On the modal embedding of intuitionistic logic: Gödel's proof of his 1933 conjecture

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Motivated by the idea that intuitionism expresses a modal notion of provability, Gödel defined in 1933 a translation of intuitionistic logic **Int** into the modal logic **S4**. He stated without proof the soundness of the translation and only conjectured its faithfulness. It took some years before McKinsey and Tarski proved the conjecture indirectly using algebraic semantics and completeness of **S4** with respect to closure algebras and of intuitionistic logic with respect to Heyting algebras. The result was later extended in various directions, most notably to embedding results for intermediate logics in modal logics between **S4** and **S5** by Dummett and Lemmon, and to the embeddings of Int into the provability logics **GL** and **Grz** of Gödel-Löb and of Grzegorczyk.

Unlike the proofs of soundness, the syntactical proofs of faithfulness of these embeddings are not entirely straightforward, as witnessed in section 9.2 of [4] for the relatively simple case of the embedding of **Int** into **S4**. In our earlier work we based our approach to such faithfulness results on the formulation of a cut-free sequent system for the logic that is the target of the embedding and offered a modular treatment by the use of labelled sequent calculi for intermediate logics and their modal companions [1, 2] and for infinitary logics [5].

It turned out, however, that Gödel's so far unknown work of 1941 in his book manuscript "Resultate Grundlagen" contains a proof of faithfulness of the translation of intuitionistic into modal logic [3]. The proof is purely syntactic and gives a converse to his translation of 1933 through a propositional version of Barr's theorem. Besides providing the topological semantics of modal logic, later used by McKinsey and Tarski to prove the same embedding result by semantic means, he obtained many other—at the time new—results by the topological semantics, among them that there is an infinity of inequivalent propositions in one variable in intuitionistic logic.

References

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