Łukasiewicz logic reasons about probability: encoding de Finetti coherence in MV-algebras

Sara Ugolini[1,*]

Artificial Intelligence Research institute (IIIA), CSIC
sara@iiia.csic.es

The interconnection between logic, algebra, and probability has played a central role in the study of reasoning since the dawn of modern logic, particularly in the groundbreaking work of Boole [3]. More recent times have seen a flourishing of formal methods and logical approaches to deal with logics capable of reasoning with probabilities. Among them, it is worth recalling the model theoretical approach mainly developed by Keisler [9] and Hoover [8], the more artificial intelligence oriented perspective initiated by Fagin, Halpern, and Megiddo in [4], and the one put forward by Hájek, Esteva, and Godo in [7]. In the latter, which we shall follow, probability is modeled by a modal operator P added to the language of Łukasiewicz logic; formulas of the form $P(\varphi)$ – for φ any classical formula – read as "φ is probable". Interestingly, the logic of [4] and a slight variant of Hájek, Esteva, and Godo's logic have been shown to be syntactically interdefinable, and hence equivalent, in the recent [1].

In joint work with Flaminio, we are concerned with an extension of Hájek, Esteva, and Godo's logic first axiomatized in [6], denoted by $\mathrm{FP}(\text{Ł}, \text{Ł})$, that has been recently proved ([5]) to be the logic of *state theory*: a generalization of probability theory for uncertain quantification on Łukasiewicz sentences, introduced by Mundici in [11]. In $\mathrm{FP}(\text{Ł}, \text{Ł})$, Łukasiewicz logic plays a twofold role: it is the *inner* logic that represents the formulas that fall under the scope of the modality P (i.e., *events*) and it is also the *outer* logic that reasons on complex probabilistic modal formulas.

We show that, roughly speaking, the modal expansion leading to the logic $\mathrm{FP}(\text{Ł}, \text{Ł})$ is not needed to formalize probabilistic reasoning within Łukasiewicz calculus. In order to do so, we use the equivalent algebraic semantics of Łukasiewicz in the sense of [2], MV-algebras. Phrased in this setting, we show that the quasi-equational theory of MV-algebras is expressive enough to encode probabilistic reasoning.

In particular, the categorical duality between rational polyhedra and finitely presented MV-algebras put forward in [10] allows us to encode within Łukasiewicz logic itself the local, finitary, probabilistic information described by the convex rational polyhedra being the geometric interpretation of de Finetti's coherence criterion.

Moreover, leveraging the categorical duality between rational polyhedra and finitely presented MV-algebras, we are able to identify a class of MV-algebras that forms a semantics for $\mathrm{FP}(\text{Ł}, \text{Ł})$. These algebras, that will be called *coherent*, form a proper subclass of finitely presented and projective MV-algebras.

Finally, exploiting the interplay between the algebraic and geometric approaches, we are able to study purely logical properties of the logic $\mathrm{FP}(\text{Ł}, \text{Ł})$, exploring the connection between logic and probability in the many-valued setting.

[*] Speaker.

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