Sahlqvist theory for fragments of Intuitionistic Logic

Damiano Fornasiere[1] and Tommaso Moraschini[2]

[1] Departament de Filosofia, Facultat de Filosofia, Universitat de Barcelona (UB), Carrer Montalegre, 6, 08001 Barcelona, Spain
damiano.fornasiere@ub.edu
[2] Departament de Filosofia, Facultat de Filosofia, Universitat de Barcelona (UB), Carrer Montalegre, 6, 08001 Barcelona, Spain
tommaso.moraschini@ub.edu

The aim of this talk is to present an extension of Sahlqvist theory [9] to the fragments of intuitionistic logic IPC associated with the varieties PSL, (b)ISL, PDL, IL and HA, of pseudocomplemented semilattices, (bounded) implicative semilatices, pseudocomplemented distributive lattices and Heyting algebras, respectively. This result will serves as a basis for another talk in this conference, namely [4].

Consider the modal language

$$\mathcal{L}_\Box \quad ::= \quad x \mid \varphi \land \psi \mid \varphi \lor \psi \mid \varphi \to \psi \mid \neg\varphi \mid \Box\varphi \mid \Diamond\varphi \mid 0 \mid 1.$$

Formulas of $\mathcal{L}_\Box$ will be assumed to have variables in a denumerable set $Var = \{x_n : n \in \mathbb{Z}^+\}$ and arbitrary elements of Var will often be denoted by x, y, and z.

Definition 1. Let φ be a formula of $\mathcal{L}_\Box$ and x a variable. An occurrence of x in φ is said to be *positive* (resp. *negative*) if the sum of negations and antecedents of implications within whose scopes it appears is even (resp. odd). Moreover, we say a x is *positive* (resp. *negative*) in φ if every occurrence of x in φ is positive (resp. negative). Lastly, φ is said to be *positive* (resp. *negative*) if every variable is positive (resp. negative) in φ.

Formulas of the form $\Box^n x$ with $x \in Var$ and $n \in \mathbb{N}$ will be called *boxed atoms*.

Definition 2. A formula of $\mathcal{L}_\Box$ is said to be

(i) a *Sahlqvist antecedent* if it is constructed from boxed atoms, negative formulas and the constants 0 and 1 using only $\land$, $\lor$ and $\Diamond$;

(ii) a *Sahlqvist implication* if either it is positive, or it is of the form $\neg\varphi$ for a Sahlqvist antecedent φ, or it is of the form $\varphi \to \psi$ for a Sahlqvist antecedent φ and a positive formula ψ.

Remark 3. When applied to modal logic, our definition of a Sahlqvist implication is intentionally redundant. For if φ is positive and ψ a Sahlqvist antecedent, then φ is equivalent to $1 \to \varphi$ and $\neg\psi$ is equivalent to $\psi \to 0$. ⊠

Definition 4. A *Sahlqvist quasiequation* is a universal sentence of the form

$$\forall \vec{x}, y, z((\varphi_1(\vec{x}) \land y \leqslant z \,\&\, \ldots \,\&\, \varphi_n(\vec{x}) \land y \leqslant z) \Longrightarrow y \leqslant z),$$

where y and z are distinct variables that do not occur in $\varphi_1, \ldots, \varphi_n$ and each φ_i is constructed from Sahlqvist implications using only $\land$, $\lor$, and $\Box$.

Remark 5. The role of Sahlqvist quasiequations is usually played by the so-called *Sahlqvist formulas*, i.e., formulas that can be constructed from Sahlqvist implications using only $\wedge, \vee$, and $\Box$. To clarify the relation between Sahlqvist quasiequations and formulas, recall that a *modal algebra* is a structure $\langle A; \wedge, \vee, \neg, \Box, 0, 1\rangle$ where $\langle A; \wedge, \vee, \neg, 0, 1\rangle$ is a Boolean algebra and for every $a, b \in A$,

$$\Box(a \wedge b) = \Box a \wedge \Box b \text{ and } \Box 1 = 1.$$

Then, a Sahlqvist quasiequation $\Phi = \forall \vec{x}, y, z((\varphi_1(\vec{x}) \wedge y \leqslant z \,\&\, \dots \,\&\, \varphi_n(\vec{x}) \wedge y \leqslant z) \Longrightarrow y \leqslant z)$ is valid in a modal algebra $\boldsymbol{A}$ if and only if $\boldsymbol{A} \vDash \varphi_1 \vee \dots \vee \varphi_n$. The focus on Sahlqvist quasiequations (as opposed to formulas) is motivated by the fact that we deal with fragments where equations have a very limited expressive power. For instance, in PSL there are only three nonequivalent equations [8], while there are infinitely many nonequivalent Sahlqvist quasiequations. ⊠

With every Kripke frame $\mathbb{X} = \langle X, R\rangle$ we can associate a modal algebra

$$\mathscr{P}_{\mathsf{M}}(\mathbb{X}) := \langle \mathscr{P}(X); \cap, \cup, \neg, \Box, \emptyset, X\rangle,$$

where $\neg$ and $\Box$ are defined for every $Y \subseteq X$ as

$$\neg Y := X \smallsetminus Y \text{ and } \Box Y := \{x \in X : \text{if } \langle x, y\rangle \in R, \text{then } y \in Y\}.$$

Conversely, with a modal algebra $\boldsymbol{A}$ we can associate a Kripke frame $\boldsymbol{A}_+ := \langle X, R\rangle$, where X is the set of ultrafilters of $\boldsymbol{A}$ and

$$R := \{\langle F, G\rangle \in X \times X : \text{for every } a \in A, \text{if } \Box a \in F, \text{then } a \in G\}.$$

Our aim is to extend the next classical version of Sahlqvist Theorem to the above-mentioned fragments of IPC.

Theorem 6. *The following conditions hold for a Sahlqvist quasiequation* Φ*:*

(i) Canonicity*: If a modal algebra* $\boldsymbol{A}$ *validates* Φ*, then also* $\mathscr{P}_{\mathsf{M}}(\boldsymbol{A}_+)$ *validates* Φ*;*

(ii) Correspondence*: There is an effectively computable first order sentence* $\mathsf{tr}(\Phi)$ *in the language of Kripke frames such that* $\mathscr{P}_{\mathsf{M}}(\mathbb{X}) \vDash \Phi$ *iff* $\mathbb{X} \vDash \mathsf{tr}(\Phi)$*, for every Kripke frame* $\mathbb{X}$*.*

In order to do so, first we extend Sahlqvist Theorem to IPC using Gödel translation of IPC into S4 [7] and its duality theoretic interpretation (see, e.g., [2]). Then, we individuate a correspondence between homomorphisms in the varieties PSL, (b)ISL, PDL, IL, and HA and appropriate partial functions between (possibly empty) posets that generalize the notion of a p-morphism typical of *Esakia duality* for Heyting algebras [5, 6]. Our approach is inspired by [1].

For a poset $\mathbb{X}$ and $Y \subseteq X$, let

$$\uparrow^{\mathbb{X}} Y := \{x \in X : \text{there exists } y \in Y \text{ s.t. } y \leqslant x\};$$
$$\downarrow^{\mathbb{X}} Y := \{x \in X : \text{there exists } y \in Y \text{ s.t. } x \leqslant y\}.$$

Definition 7. An order preserving partial function $p\colon \mathbb{X} \rightharpoonup \mathbb{Y}$ between posets is

(i) a *partial negative p-morphism* if

$$X = \downarrow^{\mathbb{X}}\{x \in X : \uparrow^{\mathbb{X}} x \subseteq \mathsf{dom}(p)\}$$

and for every $x \in \mathsf{dom}(p)$ and $y \in Y$,

$$\text{if } p(x) \leqslant^{\mathbb{Y}} y, \text{ there exists } z \in \mathsf{dom}(p) \text{ s.t. } x \leqslant^{\mathbb{X}} z \text{ and } y \leqslant^{\mathbb{Y}} p(z);$$

(ii) a *partial positive p-morphism* if for every $x \in \mathsf{dom}(p)$ and $y \in Y$,

$$\text{if } p(x) \leqslant^{\mathbb{Y}} y, \text{ there exists } z \in \mathsf{dom}(p) \text{ s.t. } x \leqslant^{\mathbb{X}} z \text{ and } y = p(z);$$

(iii) a *partial p-morphism* if it is both a partial negative p-morphism and a partial positive p-morphism.

When p is a total function, we drop the adjective *partial* in the above definitions.

With every variety K among PSL, (b)ISL, PDL, IL, and HA we associate a collection K^{∂} consisting of the class of all posets with suitable partial functions between them as follows:[1]

$\mathsf{PSL}^{\partial} :=$ the collection of posets with partial negative p-morphisms;
$\mathsf{ISL}^{\partial} :=$ the collection of posets with partial positive p-morphisms;
$\mathsf{bISL}^{\partial} :=$ the collection of posets with partial p-morphisms;
$\mathsf{PDL}^{\partial} :=$ the collection of posets with negative p-morphisms;
$\mathsf{IL}^{\partial} :=$ the collection of posets with almost total partial positive p-morphisms;
$\mathsf{HA}^{\partial} :=$ the collection of posets with p-morphisms.

We will refer to the partial functions in K^{∂} as to the *arrows* of K^{∂}. Given $\boldsymbol{A}, \boldsymbol{B} \in \mathsf{K}$ and a homomorphism $f\colon \boldsymbol{A} \to \boldsymbol{B}$, let $f_*\colon \boldsymbol{B}_* \rightharpoonup \boldsymbol{A}_*$ be the partial function between the posets of meet irreducible filters of $\boldsymbol{B}$ and of $\boldsymbol{A}$ respectively, with

$$\mathsf{dom}(f_*) := \{F \in \boldsymbol{B}_* : f^{-1}[F] \in \boldsymbol{A}_*\}$$

defined as $f_*(F) := f^{-1}[F]$ for every $F \in \mathsf{dom}(f_*)$. Conversely, given a poset $\mathbb{X}$, let $\mathsf{Up}_{\mathsf{K}}(\mathbb{X})$ be the reduct in the language of K of the Heyting algebra

$$\langle \mathsf{Up}(\mathbb{X}); \cap, \cup, \to, \emptyset, X \rangle,$$

where $\mathsf{Up}(\mathbb{X})$ is the set of upsets of $\mathbb{X}$ and $\to$ is defined by

$$U \to V := X \smallsetminus {\downarrow}(U \smallsetminus V).$$

Lastly, given an arrow $p\colon \mathbb{X} \rightharpoonup \mathbb{Y}$ in K^{∂}, let $\mathsf{Up}_{\mathsf{K}}(p)\colon \mathsf{Up}_{\mathsf{K}}(\mathbb{Y}) \to \mathsf{Up}_{\mathsf{K}}(\mathbb{X})$ be the map defined for every $U \in \mathsf{Up}_{\mathsf{K}}(\mathbb{Y})$ as $\mathsf{Up}_{\mathsf{K}}(p)(U) := X \smallsetminus {\downarrow}^{\mathbb{X}} p^{-1}[Y \smallsetminus U]$.

Remark 8. In the case of HA, the applications $(-)_*$ and $\mathsf{Up}(-)$ are the contravariant functors underlying *Esakia duality* [5, 6]. ⊠

Proposition 9. *Let* K *be a variety among* PSL, (b)ISL, PDL, IL, *and* HA. *The following conditions hold for every* $\boldsymbol{A}, \boldsymbol{B} \in \mathsf{K}$ *and every pair* $\mathbb{X}$, $\mathbb{Y}$ *of posets:*

(i) *If* $f\colon \boldsymbol{A} \to \boldsymbol{B}$ *is a homomorphism, then* $f_*\colon \boldsymbol{B}_* \rightharpoonup \boldsymbol{A}_*$ *is an arrow in* K^{∂}*;*

(ii) *If* $p\colon \mathbb{X} \rightharpoonup \mathbb{Y}$ *is an arrow in* K^{∂}, *then* $\mathsf{Up}_{\mathsf{K}}(p)\colon \mathsf{Up}_{\mathsf{K}}(\mathbb{Y}) \to \mathsf{Up}_{\mathsf{K}}(\mathbb{X})$ *is a homomorphism.*

Furthermore, if f is injective (resp. p is surjective), then f_* is surjective (resp. $\mathsf{Up}_{\mathsf{K}}(p)$ *is injective).*

[1] The collection K^{∂} need not be a category in general.

By making use of Proposition 9, one can extend Sahlqvist Theorem 6 as announced, in the following way:

Theorem 10. *The following conditions hold for every variety* K *between* PSL, $(\mathsf{b})\mathsf{ISL}$, PDL, IL *and* HA *and every Sahlqvist quasiequation* Φ *in the language of* K:

(i) Canonicity*: For every* $\boldsymbol{A} \in \mathsf{K}$, *if* $\boldsymbol{A}$ *validates* Φ, *then also* $\mathsf{Up}_{\mathsf{K}}(\boldsymbol{A}_*)$ *validates* Φ*;*

(ii) Correspondence*: There exists an effectively computable sentence* $\mathsf{tr}(\Phi)$ *in the language of posets such that* $\mathsf{Up}_{\mathsf{K}}(\mathbb{X}) \vDash \Phi$ *iff* $\mathbb{X} \vDash \mathsf{tr}(\Phi)$, *for every poset* $\mathbb{X}$.

These results are collected in the manuscript [3].

References

[1] G. Bezhanishvili and N. Bezhanishvili. An algebraic approach to canonical formulas: intuitionistic case. *The Review of Symbolic Logic*, 2(3):517–549, 2009.

[2] W. Conradie, A. Palmigiano, and Z. Zhao. Sahlqvist via translation. *Logical Methods in Computer Science*, 15(1), 2019.

[3] D. Fornasiere and T. Moraschini. Intuitionistic Sahlqvist correspondence for deductive systems. Manuscript, 2022.

[4] D. Fornasiere and T. Moraschini. Intuitionistic Sahlqvist Correspondence for Deductive Systems. Conference talk at LATD 2022.

[5] L. Esakia Topological kripke models. Doklady Akademii Nauk.

[6] L. Esakia Heyting algebras: Duality theory. Springer.

[7] K. Gödel. Eine Interpretation des intuitionistischen Aussagenkalküls. *Anzeiger der Akademie der Wissenschaften in Wien, mathematisch-naturwissenschaftlichen Klasse*, 69:65–66, 1932.

[8] G. T. Jones. Pseudo Complemented Semi-Lattices. Ph. D. Dissertation, UCLA, 1972.

[9] H. Sahlqvist. Completeness and Correspondence in First and Second Order Semantics for Modal Logic. In S. Kanger, editor, *Proceedings of the third Scandinavian logic symposium*, pages 110–143. North-Holland, Amsterdam, 1975.