Reasoning with probabilities and belief functions over Belnap–Dunn logic

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Motivation and goal. Probabilities have been developed, mostly in the context of classical logic, to model reasoning based on probabilistic information. Belief functions are a generalisation of probabilities for situations where one is not able to give the exact probability of an event, but an approximation in the terms of an upper/lower bound. They were developed based on classical reasoning to handle situations with incomplete information, but they often produce counter-intuitive results when formalising situations involving contradictory information.

In [8] the authors propose a generalisation of probabilities for reasoning based on Belnap Dunn logic BD. In this paper, we extend their work and propose a generalisation of classical belief functions which is based on BD, and provide two-layered modal logics extending BD for reasoning about probabilities and belief functions. We focus on finite structures, therefore we consider logics over a finite set of atomic propositions and finite algebras.

Representation of uncertainty

Probabilistic reasoning based on incomplete and inconsistent information. The main idea behind Belnap-Dunn logic is to treat positive and negative information independently. A BD model is a tuple $\mathcal{M} = \langle S, v^+, v^- \rangle$ where S is a finite set of states, $v^+, v^- : S \times \mathrm{Prop} \to \{0, 1\}$ are valuations encoding respectively the positive and negative information respectively. A *probabilistic model* $\mathcal{M} = \langle S, \mu, v^+, v^- \rangle$ extends a BD model with a probability measure μ on the powerset algebra $\mathcal{P}S$.

Let us call $|\varphi|^+_{\mathcal{M}} = \{s \in \Sigma : v^+(\varphi) = 1\}$ and $|\varphi|^-_{\mathcal{M}} = \{s \in \Sigma : v^-(\varphi) = 1\}$ the positive and negative extensions of φ respectively. They are mutually definable via negation: $|\varphi|^-_{\mathcal{M}} = |\neg\varphi|^+_{\mathcal{M}}$. The *non-standard probability function* based on $\mathcal{M}$ is defined as $\mathsf{p}^+_\mu(\varphi) := \mu(|\varphi|^+_{\mathcal{M}})$ and represents the positive probabilistic evidence for φ. (Positive) non-standard probabilities satisfy the following three axioms:

$$0 \leq \mathsf{p}^+(\varphi) \leq 1 \quad \{\mathsf{p}^+(\varphi) \leq \mathsf{p}^+(\psi) \mid \varphi \vdash_{\mathsf{BD}} \psi\} \quad \mathsf{p}^+(\varphi \wedge \psi) + \mathsf{p}^+(\varphi \vee \psi) = \mathsf{p}^+(\varphi) + \mathsf{p}^+(\psi).$$

We can define negative non-standard probability in a similar manner as $\mathsf{p}^-_\mu(\varphi) = \mu(|\varphi|^-_{\mathcal{M}})$, but from a formal point of view it is sufficient to work with the positive one as $\mathsf{p}^-(\varphi) = \mathsf{p}^+(\neg\varphi)$. Notice that unlike in the classical case, one can no longer prove that $\mathsf{p}^+(\varphi) + \mathsf{p}^+(\neg\varphi) = 1$.

Evidential reasoning via belief functions and Dempster-Shafer combination rule. Here, we generalise the framework introduced in [8] to belief functions. We interpret belief functions on De Morgan algebras and propose a logic to reason with belief function based on BD. Belief functions [9] allow us to reason with the lower approximation of the probability of an event rather than with its exact probability. A *belief function* $\mathsf{bel} : \mathcal{L} \to [0,1]$ on a bounded lattice is a map such that: for every $a, a_1, \dots a_k, \dots a_n \in \mathcal{L}$, we have: (1) $\mathsf{bel}(\bot) = 0$ and $\mathsf{bel}(\top) = 1$; (2) for every $a \in \mathcal{L}$, $0 \le \mathsf{bel}(a) \le 1$; (3) for every $k \ge 1$, and every $a_1, \dots, a_k \in \mathcal{L}$,

$$\mathtt{bel}\left(\bigvee_{1\le i\le k} a_i\right) \ge \sum_{\substack{J \subseteq \{1,\dots,k\} \\ J \neq \varnothing}} (-1)^{|J|+1} \cdot \mathtt{bel}\left(\bigwedge_{j \in J} a_j\right). \tag{1}$$

Recall that a *mass function* $\mathsf{m} : \mathcal{L} \to [0,1]$ on a bounded lattice $\mathcal{L}$ is a map such that: $m(\bot) = 0$ and $\sum_{a \in \mathcal{L}} m(a) = 1$. Every mass function $\mathsf{m} : \mathcal{L} \to [0,1]$ defines a belief function $\mathsf{bel}_{\mathsf{m}}$ as follows: for every $a \in \mathcal{L}$, $\mathsf{bel}_{\mathsf{m}}(a) = \sum_{b \le a} \mathsf{m}(b)$. Equivalently, for every belief function bel, one can compute its associated mass function $\mathsf{m}_{\mathsf{bel}}$ such that the previous equation holds.

Conceptually, mass of a encodes the amount of information provided exactly about a, while the belief of a represents the amount of all the evidence supporting a. Dempster-Shafer combination rule [9] provides a method to aggregate belief functions based on their associated mass functions. Let $m_1, m_2 : \mathcal{L} \to [0,1]$ be two mass functions, their aggregation $m_{1\oplus 2}$ is: $\forall a \in \mathcal{L}$,

$$m_{1\oplus 2}(a) = \frac{1}{1-K} \sum_{b \wedge c = a \neq \bot} m_1(b) m_2(c), \tag{2}$$

where $K = \sum_{b \wedge c = \bot} m_1(b) m_2(c)$. K is a normalisation term that encodes the fact that any fully contradictory information between m_1 and m_2 is ignored. For this reason the combination rule can give very counter intuitive results as demonstrated in the following example.

Example: Two disagreeing doctors. A patient has disease a, b or c and one assumes that he has only one of these diseases. A first expert thinks that the patient has disease a (resp. b and c) with probability 0.9 (resp. 0.1 and 0). This opinion is encoded via the mass function $m_1 : \mathcal{P}(\{a,b,c\}) \to [0,1]$ such that $m_1(a) = 0.9$, $m_1(b) = 0.1$ and $m_1(c) = 0$. A second expert thinks that he has disease a (resp. b and c) with probability 0 (resp. 0.1 and 0.9). This opinion is encoded via the mass function $m_2 : \mathcal{P}(\{a,b,c\}) \to [0,1]$ such that $m_2(a) = 0.9$, $m_2(b) = 0.1$ and $m_2(c) = 0$. Using (2), one gets the following aggregated mass function $\mathsf{m}_{1\oplus 2} : \mathcal{P}(\{a,b,c\}) \to [0,1]$: for every $x \in \mathcal{P}(\{a,b,c\})$, we have $\mathsf{m}_{1\oplus 2}(x) = 1$ if $x = b$, 0 otherwise. This means that $\mathsf{bel}_{1\oplus 2}(b) = 1$ and $\mathsf{bel}_{1\oplus 2}(a) = \mathsf{bel}_{1\oplus 2}(c) = 0$. Therefore while both experts agreed that b was unlikely and that it is highly likely that the patient has an other disease (a or c), one concludes that the patient must have disease b. This results follows from the fact that a, b and c are considered mutually incompatible. Notice that the term K that measure 'contradiction' is equal to 0.99 which means that most of the information given by the experts was ignored.

The same computation over the De Morgan algebra $\mathcal{D}$ generated by $\{a,b,c\}$ leads to a very different conclusion. If one considers the mass functions $m_1 : \mathcal{D} \to [0,1]$ such that $m_1(a \wedge \neg b \wedge \neg c) = 0.9$, $m_1(\neg a \wedge b \wedge \neg c) = 0.1$ and $m_1(\neg a \wedge \neg b \wedge c) = 0$ and $m_2 : \mathcal{D} \to [0,1]$ such that $m_1(a \wedge \neg b \wedge \neg c) = 0$, $m_1(\neg a \wedge b \wedge \neg c) = 0.1$ and $m_1(\neg a \wedge \neg b \wedge c) = 0.9$, one gets the following aggregated mass function $\mathsf{m}_{1\oplus 2}$ (we represent only the elements in $\mathcal{D}$ with non-zero mass):

	$\neg a \wedge b \wedge \neg c$	$a \wedge \neg a \wedge b \wedge \neg b \wedge \neg c$	$a \wedge \neg a \wedge \neg b \wedge c \wedge \neg c$	$\neg a \wedge b \wedge \neg b \wedge c \wedge \neg c$
$\mathsf{m}_{1\oplus 2}$	0.01	0.09	0.81	0.09

Therefore, one reaches the conclusion that one has strong contradictory information regarding a and c and that b is most probably not the case, since $\mathsf{m}_{1\oplus 2}(a \wedge \neg a \wedge \neg b \wedge c \wedge \neg c) = 0.81$. This tells us to search for additional information to figure out whether the patient has disease a or c. This observation leads us to think that in presence of highly conflicting information, it is more relevant to interpret belief functions over De Morgan algebras and therefore to reason with BD rather than with classical logic.

Two-layered Belnapian Logics for probabilities and belief functions

Two-layer logics for reasoning under uncertainty were introduced in [6, 7], and developed further within an abstract algebraic framework by [5] and [2]. Two-layer logics separate two layers of reasoning: the inner layer consists of a logic chosen to reason about events (often classical propositional logic interpreted over sets of possible worlds), the connecting modalities are interpreted by a chosen uncertainty measure on propositions of the inner layer (typically a probability or a belief function), and the outer layer consists of a logical framework to reason about probabilities or beliefs. The modalities apply to inner layer formulas only, to produce outer layer atomic formulas, and they never nest. Logics introduced in [6] use classical propositional logic on the lower layer, and reasoning with linear inequalities on the upper layer. [7] on the other hand uses Łukasiewicz logic on the outer layer, to capture the quantitative, many-valued reasoning about probabilities within a propositional logical language. Building on that idea, and having in mind the two-dimensionality of uncertain information (e.g. positive and negative probabilities), we have introduced a two layer modal logic to reason with non-standard probabilities in [4]. There a two-dimensional extension of Łukasiewicz logic containing an additional De Morgan negation has been proposed. Another two-dimensional extension of Łukasiewicz logic, where De Morgan negation of implication behaves differently, has been introduced in [3], and both logics (which we denote $Ł^2(\rightarrow)$ and $Ł^2(\rightarrowtriangle)$) were shown to be coNP complete using constraint tableaux calculi. We provide Hilbert-style axiomatizations for both the logics, which are finitely standard strong complete w.r.t. the twist product of the standard MV algebra $[0,1]_Ł^{\bowtie}$.

In this talk, we consider two-layered logics which use BD as the inner layer, a single unary probability modality P (or a belief modality B) applied to BD formulas, and $Ł^2(\rightarrow)$ or $Ł^2(\rightarrowtriangle)$ on the outer layer. The inner formulas are interpreted over a BD model $\mathcal{M} = \langle S, v^+, v^- \rangle$, the atomic modal formulas are interpreted in $[0,1]_Ł^{\bowtie}$ via a given probability (or belief) function on $\mathcal{P}S$ as

$$v^{\mathcal{M}}(P\varphi) = (\mathsf{p}(|\varphi|^+_{\mathcal{M}}), \mathsf{p}(|\varphi|^-_{\mathcal{M}})) \qquad v^{\mathcal{M}}(B\varphi) = (\mathsf{bel}(|\varphi|^+_{\mathcal{M}}), \mathsf{bel}(|\varphi|^-_{\mathcal{M}})),$$

and outer formulas are interpreted in the algebra $[0,1]_Ł^{\bowtie}$ following the semantics of the chosen variant of $Ł^2$.

We present the resulting two-layer logics via Hilbert-style two-layer axiomatizations of the form $\langle \mathsf{BD}, M_p, Ł^2 \rangle$, and $\langle \mathsf{BD}, M_b, Ł^2 \rangle$, and prove their completeness. Here, BD is an axiomatization of the logic BD, and M_p, M_b are sets of modal axioms and rules capturing the behaviour of the P or B modality respectively. Axioms M_p of probability for example look as follows:

$$\vdash_{Ł^2} P\neg\varphi \leftrightarrow \neg P\varphi \quad \{\vdash_{Ł^2} P\varphi \rightarrow P\psi \mid \varphi \vdash_{\mathsf{BD}} \psi\} \quad \vdash_{Ł^2} P(\varphi \vee \psi) \leftrightarrow (P\varphi \ominus P(\varphi \wedge \psi)) \oplus P\psi,$$

where $\oplus, \ominus$ are connectives definable in $Ł^2$ as in Lukasiewicz logic, corresponding (point-wise) to truncated addition/subtraction on $[0,1]$ respectively.

In the case we deal with belief functions, the first two axiom schemes for B modality stay in place. While expressing the probability axioms in Łukasiewicz logic as above is rather straightforward (see [7, 4]), formulating the belief k-monotonicity axioms is less so. We define a sequence of outer formulas γ_n in propositional letters of the inner language $p_1, \ldots, p_n$ inductively as follows:

$$\gamma_1 \coloneqq Bp_1 \quad \gamma_{n+1} \coloneqq \gamma_n \oplus (Bp_{n+1} \ominus \gamma_n[B\psi : B(\psi \wedge p_{n+1}) \mid B\psi \text{ atoms of } \gamma_n]),$$

where $\gamma_n[B\psi : B(\psi \wedge p_{n+1}) \mid B\psi \text{ modal atoms of } \gamma_n])$ is the result of replacing each modal atom $B\psi$ in γ_n with the modal atom $B(\psi \wedge p_{n+1})$ (semantically, it is a relativisation of the corresponding belief function to the sets $|p_{n+1}|^{+-}$). The n-th belief function axiom (i.e., the n-monotonicity) is expressed by substitution instances (substituting inner formulas for the atomic letters $p_1, \ldots, p_n$) of

$$\alpha_n \coloneqq \gamma_n \to B(\bigvee_{i=1}^{n} p_n).$$

Additionally to Ł^2-based logics, we present a two-layer logic for belief functions based on BD on the lower level, and two-dimensional reasoning about linear inequalities on the upper level. We will relate the two formalism by way of translation, following [1], and we will compare the resulting logic to the one introduced in [10].

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