \neg\Box\phi \wedge \phi$. Interestingly, disbelieving ignorance is not definable in terms of $\Box$ in none of the standard frames (K, T, S4, and S5), see [2]. Since we here focus on ignorance operator, we take I^w and I^u as primitive in our language.

The two-worlds property ensures that all worlds in a model have access to some world other than themselves. This allows one to avoid some counterintuitive consequences: for instance, when evaluating formulas at a one-world model $\mathcal{M}$, we get that $\mathcal{M} \models I^d\top$, $\mathcal{M} \models \neg I^w\top$, and $\mathcal{M} \models \neg I^u\top$. Indeed, it seems implausible that an agent is disbelievingly ignorant of a tautology, but she is not ignorant of its truth (neither in the sense of I^w, nor of I^u). By assuming the two-worlds property we obtain validity of formula $\neg I^d\top$.

3 Labelled sequent calculus

In this section, we shall introduce a labelled calculus *lab*WUDI, following the methodology from [4]. We enrich our language by an infinite set of variables, called *labels*: x, y, z, etc. Then, *relational atoms* have the form xRy or $x \neq y$, and *labelled formulas* have the form $x : \phi$. A *labelled sequent* has the form $\Gamma \Rightarrow \Delta$, where Γ is a multiset of relational atoms and labelled formulas and Δ is a multiset of labelled formulas.

The rules of *lab*WUDI are illustrated in Figure 1. The calculus features only one structural rule, 2w, expressing the two-worlds property. Propositional rules and the rules for $\Box$ are standard. The rules for ignorance operators have been defined based on the truth condition of the operators at ignorance models. The condition for I^d on the left is captured by a pair of rules, one of which only introducing formulas within the label of the principal formula, x[1]. Rules I^u_{R} and I^w_{R} introduce $\Box$-formulas in its premisses, needed to express the universal conditions in the negated truth conditions for I^u and I^w respectively.

Labelled sequents do not have a direct formula interpretation, and thus we need to interpret them over ignorance models to prove soundness of the calculus, which is straightforward.

Definition 3.1. Given a labelled sequent $\Gamma \Rightarrow \Delta$ and an ignorance model $\mathcal{M} = \langle W, R, v \rangle$, let $S = \{x \mid x \in \Gamma \cup \Delta\}$ and $\rho : S \to W$. We define the following relation: $\mathcal{M}, \rho \models xRy$ iff $\rho(x)R\rho(y)$; $\mathcal{M}, \rho \models x \neq y$ iff $\rho(x) \neq \rho(y)$; and $\mathcal{M}, \rho \models x : \phi$ iff $\rho(x) \models \phi$. A sequent $\Gamma \Rightarrow \Delta$ *is satisfied at* $\mathcal{M}$ *under* ρ if, if for all formulas $\phi \in \Gamma$ it holds that $\mathcal{M}, \rho \models \phi$, then there exists a

[1] In presence of 2w, rules I^d_{L1} and I^d_{L2} can be formulated as a single rule. Our choice is motivated by modularity: the rules from Figure 1 *without* 2w are adequate w.r.t. ignorance models without the two-worlds property.

$$\mathsf{init}\ \frac{}{x:p,\Gamma\Rightarrow\Delta,x:p} \qquad \bot\ \frac{}{x:\bot,\Gamma\Rightarrow\Delta} \qquad \mathsf{2w}\ \frac{xRy,x\neq y,\Gamma\Rightarrow\Delta}{\Gamma\Rightarrow\Delta}\,*$$

$$\to_{\mathsf{L}}\ \frac{\Gamma\Rightarrow\Delta,x:\phi \quad x:\psi,\Gamma\Rightarrow\Delta}{x:\phi\to\psi,\Gamma\Rightarrow\Delta} \qquad \to_{\mathsf{R}}\ \frac{x:\phi,\Gamma\Rightarrow\Delta,x:\psi}{\Gamma\Rightarrow\Delta,x:\phi\to\psi}$$

$$\Box_{\mathsf{L}}\ \frac{xRy,x:\Box\phi,y:\phi,\Gamma\Rightarrow\Delta}{xRy,x:\Box\phi,\Gamma\Rightarrow\Delta} \qquad \Box_{\mathsf{R}}\ \frac{xRy,\Gamma\Rightarrow\Delta,y:\phi}{\Gamma\Rightarrow\Delta,x:\Box\phi}\,* \qquad I^d_{\mathsf{L1}}\ \frac{x:I^d\phi,x:\phi,\Gamma\Rightarrow\Delta}{x:I^d\phi,\Gamma\Rightarrow\Delta}$$

$$I^d_{\mathsf{L2}}\ \frac{xRy,x\neq y,x:I^d\phi,\Gamma\Rightarrow\Delta,y:\phi}{xRy,x\neq y,x:I^d\phi,\Gamma\Rightarrow\Delta} \qquad I^d_{\mathsf{R}}\ \frac{\Gamma\Rightarrow\Delta,x:\phi \quad xRy,x\neq y,y:\phi,\Gamma\Rightarrow\Delta}{\Gamma\Rightarrow\Delta,x:I^d\phi}\,*$$

$$I^u_{\mathsf{L}}\ \frac{xRy,x:\phi,\Gamma\Rightarrow\Delta,y:\phi}{x:I^u\phi,\Gamma\Rightarrow\Delta}\,* \qquad I^u_{\mathsf{R}}\ \frac{\Gamma\Rightarrow\Delta,x:\phi \quad x:\Box\phi,\Gamma\Rightarrow\Delta}{\Gamma\Rightarrow\Delta,x:I^u\phi}$$

$$I^w_{\mathsf{L}}\ \frac{xRy,xRz,y:\phi,\Gamma\Rightarrow\Delta,z:\phi}{x:I^w\phi,\Gamma\Rightarrow\Delta}\,* \qquad I^w_{\mathsf{R}}\ \frac{x:\Box\neg\phi,\Gamma\Rightarrow\Delta \quad x:\Box\phi,\Gamma\Rightarrow\Delta}{\Gamma\Rightarrow\Delta,x:I^w\phi}$$

$*$: y (and z, if present) is fresh, i.e., it does not occur in $\Gamma\cup\Delta$.

Figure 1: Sequent calculus *lab*WUDI

formula $\psi\in\Delta$ such that $\mathcal{M},\rho\models\psi$. Then, $\Gamma\Rightarrow\Delta$ *is valid in* $\mathcal{M}$ if the sequent is satisfied at $\mathcal{M}$ for all ρ. Finally, $\Gamma\Rightarrow\Delta$ is valid if the sequent is valid in all models.

Theorem 3.2 (Soundness)**.** *If* $\Rightarrow x:\phi$ *is provable in lab*WUDI*, then* ϕ *is valid.*

To prove completeness of *lab*WUDI with respect to ignorance models, we show how to construct a finite countermodel from a failed and finite proof search tree, adapting to our setting the proof-or-countermodel approach to completeness for labelled calculi introduced in [5] (refer also to [6]). Thus, we first show termination of *lab*WUDI, the main difficulty being that rule 2w may lead to non-termination of root-first proof search. We shall introduce a *proof search strategy* restricting the application of rule 2w. We first define a measure for formulas.

Definition 3.3. We define the *weight of a labelled formula* as follows: $w(xRy)=w(x\neq y)$ $=0$ and, for a labelled formula $x:\phi$, we set $w(x:\chi)=w(\chi)$, where $w(\chi)$ is inductively defined as follows: $w(p)=w(\bot)=1$; $w(\phi\to\psi)=w(\phi)+w(\psi)+1$; $w(K\phi)=w(\phi)+2$; and $w(I^d\phi)=w(I^u\phi)=w(I^w\phi)=w(\phi)+3$.

Next, we define the notion of *saturated sequent*. Intuitively, a sequent is saturated if it is not an initial sequent and if all the rules have been non-redundantly applied to it. More formally, given a branch $\mathcal{B}=\{\Gamma_i\Rightarrow\Delta_i\}_{i>0}$ in a proof search tree and a sequent $\Gamma_n\Rightarrow\Delta_n$ in $\mathcal{B}$, let $\downarrow\Gamma_n=\bigcup_{i=1}^n\Gamma_i$ and $\downarrow\Delta_n=\bigcup_{i=1}^n\Delta_i$. Moreover, given two labels z and x occurring in a sequent $\Gamma_n\Rightarrow\Delta_n$, we write $\mathsf{For}(z)=\mathsf{For}(x)$ meaning that the set of formulas labelled by z and occurring in $\downarrow\Gamma_n$ coincides with the set of formulas labelled with x and occurring in $\downarrow\Gamma_n$, and similarly for $\downarrow\Delta_n$. We then associate to each rule a saturation condition. We explicitly show the one for 2w and, by means of example, the one for I^d_{R}:

(I^d_{R}) If $x:I^d\phi\in\downarrow\Delta_n$, then either $x:\phi\in\downarrow\Delta_n$ or for some y, $xRy\in\Gamma_n$, $x\neq y\in\Gamma_n$ and $y:\phi\in\downarrow\Gamma_n$.

(2w) For all x in $\downarrow \Gamma_n \cup \downarrow \Delta_n$, either $xRy \in \Gamma_n$ and $x \neq y \in \Gamma_n$ for some y, or $zRx \in \Gamma_n$ and $z \neq x \in \Gamma_n$, for some z such that $\mathsf{For}(z) = \mathsf{For}(x)$.

A labelled sequent is *saturated* if it meets the saturation conditions for all the rules, and if it not an instance of $\bot$ or init. Next, we define our *proof search strategy* as follows: given a sequent, we first apply to it rules that do not introduce bottom-up new labels, and rules that do introduce new labels, except for 2w. Once all the other rules have been applied, we apply 2w, taking care of not applying the rule to a label x if one of the two conditions described in the saturation condition is met. The saturation condition (2w) allows to prove the following:

Theorem 3.4 (Termination)**.** *Root-first proof search for a sequent $\Rightarrow x : \phi$ built in accordance with the strategy comes to an end in a finite number of steps, and each leaf of the proof-search tree contains either an initial sequent or a saturated sequent.*

To conclude, we sketch the proof of completeness:

Theorem 3.5 (Completeness)**.** *If ϕ is valid, there is a derivation of $\Rightarrow x : \phi$.*

Proof sketch. We prove the counterpositive. Suppose that $\Rightarrow x : \phi$ is *not* derivable in *lab*WUDI. By termination, if ϕ is not derivable then there is a proof search branch $\mathcal{B}$ whose upper node is occupied by a saturated sequent, $\Gamma_n \Rightarrow \Delta_n$. We construct a model $\mathcal{M}^{\mathcal{B}} = \langle \mathcal{W}^{\mathcal{B}}, \mathcal{R}^{\mathcal{B}}, \mathcal{V}^{\mathcal{B}} \rangle$ that satisfies all formulas in $\downarrow \Gamma_n$ and falsifies all formulas in $\downarrow \Delta_n$ as follows: $\mathcal{W}^{\mathcal{B}} = \{x \mid x \in \downarrow \Gamma_n \cup \downarrow \Delta_n\}$, $\mathcal{R}^{\mathcal{B}} = \{(x, y) \mid xRy \in \Gamma_n\}$ and $\mathcal{V}^{\mathcal{B}}(p) = \{x \in \mathcal{W}^{\mathcal{B}} \mid x : p \in \Gamma_n\}$. Note that distinct variables in $\downarrow \Gamma_n \cup \downarrow \Delta_n$ get mapped to distinct elements in $\mathcal{W}^{\mathcal{B}}$. As it is, $\mathcal{M}^{\mathcal{B}}$ does not satisfy the two-worlds condition. We modify the model as follows. Whenever we have a world x that has no access to worlds other than itself, by the saturation condition (2w) there needs to be a world z such that zRx and $z \neq x$ occur in Γ_n. We add $(x, z) \in \mathcal{R}^{\mathcal{B}}$, and conclude that x satisfies the two-worlds condition. To conclude the proof, one needs to show that $\mathcal{M}^{\mathcal{B}}$ satisfies formulas in $\downarrow \Gamma_n$ and falsifies all formulas in $\downarrow \Delta_n$. This is proved by induction on the weight of formulas, and by taking $\rho(x) = x$, for all $x \in \downarrow \Gamma_n \cup \downarrow \Delta_n$. The crucial case is proving that if $x : I^d\phi$ occurs in $\downarrow \Gamma_n$, then $\mathcal{M}^{\mathcal{B}}, \rho \models x : I^d\phi$. □

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