Axiomatization of Logics with Two-Layered Modal Syntax: The Protoalgebraic Case

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Two-layered modal syntax is given by three propositional languages (collections of connectives together with their arities): the *inner* one (used, in the common applications, to speak about events), the *modal* one (whose connectives are actually called *modalities*), and the *outer* one (used to speak about measures of events). Using these three languages and a fixed set of inner variables, we construct three disjoint sets of formulas:

- *inner* formulas are built from event variables using the inner language,
- *atomic outer* formulas are built by applying the modalities to inner formulas, and
- *complex* outer formulas are built from the atomic ones using the outer language.

Early examples of logics with two-layered syntax were modal logics of uncertainty stemming from Hamblin's seminal idea of reading the atomic outer formulas $P\varphi$ as 'probably φ' [16] and semantically interpreting it (in a given Kripke frame equipped with a finitely additive probability measure) as *true* iff the probability of the set of worlds where φ is true is bigger than a given threshold. This idea was later elaborated and extended by Fagin, Halpern and many others; see e.g. [5, 15].

These initial examples used classical logic to govern the behavior of formulas on both the inner and outer layers. A departure from this classical paradigm was proposed by Hájek and Harmancová in [13] and later developed by them in collaboration with Godo and Esteva in [12]. They kept classical logic as the interpretation of the inner syntactical layer of events, but proposed Łukasiewicz logic to govern the outer layer of statements on probabilities of these events, so that the truth degree of the atomic outer formula $P\varphi$ could be directly identified with the probability of the set of worlds where φ is true. Later, numerous other authors changed even the logic governing the inner layer (e.g., another fuzzy logic in order to allow for the treatment of uncertainty of vague events) or considered additional possibly non-unary modalities (e.g. for conditional probability), see e.g. [6–11, 14, 17].

This research thus gave rise to an interesting way of combining logics which allows to use one logic to reason about formulas (or rules) of another one with numerous examples described and developed in the literature. The existing bulk of literature constitutes an area of logic screaming for systematization through the development and application of uniform, general, and abstract methods. In our previous work [3] we took the first steps towards such a theory by providing an abstract notion of two-layered syntax and logic, a general semantics of *measured* Kripke frames and proved, in a rather general setting, two forms of completeness theorem most commonly appearing in the literature. Although the level of generality seemed quite sufficient back then (*finitary weakly implicative logics with unit and lattice conjunction*, see [4]), the recent development in the field shows the need for more: e.g., the inner logic in [2] and the outer logic in [1] are not weakly implicative, and in the former case they are not even equivalential.

The aim of this talk is to overcome the restrictions of [3] and present the completeness proof for an arbitrary inner logic and an arbitrary protoalgebraic outer logic.

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