Modal Information Logic: Decidability and Completeness

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The present abstract studies formal properties of Modal Information Logic (MIL), a modal logic proposed in [1] as a way of using possible-worlds semantics to model a theory of information. It does so by extending the language of propositional logic with a single binary modality defined in terms of being the supremum of two states.

First proposed in 1996, MIL has been around for some time, yet not much is known: [2, 3] pose two central open problems, namely (1) axiomatizing the logic and (2) proving (un)decidability.

While the majority of this abstract is spent on motivations and definitions, the first novel part of this abstract is concerned with these two problems. We solve both, (1) by providing an axiomatization and completeness proof and (2) by proving decidability. In proving the latter, we emphasize our method as a general heuristic on proving decidability 'via completeness' for semantically introduced logics.

If time allows, we will also be presenting the second novel part of this abstract. It is concerned with axiomatizing a kindred logic, where the supremum-modality is interpreted on join-semilattices. Besides the result being of interest per se, we believe the ideas involved in the axiomatization can be used when trying to axiomatize other logics. By higlighting these ideas, a general theme of this abstract will be a study in (Kripke) completeness.

Defining the logic

We continue by formally defining Modal Information Logic.

Definition 1 (Language)**.** The language $\mathcal{L}_M$ of Modal Information Logic is defined using a countable set of proposition letters $\mathbf{P}$ and a binary modality $\langle\text{sup}\rangle$. The formulas $\varphi \in \mathcal{L}_M$ are then given by the following BNF-grammar

$$\varphi ::= \bot \mid p \mid \neg\varphi \mid \varphi \vee \psi \mid \langle\text{sup}\rangle\varphi\psi,$$

where $p \in \mathbf{P}$ and $\bot$ is the falsum constant. $\dashv$

Modal Information Logic is defined by semantical means. That is, as the set of $\mathcal{L}_M$-validities of a class of structures, namely preorders. Formally we define as follows.

Definition 2 (Frames and models)**.** A (Kripke) *preorder-frame* for $\mathcal{L}_M$ is a pair $\mathbb{F} = (W, \leq)$ where

- W is a set; and
- $\leq$ is a preorder on W, that is: reflexive and transitive.

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A (Kripke) *preorder-model* for $\mathcal{L}_M$ is a triple $\mathbb{M} = (W, \leq, V)$ where

- $(W, \leq)$ is a preorder-frame; and
- V is a valuation on W, that is: a function $V : \mathbf{P} \to \mathcal{P}(W)$. ⊣

Having defined the structures in which to interpret $\mathcal{L}_M$-formulas, we are about to define the actual semantics. In order to do so, we provide the following definition generalizing the notion of supremum from partial orders to preorders.

Definition 3 (Supremum)**.** Given a preorder-frame $(W, \leq)$ and worlds $u, v, w \in W$, we say that w is *a supremum* of u, v and write $w \in \sup(u, v)$ iff

- w is an upper bound of u, v, i.e. $u \leq w$ and $v \leq w$; and
- $w \leq x$ for all upper bounds x.

In general, $\sup(u, v)$ denotes the set of suprema of $\{u, v\}$, and if this happens to be a singleton $\{w\}$, we may write $w = \sup(u, v)$. ⊣

Definition 4 (Semantics)**.** Given a preorder-model $\mathbb{M} = (W, \leq, V)$ and a world $w \in W$, *satisfaction* of a formula $\varphi \in \mathcal{L}_M$ at w in $\mathbb{M}$ (written $\mathbb{M}, w \Vdash \varphi$) is defined using the following recursive clauses on φ:

$$\begin{array}{lll}
\mathbb{M}, w \nVdash \bot, & & \\
\mathbb{M}, w \Vdash p & \textbf{iff} & w \in V(p), \\
\mathbb{M}, w \Vdash \neg\varphi & \textbf{iff} & \mathbb{M}, w \nVdash \varphi, \\
\mathbb{M}, w \Vdash \varphi \vee \psi & \textbf{iff} & \mathbb{M}, w \Vdash \varphi \;\; \textit{or} \;\; \mathbb{M}, w \Vdash \psi, \\
\mathbb{M}, w \Vdash \langle \sup \rangle \varphi\psi & \textbf{iff} & \text{there exist } u, v \in W \text{ such that } \mathbb{M}, u \Vdash \varphi, \; \mathbb{M}, v \Vdash \psi, \text{ and } w \in \sup(u, v).
\end{array}$$

Notions like *global truth*, *validity*, etc. are defined as usual in possible-worlds semantics. ⊣

With these notions laid out, Modal Information Logic is defined as follows:

Definition 5. Modal Information Logic is denoted by MIL_{Pre}, and defined as

$$MIL_{Pre} := \{\varphi \in \mathcal{L}_M : (W, \leq) \Vdash \varphi \;\; \textit{for all preorder-frames } (W, \leq)\}.$$

That is, MIL_{Pre} is the set of $\mathcal{L}_M$-validities on the class of all preorder-frames. ⊣

Having formally defined our logic, we end this section defining natural variations of Modal Information Logic obtained by considering kindred structures, e.g.:

MIL_{Pos}, which is the logic of poset-frames, i.e. frames $(W, \leq)$ where '$\leq$' is a partial order; and

MIL_{Sem}, which is the logic of frames $(W, \leq)$ where '$\leq$' is a join-semilattice.

Results

Having formally set out the logic and semantics, we present the results obtained. Firstly, we have shown that

Proposition 6. *MIL_{Pre} does not have the FMP w.r.t. preorder-frames.*

Proof (sketch). This is witnessed by the formula

$$\psi_N := HP\langle\text{sup}\rangle pp \wedge HP\neg\langle\text{sup}\rangle pp,$$

where

$$P\varphi := \langle\text{sup}\rangle\varphi\top$$

and $H := \neg P \neg$ is the dual of P. □

At first glance, this might make decidability appear unlikely. However, we circumvent this problem as follows. We (1) axiomatize the logic, (2) use this to show the logic to be complete with respect to another class of structures (where the ternary relation of ⟨sup⟩ won't necessarily be the supremum-relation of a preorder, but something more general), and then (3) prove that the logic enjoys the FMP on this other class of structures, from which we can deduce decidability.

That is, first, we provide an axiomatization:

Definition 7 (Axiomatization)**.** Let $\mathbf{MIL_{Pre}}$ be the least normal modal logic in the language of $\mathcal{L}_M$ containing the following axioms:

(Re.) $p \wedge q \to \langle\text{sup}\rangle pq$

(4) $PPp \to Pp$

(Co.) $\langle\text{sup}\rangle pq \to \langle\text{sup}\rangle qp$

(Dk.) $(p \wedge \langle\text{sup}\rangle qr) \to \langle\text{sup}\rangle pq$ ⊣

Theorem 8 (Completeness)**.** $\mathbf{MIL_{Pre}}$ *is sound and strongly complete w.r.t.* MIL_{Pre}*. So, in particular,* $\mathbf{MIL_{Pre}} = MIL_{Pre}$.

Further, as a corollary, we get that

Corollary 9. $MIL_{Pre} = MIL_{Pos}$.

Second, we define a class of structures $\mathcal{C}$, which is seen to be complete w.r.t. $\mathbf{MIL_{Pre}}$:

Definition 10. Let $\mathcal{C}$ be the class of pairs (W, C), where W is a set and C is a ternary relation on W satisfying the following four conditions:

(Re.f) $\forall w\,(Cwww)$

(4.f) $\forall w, v, u\,(Cwwv \wedge Cvvu \to Cwwu)$

(Co.f) $\forall w, v, u\,(Cwvu \to Cwuv)$

(Dk.f) $\forall w, v, u\,(Cwvu \to Cwwv)$ ⊣

Proposition 11. $\mathbf{MIL_{Pre}}$ *is sound and (strongly) complete w.r.t.* $\mathcal{C}$.

And, third, we show the following:

Theorem 12. $\mathbf{MIL_{Pre}}$ *admits filtration w.r.t. the class* $\mathcal{C}$*. Thus,*

$$\mathbf{MIL_{Pre}} = \text{Log}(\mathcal{C}_F),$$

where $\text{Log}(\mathcal{C}_F)$ *denotes the NML of the class of finite* $\mathcal{C}$*-frames.*

Using this, we deduce that

Corollary 13. *Modal Information Logic is decidable.*

Afterwards, if time allows, we turn our attention to axiomatizing MIL_{Sem}. We do so by syntactically defining a logic $\mathbf{MIL_{Sem}}$ extending $\mathbf{MIL_{Pre}}$ via an infinite axiom-scheme. We then show

Theorem 14 (Completeness)**.** $\mathbf{MIL_{Sem}} = MIL_{Sem}$.

When presenting this last result, we highlight some of the techniques and ideas going into it, especially (a) how the infinite extension-scheme can be intuited as capturing ever-more of the algebraic structure of a given join-semilattice, and (b) how we apply König's Lemma in the completeness proof.

References

[1] J. van Benthem. Modal logic as a theory of information. In J. Copeland, editor, *Logic and Reality. Essays on the Legacy of Arthur Prior*, pages 135–168. Clarendon Press, Oxford, 1996

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[3] J. van Benthem. Implicit and explicit stances in logic. *Journal of Philosophical Logic*, 48(3):571–601, 2019.