LATD

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LATD 2022

The Logic Algebra and Truth Degrees (LATD) conference series started as an official meeting of the working group on Mathematical Fuzzy Logic and has evolved into a wider meeting in algebraic logic and related areas. Its main goal is to foster collaboration between researchers in these areas and to promote communication and cooperation with members of neighbouring fields. Previous editions of LATD were held in Siena (2008), Prague (2010), Kanazawa (2012), Vienna (2014), Phalaborwa (2016), and Bern (2018).

MOSAIC KICK OFF CONFERENCE

The Horizon 2020 Marie Curie-Skłodowska RISE project MOSAIC (Modalities in Substructural Logics: Theory Methods and Applications) has started in September 2021 and its first meeting will be held in Paestum together with the conference LATD.

Contents

Invited talks

Interpolation Meets Cyclic Proofs 2
Bahareh Afshari

Some facts and questions around the Kuznetsov problem 3
Mamuka Jibladze

Admissibility of Π_2-Inference Rules: interpolation, model completion, and contact algebras 4
Silvio Ghilardi

Ecumenical modal logic 6
Sonia Marin

Applications of Real Valued Logics to Probabilistic Logics 7
Matteo Mio

Łukasiewicz logic, MV-algebras and AF C*-algebraic truth-degrees . . 9
Daniele Mundici

On the modal embedding of intuitionistic logic: Gödel's proof of his 1933 conjecture 10
Sara Negri

First-order fuzzy logics and their model theory 11
Carles Noguera

A Journey in Intuitionistic Modal Logic: normal and non-normal modalities 12
Nicola Olivetti

From unified correspondence to parametric correspondence 14
Alessandra Palmigiano

Łukasiewicz logic reasons about probability: encoding de Finetti coherence in MV-algebras 15
Sara Ugolini

Contributed talks

Logic Beyond Formulas: Designing Proof Systems on Graphs 18
Matteo Acclavio

On Proof Equivalence for Modal Logics 21
Matteo Acclavio and Lutz Straßburger

Polyhedral Completeness of Intermediate and Modal Logics 25
Sam Adam-Day, Nick Bezhanishvili, David Gabelaia and Vincenzo Marra

Strictly join irreducible varieties of residuated lattices 28
Paolo Aglianò and Sara Ugolini

Gödel temporal logic 31
Juan Pablo Aguilera, Martín Diéguez, David Fernández-Duque and Brett McLean

Automorphisms of Product algebras and related varieties 35
Stefano Aguzzoli and Brunella Gerla

Cut-Elimination for a Hypersequent Calculus for First-order Gödel Logic over $[0,1]$ with Δ 37
Matthias Baaz, Christian Fermüller and Norbert Preining

Logical Approximations of Qualitative Probability 41
Paolo Baldi and Hykel Hosni

Positive (Modal) Logic Beyond Distributivity 44
Nick Bezhanishvili, Anna Dmitrieva, Jim de Groot and Tommaso Moraschini

Bi-intermediate Logics of Trees and Co-trees 48
Nick Bezhanishvili, Miguel Martins and Tommaso Moraschini

Degrees of the finite model property: The antidichotomy theorem . . . 52
Guram Bezhanishvili, Nick Bezhanishvili and Tommaso Moraschini

Towards a non-integral variant of Łukasiewicz logic 55
Marta Bilkova, Petr Cintula and Carles Noguera

Reasoning with probabilities and belief functions over Belnap–Dunn logic 57
Marta Bilkova, Sabine Frittella, Daniil Kozhemiachenko, Ondrej Majer and Sajad Nazari

Embeddings of metric Boolean algebras in $\mathbb{R}^N$ 61
Stefano Bonzio and Andrea Loi

Kites and pseudo MV-algebras 63
Michal Botur and Tomasz Kowalski

Hereditary Structural Completeness over K4 67
James Carr, Nick Bezhanishvili and Tommaso Moraschini

Game semantics for constructive modal logic 71
Davide Catta, Matteo Acclavio and Lutz Straßburger

Degrees of FMP in extensions of bi-intuitionistic logic 73
Anton Chernev

The general algebraic framework for Mathematical Fuzzy Logic 76
Petr Cintula and Carles Noguera

Axiomatization of Logics with Two-Layered Modal Syntax: The Protoalgebraic Case 78
Petr Cintula and Carles Noguera

One-Variable Lattice-Valued Logics 80
Petr Cintula, George Metcalfe and Naomi Tokuda

Truthmaker Semantics for Degreeism of Vagueness 83
Shimpei Endo

Connexive implication in substructural logics 85
Davide Fazio and Gavin St. John

Can we have Constant Domain RQ? 87
Nicholas Ferenz

Intuitionistic Sahlqvist correspondence for deductive systems 89
Damiano Fornasiere and Tommaso Moraschini

Sahlqvist theory for fragments of Intuitionistic Logic 92
Damiano Fornasiere and Tommaso Moraschini

Lifting Properties from Finitely Subdirectly Irreducible Algebras 96
Wesley Fussner and George Metcalfe

A proof-theoretic approach to ignorance 99
Marianna Girlando, Ekaterina Kubyshkina and Mattia Petrolo

Structural completeness and lattice of extensions in many-valued logics with rational constants 103
Joan Gispert, Zuzana Haniková, Tommaso Moraschini and Michal Stronkowski

From implicative reducts to Mundici's functor 107
Valeria Giustarini

Multi-type modal extensions of the Lambek calculus for structural control 110
Giuseppe Greco, Apostolos Tzimoulis, Michael Moortgat and Mattia Panettiere

Probability via Łukasiewicz logic - A multi-type semantic and proof theoretical account 114
Giuseppe Greco, Krishna Manoorkar, Apostolos Tzimoulis, Sabine Frittella and Daniil Kozhemiachenko

From contact relations to modal operators, and back 118
Rafal Gruszczynski and Paula Menchon

Weak Systems Have Intractable Theorems 122
Raheleh Jalali

Abstract Model and Deduction System for Logic of Multiple Agent in Quantum Physics 126
Tomoaki Kawano

Some properties of residuated lattices using two parameters derivations 130
Darline Laure Keubeng Yemene, Lele Celestin, Stefan Schmidt and Tallee Kakeu Ariane Gabriel

Modal Information Logic: Decidability and Completeness 132
Søren Brinck Knudstorp

Finite Characterisations for Modal Formulas 136
Raoul Koudijs and Balder ten Cate

Many-valued coalgebraic logic with a semi-primal algebra of truth-degrees 141
Alexander Kurz, Bruno Teheux and Wolfgang Poiger

What is the cost of cut? 144
Timo Lang, Carlos Olarte and Elaine Pimentel

Free algebras in all subvarieties of the variety generated by the MG t-norm 148
Noemi Lubomirsky and José Patricio Dìaz Varela

The quasivariety given by the class of possibilistic Ln-valued Kripke frames 150
Miguel Andrès Marcos

Modal Nelson lattices and their associated twist structures 154
María Paula Menchón and Ricardo Oscar Rodriguez

On equational completeness theorems 158
Tommaso Moraschini

Quantificational issues in Prawitzian validity 162
Antonio Piccolomini d'Aragona

Some Proof-theoretical aspects of non-associative, non-commutative multi-modal linear logic 166
Elaine Pimentel, Stepan Kuznetsov, Andre Scedrov, Eben Blaisdell and Max Kanovich

Logics of upsets of De Morgan lattices 170
Adam Přenosil

Some Completeness Results in Derivational Modal Logic 174
Quentin Gougeon

Nelson conuclei and nuclei: the twist construction beyond involutivity . 178
Umberto Rivieccio

Modal intutionistic algebras and twist representations 181
Umberto Rivieccio

Algebras of Counterfactual Conditionals 185
Giuliano Rosella and Sara Ugolini

Modal Algebraic Models of Counterfactuals 187
Giuliano Rosella, Tommaso Flaminio and Stefano Bonzio

Strong standard completeness for S5-modal Lukasiewicz logics 190
Gabriel Savoy, Patricio Diaz Varela and Diego Castañ o

Relevant Reasoners in a Classical World 192
Igor Sedlar and Pietro Vigiani

Proof Theory for Intuitionistic Temporal Logic over Topological Dynamics 196
Thomas Studer and Lukas Zenger

Framing Faultiness Kripke Style 199
Hans van Ditmarsch, Krisztina Fruzsa and Roman Kuznets

Local Modal Product Logic is decidable 203
Amanda Vidal Wandelmer

One-sorted Program Algebras 208
Jamie Wannenburg and Igor Sedlár

Modules with Fusion and Implication based over Distributive Lattices . 211
William Javier Zuluaga Botero and Ismael Calomino

A focused linear nested system for multi-modalities 215
Bruno Xavier, Carlos Olarte and Elaine Pimentel

Invited talks

Interpolation Meets Cyclic Proofs

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The usual proof-theoretic approach to Craig interpolation is algorithmic. It works by taking a proof π of a valid implication $\phi \to \psi$ and returning, simultaneously, a formula I (in the common vocabulary of ϕ and ψ) and proofs π_0 and π_1, respectively, of the implications $\phi \to I$ and $I \to \psi$, hence establishing that I is an interpolant for $\phi \to \psi$. The construction and verification of the procedure relies heavily on the salient qualities of the utilised proof system, such as cut-free completeness, form of the initial sequents or, more generally, the extent to which the vocabulary is preserved transiting from premise(s) to the conclusion of a rule.

In this talk, we will look at how the proof-theoretic method can be applied to the realm of cyclic proofs, wherein proofs are no longer finite trees but finite *graphs*. In particular, we will re-visit the question of uniform interpolation for the modal μ-calculus and its deep connection with completeness.

SOME FACTS AND QUESTIONS AROUND THE KUZNETSOV PROBLEM

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In 1974, A. V. Kuznetsov asked whether every superintuitionistic logic is the logic of some class of topological spaces. The answer is still not known. The aim of the talk is to review some results related to it, state some of its reformulations, generalizations and particular cases that might be interesting to work on, explain some difficulties that we encountered when thinking about it, and outline some approaches to it that in our opinion might be promising.

Mostly we will focus on three themes. The first one is about possible use for the Kuznetsov problem of a Kripke incomplete superintuitionistic logic discovered in 1977 by V. Shehtman. This logic combines the Gabbay-De Jongh formula $\mathbf{bb}_2$ that on finite Kripke frames restricts possible branching to not more than two, and an intuitionistic analog of a Kripke incomplete S4-logic constructed by Fine. We are going in particular to consider the effect of these on various classes of topological spaces, including first countable, metrizable, hereditarily irresolvable, scattered and Stone spaces.

The second theme concerns versions of the Kuznetsov question for some semantics other than topological. For example, from the point of view of algebraic semantics a natural question is whether every variety of Heyting algebras is generated by complete Heyting algebras. Dually complete Heyting algebras correspond to extremally order-disconnected Esakia spaces, and in particular we will try to outline what do $\mathbf{bb}_2$ and the Shehtman axiom express for Esakia spaces.

Finally we will discuss some senses in which one might try to "stay closer" to the Kripke semantics. This includes consideration of Scott topologies on directed complete partially ordered sets, Beth and Dragalin semantics that realize complete Heyting algebras as algebras of particular upper sets (generated subframes) in an intuitionistic Kripke frame, and the analog of the Kuznetsov question for bi-Heyting algebras.

Our recent work, as well as related results by some other authors will be reviewed.

Admissibility of Π_2-Inference Rules: interpolation, model completion, and contact algebras

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The use of non-standard rules has a long tradition in modal logic starting from the pioneering work of Gabbay [5], who introduced a non-standard rule for irreflexivity. While standard inference rules can be identified with universally quantified Horn formulas, non-standard rules correspond to formulas that allow extra universally quantified variables in their premises. Non-standard rules have been employed in temporal logic in the context of branching time logic [3] and for axiomatization problems [6] concerning the logic of the real line in the language with the Since and Until modalities. General completeness results for modal languages that are sufficiently expressive to define the so-called difference modality have been obtained in [13]. For the use of the non-standard density rule in many-valued logics we refer to [10] and [11].

Recently, there has been a renewed interest in non-standard rules in the context of the region-based theories of space [12]. One of the key algebraic structures in these theories is that of *contact algebras*. These algebras form a discriminator variety, see, e.g., [2]. Compingent algebras are contact algebras satisfying two $\forall\exists$-sentences (aka Π_2-sentences) [2, 4]. De Vries [4] established a duality between complete compingent algebras and compact Hausdorff spaces. This duality led to new logical calculi for compact Hausdorff spaces in [1] for a two-sorted modal language and in [2] for a uni-modal language with a strict implication. Key to these approaches is a development of logical calculi corresponding to contact algebras. In [2] such a calculus is called the *strict symmetric implication calculus* and is denoted by $\mathsf{S^2IC}$. The extra Π_2-axioms of compingent algebras then correspond to non-standard Π_2-rules, which turn out to be admissible in $\mathsf{S^2IC}$. This generates a natural question of investigating admissibility of Π_2-rules in $\mathsf{S^2IC}$ studied in [2] and in general in logical calculi corresponding to varieties of modal algebras. In fact, rather little is known about the problem of recognizing *admissibility* for such non-standard rules, although this problem has already been raised in [13]. This is the question that we address in this paper.

We undertake a systematic study of admissibility of Π_2-rules. We show that *there are tools already available in the literature on modal logic* that can be fruitfully employed for this aim: these tools include algorithms for deciding conservativity [7, 9], as well as algorithms for computing local and global interpolants. We devise three different strategies for recognizing admissibility of Π_2-rules over some system $\mathcal{S}$. The definition of Π_2-rules that we consider is taken from [2] and is close to that of Balbiani et al. [1].

The first strategy applies to a logic $\mathcal{S}$ with the interpolation property. We show that Π_2-rules are effectively recognizable in $\mathcal{S}$ in case $\mathcal{S}$ has the interpolation property and conservativity

*Speaker.

is decidable in $\mathcal{S}$. The second strategy applies to logics admitting local and global uniform interpolants, respectively. Global interpolants are strictly related to model completions and to axiomatizations of existentially closed structures [8], thus establishing a direct connection between Π_2-rules and model-theoretic machinery. Directly exploiting this connection leads to our third strategy. We apply the third strategy to our main case study to show admissibility of various Π_2-rules in $\mathsf{S^2IC}$, thus recovering admissibility results from [2] as special cases (we also show that the admissibility problem for $\mathsf{S^2IC}$ is co-NExpTime-complete). The model completion we use to this aim is that of the theory of contact algebras. Finally, we prove the technically most challenging result of our contribution: that the model completion of contact algebras is finitely axiomatizable. As a consequence of this result we obtain a finite basis for admissible Π_2-rules in $\mathsf{S^2IC}$.

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Ecumenical modal logic

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Recent works about systems where connectives from classical and intuitionistic logics can co-exist in peace warmed the discussion of proof systems for combining logics, called Ecumenical systems by Prawitz and others [5, 4].

In Prawitz' system, the classical logician and the intuitionistic logician would share the universal quantifier, conjunction, negation, and the constant for the absurd, but they would each have their own existential quantifier, disjunction, and implication, with different meanings.

We extended this discussion to alethic K-modalities: using Simpson's meta-logical characterization, necessity is shown to be independent of the viewer, while possibility can be either intuitionistic or classical [1].

We furthermore proposed an internal and pure calculus for ecumenical modalities, where every basic object of the calculus can be read as a formula in the language of the ecumenical modal logic [2, 3].

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Applications of Real Valued Logics to Probabilistic Logics

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Real–valued logics are logical formalisms whose formulas are semantically interpreted as real numbers ($\mathbb{R}$) like 0, $\frac{1}{2}$ and $\sqrt{2}$. By contrast, in Boolean logic, the truth values may only be *true* (1) or *false* (0). Examples of real–valued logics include many well–known fuzzy logics (typically interpreted in $[0,1] \subseteq \mathbb{R}$) investigated for decades in the field of mathematical logic [3], such as e.g. Łukasiewicz logic dating back to the 1930's. More examples can be found in the recent literature in computer science, where real–valued logics have been considered as formalisms for expressing and verifying properties of computer programs and interacting systems (see, e.g., [4, 5]).

In this invited talk at the LATD 2022 ("Logic, Algebra and Truth Degrees") conference, I will discuss how some old standing problems in theoretical computer science could be approached using ideas, methods and techniques developed in the field of mathematical fuzzy logic. The main problem I will discuss dates back at least to the 1982 article of Lehman and Shelah [1]:

Problem: is the SAT problem of the probabilistic logic *pCTL* decidable?

The logic pCTL ("probabilistic Computation Tree Logic" [2, §10.2]) is a logical formalism, with a Boolean semantics (*true*, *false*), for expressing properties of probabilistic transition systems (a.k.a, discrete-time Markov chains). It has important applications as a tool for specifying and verifying properties of computer programs that can use randomisation as in *probabilistic programming* [6]. It is thus remarkable that the answer to the basic problem above is still unknown.

I will argue that progress could be made by studying the problem above for a more expressive (i.e., capable of interpreting pCTL) probabilistic logic having a real–valued semantics instead of a Boolean semantics. The basic intuition is that a real–valued semantics allows for a cleaner mathematical treatment of the problems under consideration.

I will present some of the contributions I have obtained with my coauthors along this line of research. An expressive fixed–point logic, called Łukasiewicz μ-calculus [7, 9], which is capable of interpreting pCTL. A simple real–valued modal logic called *Riesz modal logic* [8], interpreted over probabilistic transition systems, which allows for an elegant, sound and complete axiomatisation. And a hypersequent calculus proof system, sound and complete for the Riesz modal logic, admitting a CUT–elimination theorem [10].

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Łukasiewicz logic, MV-algebras and AF C*-algebraic truth-degrees

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As shown in [2], Łukasiewicz logic $Ł_\infty$ is the only logic arising from a *continuous* $[0,1]$-valued function on the square $[0,1]^2$, having the bare minimum properties of what is usually meant by an implication on a partially ordered set of truth-degrees with a top element. Furthermore, in [4] it is shown that $Ł_\infty$ -formulas code Murray-von Neumann equivalence classes of projections on those *approximately finite dimensional (AF) C*-algebras* (i.e., limits of sequences of finite-dimensional C*-algebras) whose Grothendieck group K_0 is lattice ordered. Many, if not most, preeminent AF algebras in the literature on AF-algebras have this property. For these AF-algebras, Elliott classification [1] and the Γ functor yield a one-one correspondence with countable MV-algebras, the algebras of Łukasiewicz logic. The AF algebra $\mathfrak{M}$ corresponding to the free MV-algebra F_ω on countably many generators inherits from F_ω many properties, [3]. Several uniform and non-uniform recognition problems for projections in these C*-algebras can be decided using the NP-complete logic-algorithmic machinery of Łukasiewicz logic. As shown in [4], in many relevant cases these problems turn out to be polytime decidable.

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On the modal embedding of intuitionistic logic: Gödel's proof of his 1933 conjecture

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Motivated by the idea that intuitionism expresses a modal notion of provability, Gödel defined in 1933 a translation of intuitionistic logic **Int** into the modal logic **S4**. He stated without proof the soundness of the translation and only conjectured its faithfulness. It took some years before McKinsey and Tarski proved the conjecture indirectly using algebraic semantics and completeness of **S4** with respect to closure algebras and of intuitionistic logic with respect to Heyting algebras. The result was later extended in various directions, most notably to embedding results for intermediate logics in modal logics between **S4** and **S5** by Dummett and Lemmon, and to the embeddings of Int into the provability logics **GL** and **Grz** of Gödel-Löb and of Grzegorczyk.

Unlike the proofs of soundness, the syntactical proofs of faithfulness of these embeddings are not entirely straightforward, as witnessed in section 9.2 of [4] for the relatively simple case of the embedding of **Int** into **S4**. In our earlier work we based our approach to such faithfulness results on the formulation of a cut-free sequent system for the logic that is the target of the embedding and offered a modular treatment by the use of labelled sequent calculi for intermediate logics and their modal companions [1, 2] and for infinitary logics [5].

It turned out, however, that Gödel's so far unknown work of 1941 in his book manuscript "Resultate Grundlagen" contains a proof of faithfulness of the translation of intuitionistic into modal logic [3]. The proof is purely syntactic and gives a converse to his translation of 1933 through a propositional version of Barr's theorem. Besides providing the topological semantics of modal logic, later used by McKinsey and Tarski to prove the same embedding result by semantic means, he obtained many other—at the time new—results by the topological semantics, among them that there is an infinity of inequivalent propositions in one variable in intuitionistic logic.

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First-order fuzzy logics and their model theory

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A coherent commitment to fuzzy logic as a viable tool for modeling reasoning with graded predicates needs to go beyond the expressive power provided by propositional and modal languages, and consider as well first-order formalisms. After more than three decades of slow development, the study of first-order fuzzy logics is gaining momentum and giving rise to a corresponding model theory that deviates in crucial aspects from the classical model theory. The aim of this talk is to survey the current state of art and some of its challenges.

A Journey in Intuitionistic Modal Logic: normal and non-normal modalities

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Modal extensions of intuitionistic logic have a long history going back to the work by Fitch in the 40' [6]. Two traditions are now consolidated, called respectively Intuitionistic Modal Logic and Constructive Modal logic. Each of the two has its own motivation and is more natural than the other from some standpoint. In the former tradition originated by Fischer-Servi [5] and systematized by Simpson [9], the basic system is IK, whereas in the tradition of constructive modal logics the two basic systems are Wijesekera' systems WK [10] and the system CK by Bellin et als. [1]. Constructive modal logic are non-normal modal logics. In the classical setting, non-normal modal logics have been studied for a long time for several purposes (see [2], [8]). The observation that constructive modal logics are non-normal and the interest in non-normal modalities in itself leads to the question: which are the intuitionistic analog of classical non-normal modal logic?

It turns out that the framework of intuitionistic non-normal modal logic is richer than the classical one. In particular different interactions between the two modalities $\Box$ and $\Diamond$ give rise to distinct systems; some of them do not have a counterpart in the classical case. The resulting picture is a lattice of 24 non-normal modal logics with an intuitionistic base each of them determined by a cut-free sequent calculus.

Similarly to classical non-normal modal logics, all systems of non-normal intuitionistic modal logic are characterized by a simple neighbourhood semantics. Moreover the neighbourhood semantics helps to understand also Constructive modal logics CK and WK, as it covers also these systems.

The interest of the neighbourhood semantics for constructive modal logic can also be justified from a proof-theoretical perspective, as it witnessed by some recently introduced *unprovability calculi* for these logics. In these calculi, each derivation precisely corresponds to one neighbourhood countermodel, whereas there is no direct correspondence with relational models. This fact confirms the usefulness and the naturalness of neighbourhood semantics for analysing intuitionistic modal logics.

[Joint work with Tiziano Dalmonte and Charles Grellois.]

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From unified correspondence to parametric correspondence

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In this talk, we discuss a research program aimed at establishing systematic connections among the first-order correspondents of Sahlqvist/inductive formulas/inequalities across various relational semantic settings. We will focus on modal reduction principles, and the relational settings we will discuss include crisp and many-valued Kripke frames, and crisp and many-valued polarity-based frames (aka enriched formal contexts). Building on unified correspondence theory, we will discuss a theoretical environment which makes it possible to: (a) compare and inter-relate the various frame correspondents (in different relational settings) of any given Sahlqvist modal reduction principle; (b) recognize when first-order sentences in the frame- correspondence languages of different types of relational structures encode the same "modal content"; (c) meaningfully transfer and represent well known relational properties such as reflexivity, transitivity, symmetry, seriality, confluence, density, across different semantic contexts. These results can be understood as a first step in a research program aimed at making correspondence theory not just (methodologically) unified, but also (effectively) parametric.

Łukasiewicz logic reasons about probability: encoding de Finetti coherence in MV-algebras

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The interconnection between logic, algebra, and probability has played a central role in the study of reasoning since the dawn of modern logic, particularly in the groundbreaking work of Boole [3]. More recent times have seen a flourishing of formal methods and logical approaches to deal with logics capable of reasoning with probabilities. Among them, it is worth recalling the model theoretical approach mainly developed by Keisler [9] and Hoover [8], the more artificial intelligence oriented perspective initiated by Fagin, Halpern, and Megiddo in [4], and the one put forward by Hájek, Esteva, and Godo in [7]. In the latter, which we shall follow, probability is modeled by a modal operator P added to the language of Łukasiewicz logic; formulas of the form $P(\varphi)$ – for φ any classical formula – read as "φ is probable". Interestingly, the logic of [4] and a slight variant of Hájek, Esteva, and Godo's logic have been shown to be syntactically interdefinable, and hence equivalent, in the recent [1].

In joint work with Flaminio, we are concerned with an extension of Hájek, Esteva, and Godo's logic first axiomatized in [6], denoted by $\mathrm{FP}(Ł, Ł)$, that has been recently proved ([5]) to be the logic of *state theory*: a generalization of probability theory for uncertain quantification on Łukasiewicz sentences, introduced by Mundici in [11]. In $\mathrm{FP}(Ł, Ł)$, Łukasiewicz logic plays a twofold role: it is the *inner* logic that represents the formulas that fall under the scope of the modality P (i.e., *events*) and it is also the *outer* logic that reasons on complex probabilistic modal formulas.

We show that, roughly speaking, the modal expansion leading to the logic $\mathrm{FP}(Ł, Ł)$ is not needed to formalize probabilistic reasoning within Łukasiewicz calculus. In order to do so, we use the equivalent algebraic semantics of Łukasiewicz in the sense of [2], MV-algebras. Phrased in this setting, we show that the quasi-equational theory of MV-algebras is expressive enough to encode probabilistic reasoning.

In particular, the categorical duality between rational polyhedra and finitely presented MV-algebras put forward in [10] allows us to encode within Łukasiewicz logic itself the local, finitary, probabilistic information described by the convex rational polyhedra being the geometric interpretation of de Finetti's coherence criterion.

Moreover, leveraging the categorical duality between rational polyhedra and finitely presented MV-algebras, we are able to identify a class of MV-algebras that forms a semantics for $\mathrm{FP}(Ł, Ł)$. These algebras, that will be called *coherent*, form a proper subclass of finitely presented and projective MV-algebras.

Finally, exploiting the interplay between the algebraic and geometric approaches, we are able to study purely logical properties of the logic $\mathrm{FP}(Ł, Ł)$, exploring the connection between logic and probability in the many-valued setting.

[*] Speaker.

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Contributed talks

Logic Beyond Formulas: Designing Proof Systems on Graphs

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Keeping track of relations between objects or events is essential in modeling processes and in verifying their security and privacy properties. For this purpose, relations are encoded by means of formulas in order to use proof theoretical results to design verification tools.

The "happens before" relation [12], providing a partial order between events to express when an event precede another, is crucial when studying distributed systems. Its restriction to series-parallel orders have received a special attention [16, 9, 5], giving rise to a family of non-commutative logics, including pomset logic [14] and BV [10].

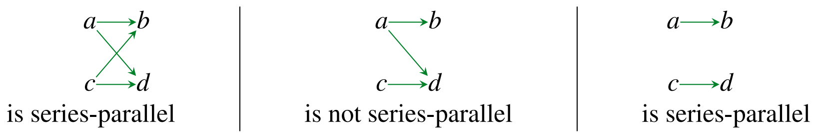
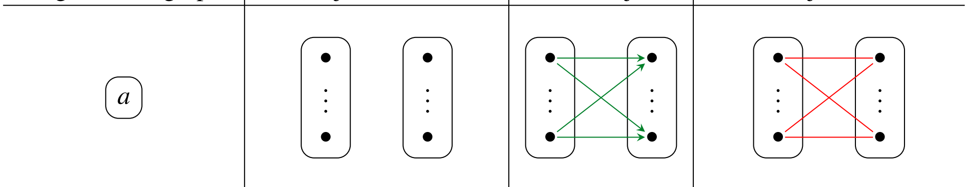


Figure 1: Three partial orders represented by their Hessse diagrams.

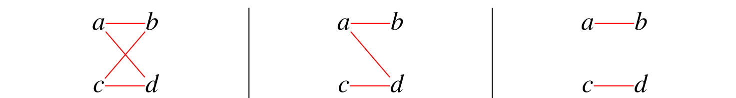
However, relations admitting no series-parallel decomposition [7] cannot be directly treated by the same proof theoretical methods since they require the use of encodings, which create a gap between meaning (semantics) and formal representation (syntax). In fact, the natural correspondence between graphs and formulas provided by the operations below [14, 10] fails as soon as simple topological conditions are not met in the graph representing a relation.

Propositional atom a (single vertex graph)	Disjunction of A and B (disjoint union)	A before B (directed join)	Conjunction of A and B (join)
$[\![a]\!]$	$[\![A]\!] \parr [\![B]\!]$	$[\![A]\!] \lhd [\![B]\!]$	$[\![A]\!] \otimes [\![B]\!]$



(1)

By means of example, consider four processes a, b, c and d where communication between some processes is forbidden because of certain conflicts of interest [6]. Thus, the following pairs cannot communicate: a and b, a and d, and c and d, as shown in the graph below in the center where the edges represent the impossibility of communication between processes.

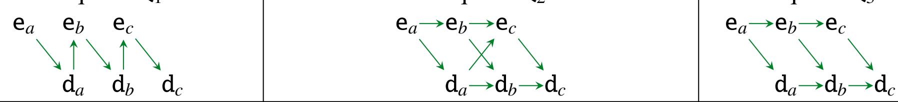


$(a \parr c) \otimes (b \parr d)$	no formula	$(a \otimes b) \parr (c \otimes d)$

(2)

Another example is given by the causality patterns for n-queues, where n is the bound on the number of elements that can be enqueued. These patterns can be represented by the graphs below, where nodes labelled by e_x and d_x respectively represent the enqueuing and dequeuing of the element x (we only represent the first three elements a, b, and c inserted into the queue), and edges represent the "happens before" relation. Among these graphs, only Q_1 and Q_2 are series-parallel graphs and can be directly encoded as formulas. In fact, the graph Q_3, and more in general the causality patterns for n-queues with $n > 2$, cannot.

a 1-queue Q_1	a 2-queue Q_2	a 3-queue Q_3



$\mathsf{e}_a \triangleleft \mathsf{d}_a \triangleleft \mathsf{e}_b \triangleleft \mathsf{d}_b \triangleleft \mathsf{e}_c \triangleleft \mathsf{d}_c$	$\mathsf{e}_a \triangleleft (\mathsf{d}_a \parr \mathsf{e}_b) \triangleleft (\mathsf{d}_b \parr \mathsf{e}_c) \triangleleft \mathsf{d}_c$	no formula	(3)

This contribution, based on joint works with Straßburger, Horne and Mauw [3, 2, 1], is an introduction on the proof theory of proof systems operating on graphs instead of formulas. This line of work aims at defining proof theoretical tools able to directly handle non series-parallel relations as primitive objects of a logic.

In order to design such systems, we use results on graph modular decomposition [13] allowing us to associate abstract syntax trees to graphs, and therefore to generalize the notions of connective and subformula which are fundamental to express desirable proof theoretical notions. After defining a (linear) implication $\multimap$, we define proof systems meeting certain basic desiderata such as the derivability of the general identity ($G \multimap G$ is provable for any graph G), the transitivity of implication (if $G \multimap H$ and $H \multimap K$ are provable, then $G \multimap K$ also is), and analyticity (if G is provable, then G admits a proof containing only its "subformulas" of G) . To this end, we use the open deduction [11] proof formalism (see Figure 2 for an example) based on deep inference [4] since, as observed for the non-commutative logic BV [10, 15], it is not possible to define an analytic sequent calculus for these logics.

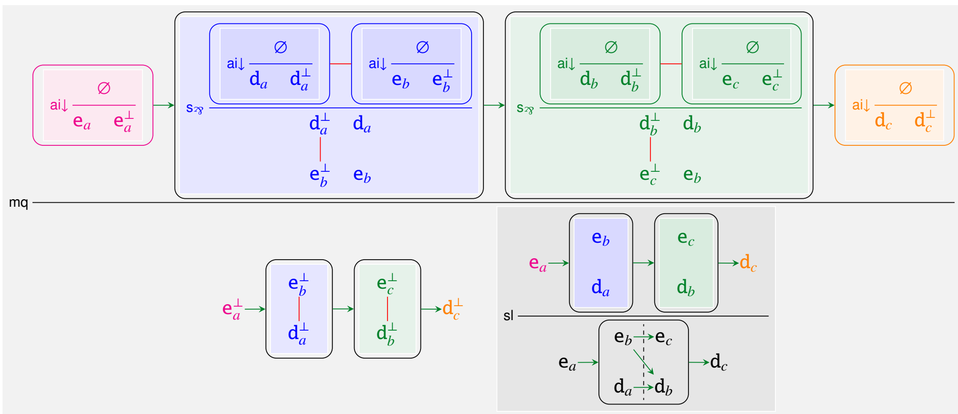


Figure 2: A proof of the graph $\mathsf{Q}_3 \multimap \mathsf{Q}_2$ in the system $\mathsf{GV}^{\mathsf{sl}}$ serving as proof that 3-queues can simulate behaviours of 2-queues. The rule sl slices a directed graph into a "before" and an "after" part by introducing additional directed edges. The rule mq merges the modules of two copies of the same directed graph.

We present the system GS, handling undirected graphs as the ones in Equation 2, and the systems GV and $\mathsf{GV}^{\mathsf{sl}}$ handling graphs with both directed and undirected edges. The system GS defines a conservative extension of the *multiplicative linear logic with mix* [8], while the systems GV and $\mathsf{GV}^{\mathsf{sl}}$ defines conservative extensions of both the graphical logic defined by GS and the non-commutative logic BV. We present the technique developed to prove these results, including the challenges we encountered in proving the analogous of cut-elimination for deep inference systems in the graphical setting. We conclude by recalling related results in proof theory and concurrency theory, their possible applications to verification thanks to their more expressive power, and giving an overview on the the ongoing researches on the topic.

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On Proof Equivalence for Modal Logics

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The proof theory of modal logics has seen enormous progress during the last three decades. In the couse of the years, several proof systems have been defined for modal logics: nested sequents [8, 23, 19], hyper sequents [7, 17] and labeled systems [22, 21]. Moreover, we understand the relation between display calculus and nested sequents [12] and hyper sequents [13], and we have focused proof systems for classical and intuitionistic modal logics [20, 9, 10]. However, none of these formalisms provide an answer the question

When are two proofs the same?

The standard approach to this question is to define a proof as an equivalence class of sequent calculus derivations modulo these permutations. In this regard, focused proof systems considerably reduce the rule permutations in the sequent calculi by grouping rules into phases and could be considered a good candidate for a notion of proof equivalence. Nevertheless, because of the sequential nature of the focused systems, derivations differing for the order of phases are still considered different proofs even if they could be transformed the one into the other via rule permutations, making this approach unsatisfactory to answer this question.

In this talk we propose an answer to this question by defining a notion of proof equivalence for some modal logics based on the syntax of *combinatorial proofs* presented in [4, 2, 5]. For these logics we define the following notion of proof identity:

Two proofs are the same iff they have the same combinatorial proof.

Combinatorial proof are a proof system introduced by Hughes in [15, 16], providing a proof system to address the question of proof identity for classical propositional logic and Hilbert's 24th problem [26, 25]. They capture rules permutations, among which as the ones required in the cut-elimination procedure of sequent calculus, and allow to compare proofs in different proof formalisms (see Figure 1), such as sequent calculus [16, 15], calculus of structures [24], resolution calculus and analytic tableaux [3].

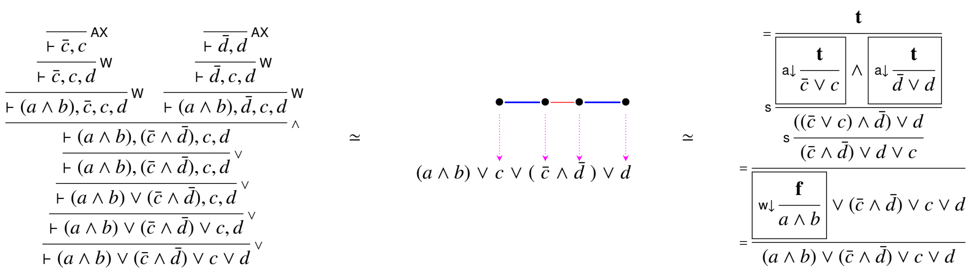

Figure 1: A sequent calculus proof, a combinatorial proof, and a deep inference derivation representing the same proof.

A combinatorial proof is defined as a specific graph homomorphism (the dotted lines in Figure 2) from a graph provided with a partition on its vertices (the blue edges in Figure 2) satisfying certain topological conditions, to a graph encoding a formula (represented by the formula itself in Figure 2).

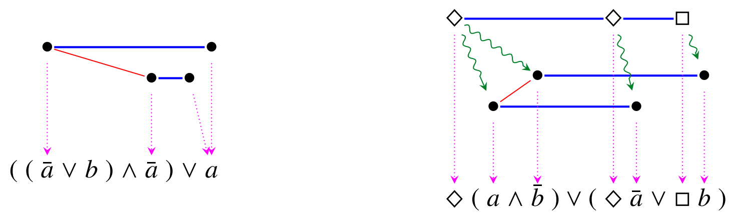

Figure 2: A combinatorial proof of Pierce's law $((\bar{a} \vee b) \wedge \bar{a}) \vee a$ and a combinatorial proof the formula $\Diamond(a \wedge \bar{b}) \vee \Diamond\bar{a} \vee \Box b$, that is, the proof of the axiom $\mathbf{K} := \Box(a \supset b) \supset (\Box a \supset \Box b)$.

More precisely, the graph homomorphism captures the resource management part of a proof, that is, resources erasing and duplications. In case of modalities satisfying the the modal axioms **T** and **4**, the homomorphism also capture these transformations. The partitioned graphs represents a linear proof, that is, a proof where each atom is used exactly once. For modal logic, the partition also gather the modalities corresponding to the same rule for the modal axioms **K** and **D**. The partition satisfies specific topological conditions, guaranteeing the possibility of reconstruct a correct derivation using the information contained by the graph.

In this talk we present combinatorial proofs for the modal logics S4-plane and, more in general, non-normal monotonic logics (see the S4-tesseract in [18]). To show soundness and completeness results, we prove a decomposition theorem for these logics by defining hybrid sequent calculi making use of certain deep inference rules [6]. We then prove that the linear part and the resource management part of the proof constructed using this theorem are in correspondence with the partitioned graph and the graph homomorphism of a combinatorial proof. To conclude, we prove that the topological conditions characterising combinatorial proofs can be checked in polynomial time with respect to the size of the graphs, that is, combinatorial proofs are a proof system in the sense of Cook Reckhow [11].

We then present the syntax of combinatorial proofs for intuitionistic logic [14], which relies on similar methods, but different formula encodings and topological conditions, and the combinatorial proofs for the constructive modal logic of the S4-plane [2]. For these combinatorial proofs, we prove a full completeness result (not achievable in the classical setting), and we highlight their relations with winning innocent strategies from game semantics [1].

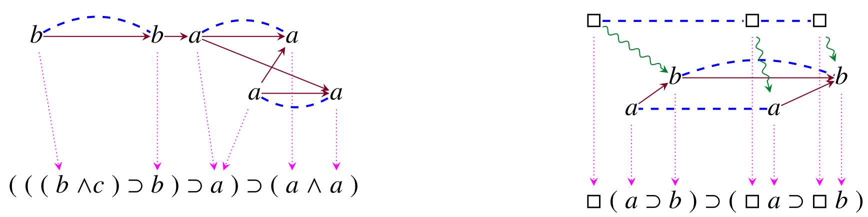

Figure 3: A combinatorial proof of the formula $(((b \wedge c) \supset b) \supset a) \supset (a \wedge a)$ and a combinatorial proof of the modal axiom $\mathbf{K} := \Box(a \supset b) \supset (\Box a \supset \Box b)$.

We conclude the presentation with some remarks on the proof equivalence for modal logics induced by this syntax. In particular, we show which rule permutations of the sequent calculus are captured by the proof equivalence defined by the combinatorial proofs syntax, and we explain why certain rule permutations make the complexity of the equivalence problem for these logic non-polynomial. As consequence of this later result, we rule out the possibility of a syntax which, at the same time, is a proof system (in the sense of [11]) and captures all rule permutations.

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Polyhedral Completeness of Intermediate and Modal Logics

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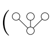
In this talk we investigate a recent semantics for modal and intermediate logics using polyhedra. The starting point is that the collection of open subpolyhedra of a compact polyhedron (of any dimension) forms a Heyting algebra [1, 5, 7]. Precursors of this work are [2], [4] and [3].

Let P be an n-dimensional compact polyhedron. By an *open subpolyhedron* of P we mean a subset of P whose complementary subset in P is a compact polyhedron. Under inclusion order, the poset $\mathsf{Sub}(P)$ of all open subpolyhedra of P is a Heyting algebra [5]. For a propositional formula φ, we say that $P \models \varphi$ if $\mathsf{Sub}(P) \models \varphi$ (i.e., φ is valid in the Heyting algebra $\mathsf{Sub}(P)$). For a class $\mathcal{P}$ of polyhedra we write $\mathcal{P} \models \varphi$ if $P \models \varphi$ for each $P \in \mathcal{P}$.

In this abstract, we think of posets as both Kripke frames and topological spaces given by the Alexandrov topology of upwards-closed subsets. An important connector between polyhedra and posets is the notion of a polyhedral map. Let P be a polyhedron and F be a poset. A function $f : P \to F$ is a *polyhedral map* if the preimage of any open set in F is an open subpolyhedron of P.

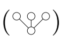
Lemma 1. *If $f : P \to F$ is polyhedral and open, then the logic of P is contained in the logic of F.*

The purpose of this talk is to give a characterisation of the logic of *convex* polyhedra. We focus on the intermediate logic side in this abstract; analogous results hold for modal logic. Our logic PL is axiomatised by Jankov-Fine formulas. To every finite rooted poset Q, we associate a formula $\chi(Q)$, the *Jankov-Fine* formula of Q (also called its *Jankov-De Jongh formula*). This has the property that $F \models \chi(Q)$ if and only if there is no surjective p-morphism f from an open subset $U \subseteq F$ onto Q [6, §9]. Our logic PL is then axiomatised by adding to intuitionistic propositional calculus the Jankov-Fine formulas of two simple posets:

$\mathsf{PL} = \mathsf{IPC} + \chi($  $) + \chi($  $)$

Theorem 2. PL *is the logic of all convex polyhedra.*

Moreover, we obtain the more fine-grained characterisation of the logic of convex polyhedra of dimension at most n by extending the logic of bounded depth n [6, §9]. Let:

$\mathsf{PL}_n = \mathsf{BD}_n + \chi($  $) + \chi($  $)$

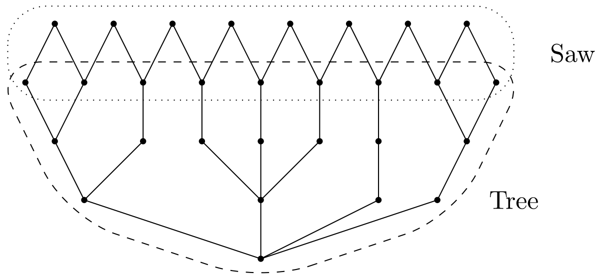


Figure 1: An example sawed tree

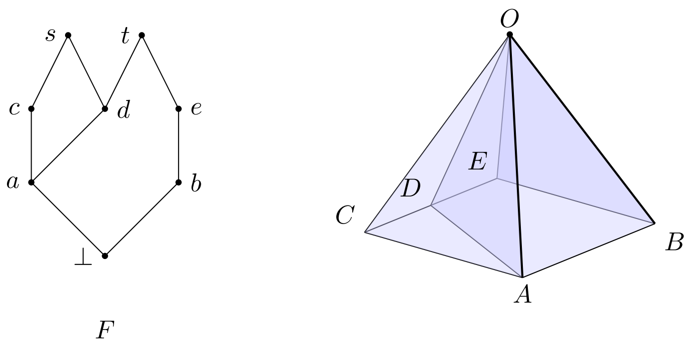


Figure 2: An example of convex geometric realisation

Theorem 3. PL_n *is the logic of all convex polyhedra of dimension at most* n.

We prove Theorem 3 first. This splits into the soundness and completeness direction. For the soundness direction, we show using a geometric argument that the logic that every convex n-dimensional polyhedron is the same, and that PL_n is valid on the simplest example of these: the n-simplex.

The completeness direction splits into two steps. In the first step, we show that any poset F validating PL_n is the p-morphic image of a frame which has a special form, called a *sawed tree*. This consists of a planar tree of uniform height with a 'saw structure' on top. See Figure 1 for an example. In the second step, we show how to realise any sawed tree of height n as a convex polyhedra. This convex polyhedron comes equipped with an open polyhedral map onto the original sawed tree. By Lemma 1, this means that the logic of the convex polyhedron is contained in the logic of F. See Figure 2 for an example of this process of geometric realisation.

Finally, with Theorem 3 established, we make use of a result of Zakharyaschev, which entails that PL is the logic of its finite frames (i.e. it has the finite model property) [8, Corollary 0.11]. This then completes the proof of Theorem 2.

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Strictly join irreducible varieties of residuated lattices

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Substructural logics constitute a large class of logical systems algebraizable in the sense of Blok-Pigozzi, where the semantical characterization of provability of the Lindenbaum-Tarski algebraization extends to a characterization of logical deducibility via the algebraic equational consequence (see [4] for a detailed investigation). Substructural logics encompass classical logic, intermediate logics, fuzzy logics, relevance logics and many other systems, all seen as logical extensions of the Full Lambek calculus $\mathcal{FL}$. As a consequence of algebraizability, all extensions of $\mathcal{FL}$ are also algebraizable, and the lattice of axiomatic extensions is dually isomorphic to the subvariety lattice of the algebraic semantics, given by the variety of FL-algebras. In this work we are interested in the positive fragment of $\mathcal{FL}$ (the system obtained by removing the constant 0, and consequently negation, from the language), $\mathcal{FL}^+$, whose corresponding algebraic semantics is given by the variety of residuated lattices RL.

Our investigation will be carried on in the algebraic framework, and goes in the direction of gaining a better understanding of the lattice of subvarieties of residuated lattices (thus, equivalently, the lattice of axiomatic extensions of the corresponding logics). In particular we study properties, and in some relevant cases we find characterizations, of those varieties that in the lattice of subvarieties are join irreducible or strictly join irreducible. Kihara and Ono showed that, in presence of integrality and commutativity, join irreducibility of a variety is characterized by both a logical property, Halldén completeness, and by an algebraic property of the generating algebras. A substructural logic $\mathcal{L}$ has the **disjunction property** if whenever $\varphi \vee \psi$ is a theorem of $\mathcal{L}$, in symbols $\mathcal{L} \vdash \varphi \vee \psi$, then either $\mathcal{L} \vdash \varphi$ or $\mathcal{L} \vdash \psi$. Likewise a commutative and integral residuated lattice $\mathbf{A}$ is **well-connected** if 1 is join irreducible, i.e. $a \vee b = 1$ implies either $a = 1$ or $b = 1$. A weaker property is Halldén completeness; a logic L is **Halldén complete** if it has the disjunction property w.r.t. to any pair of formulas that have no variables in common. Classical logic is Halldén complete but does not have the disjunction property, thus differentiating the two concepts. As shown in [5] these concepts are connected in commutative integral residuated lattices.

Theorem 0.1. *(Theorem 2.5 in [5]) For a variety V of commutative and integral residuated lattices the following are equivalent:*

1. *$\mathcal{L}_{\mathsf{V}}$ is Halldén complete;*
2. *V is join irreducible;*
3. *$\mathsf{V} = \mathsf{V}(\mathbf{A})$ for some well-connected algebra $\mathbf{A}$.*

How can we extend the definition of well-connected to the nonintegral case? The solution proposed in [5] (and later followed in [2]) is to define a residuated lattice $\mathbf{A}$ to be **well-connected** if 1 is **join prime** in $\mathbf{A}$, i.e. $a \vee b \geq 1$ implies $a \geq 1$ or $b \geq 1$.

We observe straight away that neither integrality nor commutativity are needed to prove that (3) implies (2).

Lemma 0.2. *Let* V *be a variety of residuated lattices; if* $\mathsf{V} = \mathbf{V}(\mathbf{A})$ *for some well-connected algebra* $\mathbf{A} \in \mathsf{V}$ *then* V *is join irreducible.*

The other implications in the general case however do not hold; an analysis of the Kihara-Ono construction reveals at once that there are two critical points. If V is a variety of commutative residuated lattices then:

- every subdirectly irreducible algebra in V is well-connected ([5], Lemma 2.2);
- if W, Z are subvarieties of V axiomatized (relative to V) by $p \geq 1$ and $q \geq 1$ (and we make sure that p and q have no variables in common), then $\mathsf{W} \vee \mathsf{Z}$ is axiomatized relative to V by $p \vee q \geq 1$ ([5], Lemma 2.1).

Both statements are false if we remove commutativity; for the first it is easy to find a finite and integral residuated lattice that is simple but not well-connected (for instance the example below Lemma 3.60 in [4]), while the second fails fore more general reasons discussed at length in [3].

However it is possible to prove a similar result for non-integral, non-commutative subvarieties of RL, characterizing join irreducibility in a large class of residuated lattices, that include for instance all normal varieties, representable varieties, and ℓ-groups. To do so we will adapt to our purpose part of the theory developed in [3] about satisfaction of formulas generated by iterated conjugates.

We define a set $B^n(x, y)$ of equations in two variables x, y for all $n \in \mathbb{N}$ in the following way; let Γ^n be the set of iterated conjugates of *length* n (i.e. a composition of n left and right conjugates) over the appropriate language, with $\Gamma^0 = \{l_1\}$ (for a more general definition, here not needed, see [3], page 229). For all $n \in \mathbb{N}$

$$B^n(x, y) = \{\gamma_1(x) \vee \gamma_2(y) \approx 1 \,:\, \gamma_1, \gamma_2 \in \Gamma^n\}.$$

Let $\mathbf{A}$ be a residuated lattice and $a, b \in A$; we say that **A satisfies** $B^n(a, b)$, in symbols $\mathbf{A} \vDash B^n(a, b)$ if $\mathbf{A}, a, b \vDash B^n(x, y)$. i.e. $\gamma_1(a) \vee \gamma_2(b) = 1$ for all $\gamma_1, \gamma_2 \in \Gamma^n(\mathbf{A})$. We say that $\mathbf{A}$ satisfies $(G_{n,k})$ if for all $a, b \in A$, if $\mathbf{A} \vDash B^n(a, b)$, then $\mathbf{A} \vDash B^k(a, b)$.

This lemma is implicit in [3].

Lemma 0.3. *Let* V *be a variety of residuated lattices and let* $p(x_1, \dots, x_n) \geq 1$, $q(y_1, \dots, y_m) \geq 1$ *be two inequalities not holding in* V. *If* W *and* Z *are the subvarieties axiomatized by* $p \wedge 1 \approx 1$ *and* $q \wedge 1 \approx 1$ *respectively, then* $\mathsf{W} \vee \mathsf{Z}$ *is axiomatized by the set* $B(p, q) = \bigcup_{n \in \mathbb{N}} B^n(p, q)$. *Moreover if* V *satisfies* $(G_{l,l+1})$ *for some* $l \in \mathbb{N}$ *then* $\mathsf{W} \vee \mathsf{Z}$ *is axiomatized by the finite set* $B^l(p, q)$.

We say that a residuated lattice $\mathbf{A}$ is $\mathbf{\Gamma^n}$**-connected** if for all $a, b \in A$, if $\gamma_1(a) \vee \gamma_2(b) = 1$ for all $\gamma_1, \gamma_2 \in \Gamma_n(\mathbf{A})$, then either $a \geq 1$ or $b \geq 1$.

Lemma 0.4. *Let* V *be a variety of residuated lattices that satisfies* $(G_{n,n+1})$. *Then every subdirectly irreducible algebra in* V *is* Γ^n*-connected.*

Finally let's complete the connection with logic. Let $\mathcal{L}$ be a substructural logic over $\mathcal{FL}^+$; given any two axiomatic extensions $\mathcal{L}_1$ and $\mathcal{L}_2$ axiomatized by formulas ϕ and respectively, for any n Lemma 0.3 implicitly gives a set of formulas $B^n_{\mathcal{L}}(\phi, \psi)$ such that $B_{\mathcal{L}}(\phi, \psi) = \bigcup_{n \in \mathbb{N}} B^n_{\mathcal{L}}(\phi, \psi)$ axiomatizes the intersection $\mathcal{L}_1 \cap \mathcal{L}_2$, corresponding to the join of the varieties $\mathsf{V}_{\mathcal{L}_1} \vee \mathsf{V}_{\mathcal{L}_2}$. We say that $\mathcal{L}$ is $\mathbf{\Gamma^n}$**-complete** if for all formulas ϕ andψ which have no variables in common, if $\mathcal{L} \vdash B^n_{\mathcal{L}}(\phi,\psi)$ then either $\mathcal{L} \vdash \phi$ or $\mathcal{L} \vdash \psi$.

Theorem 0.5. *Let* V *be a variety of residuated lattices satisfying* $(G_{n,n+1})$ *for some* $n \in \mathbb{N}$*; then the following are equivalent.*

1. $\mathcal{L}_{\mathsf{V}}$ *is* Γ^n*-complete;*
2. V *is join irreducible;*
3. $\mathsf{V} = \mathbf{V}(\mathbf{A})$ *for some* Γ^n*-connected algebra* $\mathbf{A}$.

Next we point out a corollary of Lemma 0.4 and Theorem 0.5.

Corollary 0.6. *Let* V *be a variety of residuated lattices satisfying* $(G_{n,n+1})$ *for some* $n \in \mathbb{N}$*. If there is a subdirectly irreducible algebra* $\mathbf{A}$ *with* $\mathsf{V} = \mathbf{V}(\mathbf{A})$*, then* V *is join irreducible.*

This is the analogue of Lemma 2.6(2) in [5] and the authors asked if it was possible to invert it; it turns out that our (more general) version is indeed invertible, thus answering their question as well.

Theorem 0.7. *Let* V *be a variety of residuated lattices that satisfies* $(G_{n,n+1})$ *for some* $n \in \mathbb{N}$*; if* V *is join irreducible, then there is a subdirectly irreducible algebra* $\mathbf{B} \in \mathsf{V}$ *such that* $\mathbf{V}(\mathbf{B}) = \mathsf{V}$.

We observe that the above results can (and have been) used to characterize all strictly join irreducible varieties of basic hoops and all linear varieties of basic hoops. Finally we point out that all the material covered in this abstract has appeared in [1].

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Gödel temporal logic

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Abstract

We present investigations of a non-classical version of linear temporal logic (with next, eventually, and henceforth modalities) whose propositional fragment is Gödel–Dummett logic (which is well known both as a superintuitionistic logic and a t-norm fuzzy logic). We define the logic using two natural semantics—a real-valued semantics and a semantics where truth values are captured by a linear Kripke frame—and can show that these indeed define one and the same logic. Although this Gödel temporal logic does not have any form of the finite model property for these two semantics, we are able to prove decidability of the validity problem. The proof makes use of quasimodels, which are a variation on Kripke models where time can be nondeterministic. We can show that every falsifiable formula is falsifiable on a finite quasimodel, which yields decidability. We then strengthen this result to PSPACE-complete. Further, we provide a deductive calculus for Gödel temporal logic with a finite number of axioms and deduction rules, and can show this calculus to be sound and complete for the above-mentioned semantics.

1 Introduction

The importance of temporal logics and, independently, of fuzzy logics in computer science is well established. The potential usefulness of their combination is clear: for instance, it would provide a natural framework for the specification of programs dealing with vague data. Subclassical temporal logics have mostly been studied in the context of here-and-there logic, which allows for three truth values and is the basis for temporal answer set programming [1, 2, 3].

One may, however, be concerned that infinite-valued temporal logics could lead to an explosion in computational complexity, as has been known to happen when combining fuzzy logic with transitive modal logics: these combinations are often undecidable [11], or decidable with only an exponential upper bound being known [4]. As we will see, this need not be the case: the combination of Gödel–Dummett logic with linear temporal logic, which we call Gödel temporal logic (GTL), remains PSPACE-complete, the minimal possible complexity given that classical LTL embeds into it. This is true even when the logic is enriched with the dual implication [10], which has been argued in [5] to be useful for reasoning with incomplete or inconsistent information.

The decidability of GTL is already surprising, as it does not enjoy the finite model property. In fact, GTL possesses two natural semantics, corresponding to whether it is viewed as a fuzzy logic or a superintuitionistic logic. As a fuzzy logic, propositions take values in $[0, 1]$, and truth values of compound propositions are defined using standard operations on the real line. As a superintuitionistic logic, models consist of Kripke structures equipped with a partial order to

*Speaker.

interpret implication intuitionistically and a function to interpret the LTL tenses. Remarkably, the two semantics give rise to the same set of valid formulas.

To overcome the failure of the finite model property, we introduce quasimodels, which do enjoy their own version of the finite model property. Quasimodels are not 'true' models in that the functionality of the 'next' relation is lost, but they give rise to standard Kripke models by unwinding. Similar structures were used to prove upper complexity bounds for dynamic topological logic [6, 7] and intuitionistic temporal logic [8], but they are particularly effective in the setting of Gödel temporal logic, as they yield an optimal PSPACE upper bound.

Finally, we provide a deductive calculus for Gödel temporal logic with a finite number of axioms and deduction rules, and can show this calculus to be sound and complete for the above-mentioned semantics.

2 Syntax and semantics

Fix a countably infinite set $\mathbb{P}$ of propositional variables. Then the **Gödel temporal language** $\mathcal{L}$ is defined by the grammar (in Backus–Naur form):

$$\varphi, \psi := p \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \varphi \Rightarrow \psi \mid \varphi \Leftarrow \psi \mid \bigcirc\varphi \mid \Diamond\varphi \mid \Box\varphi,$$

where $p \in \mathbb{P}$. Here, $\bigcirc$ is read as 'next', $\Diamond$ as 'eventually', and $\Box$ as 'henceforth'. The connective $\Leftarrow$ is coimplication and represents the operator dual to implication [12]. We also use $\bot$ as a shorthand for $p \Leftarrow p$ and $\neg\varphi$ as a shorthand for $\varphi \Rightarrow \bot$.

We now introduce the first of our semantics for the Gödel temporal language: real semantics, which views $\mathcal{L}$ as a fuzzy logic (enriched with temporal modalities). In the definition, $[0, 1]$ denotes the real unit interval.

Definition 1 (real semantics)**.** A **flow** is a pair $\mathcal{T} = (T, S)$, where T is a set and $S \colon T \to T$ is a function. A **real valuation** on $\mathcal{T}$ is a function $V \colon \mathcal{L} \times T \to [0, 1]$ such that, for all $t \in T$, the following equalities hold.

$$\begin{array}{lcl} V(\varphi \wedge \psi, t) = \min\{V(\varphi, t)V(\psi, t)\} & & V(\varphi \vee \psi, t) = \max\{V(\varphi, t), V(\psi, t)\} \\ V(\varphi \Rightarrow \psi, t) = \begin{cases} 1 \text{ if } V(\varphi, t) \le V(\psi, t) \\ V(\psi, t) \text{ otherwise} \end{cases} & & V(\varphi \Leftarrow \psi, t) = \begin{cases} 0 \text{ if } V(\varphi, t) \le V(\psi, t) \\ V(\varphi, t) \text{ otherwise} \end{cases} \\ & V(\bigcirc\varphi, t) = V(\varphi, S(t)) & \\ V(\Diamond\varphi, t) = \sup_{n<\omega} V(\varphi, S^n(t)) & & V(\Box\varphi, t) = \inf_{n<\omega} V(\varphi, S^n(t)) \end{array}$$

A flow $\mathcal{T}$ equipped with a valuation V is a **real (Gödel temporal) model**.

The second semantics, Kripke semantics, views $\mathcal{L}$ as an intuitionistic logic (temporally enriched). Below, define $\vec{S}(w, t) = (w, S(t))$.

Definition 2 (Kripke semantics)**.** A **(Gödel temporal) Kripke frame** is a quadruple $\mathcal{F} = (W, T, \le, S)$ where $(W, \le)$ is a linearly ordered set and (T, S) is a flow. A **Kripke valuation** on $\mathcal{F}$ is a function $[\![\cdot]\!] : \mathcal{L} \to 2^{W \times T}$ such that, for each $p \in \mathbb{P}$, the set $[\![p]\!]$ is *downward closed* in its first coordinate, and the following equalities hold.

$$\begin{array}{c} [\![\varphi \wedge \psi]\!] = [\![\varphi]\!] \cap [\![\psi]\!] \qquad [\![\varphi \vee \psi]\!] = [\![\varphi]\!] \cup [\![\psi]\!] \\ [\![\varphi \Rightarrow \psi]\!] = \{(w, t) \in W \times T \mid \forall v \le w((v, t) \in [\![\varphi]\!] \text{ implies } (v, t) \in [\![\psi]\!])\} \\ [\![\varphi \Leftarrow \psi]\!] = \{(w, t) \in W \times T \mid \exists v \ge w((v, t) \in [\![\varphi]\!] \text{ and } (v, t) \notin [\![\psi]\!])\} \\ [\![\bigcirc\varphi]\!] = \vec{S}^{-1}[\![\varphi]\!] \qquad [\![\Diamond\varphi]\!] = \bigcup_{n<\omega} \vec{S}^{-n}[\![\varphi]\!] \qquad [\![\Box\varphi]\!] = \bigcap_{n<\omega} \vec{S}^{-n}[\![\varphi]\!] \end{array}$$

A Kripke frame $\mathcal{F}$ equipped with a valuation $[\![\cdot]\!]$ is a **(Gödel temporal) Kripke model**.

Definition 3 (validity)**.** A formula φ is **valid** with respect to the real semantics if $V(\varphi, t) = 1$ at all times t in all real models, otherwise φ if **falsifiable**.

A formula φ is **valid** with respect to the Kripke semantics if $[\![\varphi]\!] = W \times T$ in all Kripke models, otherwise φ if **falsifiable**.

We define the logic $\mathsf{GTL}_{\mathbb{R}}$ to be the set of $\mathcal{L}$-formulas that are valid over the class of all flows and the logic $\mathsf{GTL}_{\mathrm{K}}$ to be the set of $\mathcal{L}$-formulas that are valid over the class of all Kripke frames.

3 Results

Using model-theoretic arguments, we can prove the following result.

Theorem 1. *Validity over real and Kripke semantics coincide, that is:* $\mathsf{GTL}_{\mathbb{R}} = \mathsf{GTL}_{\mathrm{K}}$.

We now turn to the question of decidability/complexity of this set of validities. As we mentioned, finite model properties fail; we now make this precise.

Definition 4. The **strong finite model property** is the statement that if $\varphi \in \mathcal{L}$ is falsifiable on a Kripke model, then it is falsifiable on a Kripke model $\mathcal{F} = (W, T, \leq, S, [\![\cdot]\!])$ where both W and T are finite.

The **order finite model property** is the statement that if $\varphi \in \mathcal{L}$ is falsifiable on a Kripke model, then it is falsifiable on a Kripke model $\mathcal{F} = (W, T, \leq, S, [\![\cdot]\!])$ where W is finite.

The **temporal finite model property** is the statement that if $\varphi \in \mathcal{L}$ is falsifiable on a Kripke model, then it is falsifiable on a Kripke model $\mathcal{F} = (W, T, \leq, S, [\![\cdot]\!])$ where T is finite.

Proposition 2. *None of the finite model properties listed in Definition 4 hold. In particular,* $\Diamond(p \Rightarrow \bigcirc p)$ *is falsifiable, yet it is valid over the class of finite Kripke models.*

Figure 1: Left: A Kripke model falsifying $\Diamond(p \Rightarrow \bigcirc p)$; right: W and T are necessarily infinite.

However, by defining and utilising *quasimodels*, we can prove the following.

Theorem 3. *The decision problem of testing validity for* GTL *is decidable.*

Theorem 4. *The decision problem of testing validity for* GTL *is* PSPACE*-complete.*

Finally, we prove the soundness and completeness of the following deductive system.

1. **All (substitution instances of) intuitionistic tautologies**
2. **Axioms and rules of H-B logic (cf. [9]):**

$$\varphi \Rightarrow (\psi \vee (\varphi \Leftarrow \psi)) \qquad \frac{\varphi \Rightarrow \psi}{(\varphi \Leftarrow \theta) \Rightarrow (\psi \Leftarrow \theta)} \qquad \frac{\varphi \Rightarrow \psi \vee \gamma}{(\varphi \Leftarrow \psi) \Rightarrow \gamma}$$

3. **Linearity axioms:** $(\varphi \Rightarrow \psi) \vee (\psi \Rightarrow \varphi) \qquad \neg((\varphi \Leftarrow \psi) \wedge (\psi \Leftarrow \varphi))$
4. **Temporal axioms:**

(a) $\neg \bigcirc \bot$

(b) $\bigcirc(\varphi \vee \psi) \Rightarrow (\bigcirc\varphi \vee \bigcirc\psi)$

(c) $(\bigcirc\varphi \wedge \bigcirc\psi) \Rightarrow \bigcirc(\varphi \wedge \psi)$

(d) $\bigcirc(\varphi \Rightarrow \psi) \Leftrightarrow (\bigcirc\varphi \Rightarrow \bigcirc\psi)$

(e) $\Box(\varphi \Rightarrow \psi) \Rightarrow (\Box\varphi \Rightarrow \Box\psi)$

(f) $\Box(\varphi \Rightarrow \psi) \Rightarrow (\Diamond\varphi \Rightarrow \Diamond\psi)$

(g) $\Box\varphi \Rightarrow \varphi \wedge \bigcirc\Box\varphi$

(h) $\varphi \vee \bigcirc\Diamond\varphi \Rightarrow \Diamond\varphi$

(i) $\Box(\varphi \Rightarrow \bigcirc\varphi) \Rightarrow (\varphi \Rightarrow \Box\varphi)$

(j) $\Box(\bigcirc\varphi \Rightarrow \varphi) \Rightarrow (\Diamond\varphi \Rightarrow \varphi)$

5. **Back–up confluence axiom:** $\bigcirc(\varphi \Leftarrow \psi) \Rightarrow (\bigcirc\varphi \Leftarrow \bigcirc\psi)$

6. **Standard modal rules:**

(a) $\dfrac{\varphi,\ \varphi \Rightarrow \psi}{\psi}$ (b) $\dfrac{\varphi}{\bigcirc\varphi}$ (c) $\dfrac{\varphi}{\Box\varphi}$

Theorem 5. *The smallest set of $\mathcal{L}$-formulas closed under the above axioms and rules is the set* $\mathsf{GTL}_{\mathbb{R}}$ $(= \mathsf{GTL}_{\mathrm{K}})$ *of Gödel temporal logic validities.*

The proof works by building a canonical quasimodel falsifying a given unprovable formula.

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Automorphisms of Product algebras and related varieties

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1 Abstract

Product logic, together with Łukasiewicz and Gödel logics, is considered one of the major truth-functional fuzzy propositional logics. As a matter of fact, all the three major fuzzy logics are schematic extensions of Esteva and Godo's residuated t-norm based logic MTL, whose algebraic semantics is given by the variety of $\mathbb{MTL}$-algebras (see [5]).

The Lindenbaum-Tarski equivalent algebraic semantics of Product logic is given by the variety of Product algebras $\mathbb{P}$. The hierarchy of subvarieties of $\mathbb{MTL}$-algebras forms a huge lattice, dually isomorphic with the lattice of schematic extensions of MTL, ordered by strength. This hierarchy contains $\mathbb{P}$ as a subvariety along with a few subvarieties which turn out to be categorically equivalent with $\mathbb{P}$.

In turns, the subcategories of these varieties formed by their directly indecomposable algebras are equivalent to the category of cancellative hoops, which coincide with negative cones, or equivalently positive cones, of Abelian lattice-ordered groups.

We shall use these equivalences to import results from Abelian lattice-ordered groups and cancellative hoops to product algebras and its categorically equivalent varieties.

A long standing topic in group theory is the study of the group of automorphisms of a group. Automorphisms of algebras in varieties constituting the algebraic semantics of a logic are relevant for logic, too. As a matter of fact, automorphisms of the n-letter Lindenbaum algebra of such a logic (that is, the n-generated free algebra in the corresponding variety) do coincide, up to logical equivalence, with invertible substitutions, and as such have interesting logical properties. For instance, each automorphism preserves tautologies.

In this paper we shall study and describe the groups of automorphisms of finitely generated free Product algebras and of equivalent varieties, namely, the variety generated by perfect MV-algebras, and the variety generated by the t-norm obtained by (connected) rotation of the product t-norm (with an added constant 1/2).

In the literature there are already characterisations of the group of automorphisms of free algebras of logics (that is, invertible substitutions) through self-maps of their finite *prime* spectra, for locally finite varieties of MTL ([1], [2], [3]), or through combinatorial isomorphisms of unimodular simplicial complexes encoding the actions over the *maximal* spectra, for free MV-algebras and cancellative hoops ([4], [6], [7]). As far as we know this is the first work where the two approaches — the purely combinatorial on the finite posets of prime spectra, and the geometric-combinatorial of simplicial complexes over the maximal spectra — are combined in determining the structure of the automorphism groups.

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Cut-elimination for a Hypersequent Calculus for First-order Gödel Logic over $[0,1]$ with $\triangle$

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The family of Gödel logics has originally been introduced by Gödel [9] for the purpose of showing that intuitionistic logic cannot be characterized by finite truth tables. They were first studied in detail by Dummett [8]. Takeuti and Titani [10] based their "intuitionistic fuzzy set theory" on the first-order Gödel logic with truth values from real unit interval $[0,1]$. Nowadays Gödel logics are studied intensively in the context of mathematical fuzzy logic [4]. We will restrict attention to the version $\mathbf{G}^{\forall\triangle}_{[0,1]}$ of first-order Gödel logic over $[0,1]$, where the usual logical connectives are augmented by the projection operator $\triangle$ [1].

We work in a usual first-order language $\mathscr{L}$ with free $(a,\ b,\ \ldots)$ and bound $(x,\ y,\ \ldots)$ variables, predicate and function symbols, logical connectives $\vee$, $\wedge$, $\rightarrow$, a propositional constant $\bot$, quantifiers $\forall$, $\exists$, and a unary operator $\triangle$. Terms and formulas are defined in the usual way. We use $\neg$ as a defined connective; $\neg A \equiv A \rightarrow \bot$.

Definition 1 (Semantics of $\mathbf{G}^{\forall\triangle}_{[0,1]}$)**.** An *interpretation* $\mathfrak{I}$ into $[0,1]$ consists of

1. a nonempty set $|\mathfrak{I}|$, the 'universe' of $\mathfrak{I}$,
2. for each k-ary predicate symbol P, a function $P^{\mathfrak{I}} : |\mathfrak{I}| \rightarrow [0,1]$,
3. for each k-ary function symbol f, a function $f^{\mathfrak{I}} : |\mathfrak{I}| \rightarrow |\mathfrak{I}|$.
4. for each free variable a, a value $a^{\mathfrak{I}} \in [0,1]$.

Let $\mathscr{L}^{\mathfrak{I}}$ be the language $\mathscr{L}$ extended by constant symbols for the elements of $|\mathfrak{I}|$ (so that $d^{\mathfrak{I}} = d$).

Any interpretation $\mathfrak{I}$ extends to an evaluation function yielding a value $\mathfrak{I}(A)$ for any formula A of $\mathscr{L}^{\mathfrak{I}}$. For terms $t = f(u_1, \ldots, u_k)$ we define $\mathfrak{I}(t) = f^{\mathfrak{I}}(\mathfrak{I}(u_1), \ldots, \mathfrak{I}(u_k))$, for atomic formulas $A \equiv P(t_1, \ldots, t_n)$, we define $\mathfrak{I}(A) = P^{\mathfrak{I}}(\mathfrak{I}(t_1), \ldots, \mathfrak{I}(t_n))$, and for composite formulas A we define $\mathfrak{I}(A)$ naturally by:

$$\mathfrak{I}(\bot) = 0 \tag{1}$$

$$\mathfrak{I}(A \wedge B) = \min(\mathfrak{I}(A), \mathfrak{I}(B)) \tag{2}$$

$$\mathfrak{I}(A \vee B) = \max(\mathfrak{I}(A), \mathfrak{I}(B)) \tag{3}$$

$$\mathfrak{I}(A \rightarrow B) = \begin{cases} 1 & \text{if } \mathfrak{I}(A) \leq \mathfrak{I}(B) \\ \mathfrak{I}(B) & \text{if } \mathfrak{I}(A) > \mathfrak{I}(B) \end{cases} \tag{4}$$

$$\mathfrak{I}(\triangle A) = \begin{cases} 1 & \text{if } \mathfrak{I}(A) = 1 \\ 0 & \text{if } \mathfrak{I}(A) < 1 \end{cases} \tag{5}$$

$$\mathfrak{I}(\forall x\, A(x)) = \inf\{\mathfrak{I}(A(u)) : u \in |\mathfrak{I}|\} \tag{6}$$

$$\mathfrak{I}(\exists x\, A(x)) = \sup\{\mathfrak{I}(A(u)) : u \in |\mathfrak{I}|\} \tag{7}$$

From a proof-theoretic perspective, several versions of hypersequent calculi for Gödel logics have been proposed, including systems for first-order logics [2, 3, 6] and systems with $\triangle$ [7]. In [5] the hypersequent calculus **HGIF** is shown to be complete for first-order $[0,1]$-based Gödel logic with $\triangle$. In this contribution we settle the problem of cut-elimination for **HGIF**.

Hypersequents are finite multisets of single-conclusion sequents, written

$$\Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n.$$

The calculus **HGIF** is defined as follows.

Axioms:

$$A \Rightarrow A \qquad \bot \Rightarrow$$

Internal structural rules:

$$\frac{G \mid \Gamma \Rightarrow \Delta}{G \mid A, \Gamma \Rightarrow \Delta}\ iw\Rightarrow \qquad \frac{G \mid \Gamma \Rightarrow}{G \mid \Gamma \Rightarrow A}\ \Rightarrow iw \qquad \frac{G \mid A, A, \Gamma \Rightarrow \Delta}{G \mid A, \Gamma \Rightarrow \Delta}\ ic\Rightarrow$$

External structural rules:

$$\frac{G}{G \mid \Gamma \Rightarrow \Delta}\ ew \qquad \frac{G \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta}{G \mid \Gamma \Rightarrow \Delta}\ ec$$

Logical rules:

$$\frac{G \mid \Gamma \Rightarrow A}{G \mid \neg A, \Gamma \Rightarrow}\ \neg\Rightarrow \qquad \frac{G \mid A, \Gamma \Rightarrow}{G \mid \Gamma \Rightarrow \neg A}\ \Rightarrow\neg$$

$$\frac{G \mid A, \Gamma \Rightarrow \Delta \quad G \mid B, \Gamma \Rightarrow \Delta}{G \mid A \vee B, \Gamma \Rightarrow \Delta}\ \vee\Rightarrow \qquad \frac{G \mid \Gamma \Rightarrow A \quad G \mid \Gamma \Rightarrow B}{G \mid \Gamma \Rightarrow A \wedge B}\ \Rightarrow\wedge$$

$$\frac{G \mid \Gamma \Rightarrow A}{G \mid \Gamma \Rightarrow A \vee B}\ \Rightarrow\vee_1 \qquad \frac{G \mid A, \Gamma \Rightarrow \Delta}{G \mid A \wedge B, \Gamma \Rightarrow \Delta}\ \wedge\Rightarrow_1$$

$$\frac{G \mid \Gamma \Rightarrow B}{G \mid \Gamma \Rightarrow A \vee B}\ \Rightarrow\vee_2 \qquad \frac{G \mid B, \Gamma \Rightarrow \Delta}{G \mid A \wedge B, \Gamma \Rightarrow \Delta}\ \wedge\Rightarrow_2$$

$$\frac{G \mid \Gamma_1 \Rightarrow A \quad G \mid B, \Gamma_2 \Rightarrow \Delta}{G \mid A \to B, \Gamma_1, \Gamma_2 \Rightarrow \Delta}\ \to\Rightarrow \qquad \frac{G \mid A, \Gamma \Rightarrow B}{G \mid \Gamma \Rightarrow A \to B}\ \Rightarrow\to$$

$$\frac{G \mid A(t), \Gamma \Rightarrow \Delta}{G \mid (\forall x)A(x), \Gamma \Rightarrow \Delta}\ \forall\Rightarrow \qquad \frac{G \mid \Gamma \Rightarrow A(a)}{G \mid \Gamma \Rightarrow (\forall x)A(x)}\ \Rightarrow\forall$$

$$\frac{G \mid A(a), \Gamma \Rightarrow \Delta}{G \mid (\exists x)A(x), \Gamma \Rightarrow \Delta}\ \exists\Rightarrow \qquad \frac{G \mid \Gamma \Rightarrow A(t)}{G \mid \Gamma \Rightarrow (\exists x)A(x)}\ \Rightarrow\exists$$

The rules $(\Rightarrow\forall)$ and $(\exists\Rightarrow)$ are subject to eigenvariable conditions: the free variable a must not occur in the lower hypersequent.

Rules for $\triangle$:

$$\frac{G \mid \quad A, \Gamma \Rightarrow \Delta}{G \mid \triangle A, \Gamma \Rightarrow \Delta}\ \triangle\Rightarrow \qquad \frac{G \mid \triangle\Gamma \Rightarrow \quad A}{G \mid \triangle\Gamma \Rightarrow \triangle A}\ \Rightarrow\triangle$$

$$\frac{G \mid \triangle\Gamma, \Gamma' \Rightarrow \Delta}{G \mid \triangle\Gamma \Rightarrow \mid \Gamma' \Rightarrow \Delta}\ \triangle cl$$

Communication:

$$\frac{G \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta \quad G \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta'}{G \mid \Gamma_1 \Rightarrow \Delta \mid \Gamma_2 \Rightarrow \Delta'}\ cm$$

Cut:

$$\frac{G \mid \Gamma \Rightarrow A \quad G \mid A, \Pi \Rightarrow \Lambda}{G \mid \Gamma, \Pi \Rightarrow \Lambda} \; cut$$

Our main result is the following:

Theorem 2 (Cut-Elimination)**.** *Every proof in* **HGIF** *of some hypersequent σ can be transformed into a proof of σ that does not contain applications of* (*cut*)*.*

The problem of cut-elimination in hypersequent calculi is that Gentzen's original method is not suitable due to the lack of a definable mix-rule. This implies that induction on the height and size of the cut-formula does not lead to the desired result, as the contraction rule appears as obstacle. We therefore adopt the so-called Schütte-Tait procedure, where cut-elimination proceeds by iteratively removing the maximal cuts; i.e., applications of *cut*, where the cut-formula is of maximal size. This method of cut-elimination is based on the reduction of one side of the cut without moving the cut-formula. Our adaption of this procedure is that not the highest maximal cut is reduced, but the highest cut with a specific cut-formula is reduced top-down with possibly multiplying the occurrences, but not the number of other maximal cut formulas.

Spelling out details of the cut-elimination procedure requires quite a few technical preparations. Obviously this abstract is not the right place to do so. However we formulate a few interesting corollaries that follow straightforwardly from the proof of Theorem 2.

Corollary 3 (Mid-Hypersequent Theorem)**.** *Let the end-hypersequent of a cut-free proof π contain prenex formulas only. There is a hypersequent σ in π such that, besides structural inferences, all inferences in π ocurring above σ are propositional and all inferences below σ are quantificational.*

Corollary 4. *The prenex fragment of $\mathbf{G}^{\forall\triangle}_{[0,1]}$ admits Skolemization and interpolation.*

The following corollary in essence entails a version of Herbrand's Theorem:

Corollary 5. *Let A be quantifier-free, then the following rule is admissible in* **HGIF***:*

$$\frac{\Rightarrow \triangle\exists x A(x)}{\Rightarrow \exists x \triangle\, A(x)}$$

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Logical Approximations of Qualitative Probability

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Comparative structures provide a natural bridge between the logical and probabilistic representation of uncertainty, of relevance both for the foundations of probability and statistics [5], and AI[6]. Formally, a comparative structure is a pair $(\mathcal{A}, \preceq)$ where $\mathcal{A}$ is a boolean algebra and $\preceq$ is interpreted as a *qualitative probability (relation)* on $\mathcal{A}$, i.e. we write $\theta \preceq \phi$ to say that θ is no-more-probable-than ϕ, for any $\theta, \phi \in \mathcal{A}$. The relations $\theta \approx \phi$ and $\theta \prec \phi$ are defined from $\preceq$ as usual.

Definition 1 (Comparative structure). $(\mathcal{A}, \preceq)$ *is a* comparative structure *if*

1. $\preceq$ *is a total preorder over* $\mathcal{A}$*;*
2. $\bot \prec \top$*;*
3. *if* $\alpha \sqsubseteq \beta$ *then* $\alpha \preceq \beta$ *and*
4. *if* $\alpha \wedge \gamma = \bot$ *and* $\beta \wedge \gamma = \bot$ *then*

$$\alpha \preceq \beta \text{ if and only if } \alpha \vee \gamma \preceq \beta \vee \gamma.$$

Recall that by $\sqsubseteq$ we denote the lattice order of the Boolean algebra, to be distinguished by $\preceq$. The definition above is essentially due to [4] who introduced condition *4.* as the qualitative counterpart of additivity.

De Finetti thought of comparative structures as the logical core of uncertain reasoning, and conjectured that they would be representable by usual probability functions. Let us recall the following.

Definition 2 ((Almost) Representability). *A comparative structure* $(\mathcal{A}, \preceq)$ *is said to be :*

- representable *if there exists a finitely additive probability* P *such that* $\alpha \preceq \beta$ *iff* $P(\alpha) \leq P(\beta)$*;*
- almost representable *if there exists a finitely additive probability* P *such that* $\alpha \preceq \beta$ *implies* $P(\alpha) \leq P(\beta)$*.*

However, even almost representability fails to hold for general comparative structures, as shown in 1959 by [8]. Since then, various authors have proposed additional suitable axioms for establishing (almost) representability, e.g. [10, 9, 11, 7]. All these approaches present however drawbacks, in that they make make strong idealizing assumptions, e.g.:

- $\preceq$ has to contain the classical relation $\sqsubseteq$. But finding out whether $\sqsubseteq$ holds is non-feasible, according to standard assumptions in computational complexity.
- The axioms that need to be added to Definition 1, to obtain representability either postulate [10] that there are arbitrarily fine-grained events to be compared, or impose conditions which are hard to interpret intuitively [9, 11]

*Speaker.

In this work, we address these problems, by developing a sequence of comparative structures, which are meant to be *approximations* of almost representable comparative structures. While each structure in the sequence is not by itself representable, we attain the result only in the limit, provided that the sequence satisfies certain conditions.

Our framework is crucially based on Depth-Bounded Boolean Logics [3, 2], instead of classical logic. These logics are centered around the idea of limiting the applications of the bivalence principle, which holds unboundedly for classical logic. In natural deduction-style the principle may be presented as follows:

$$\dfrac{\begin{array}{cc}[\varphi] & [\neg\varphi] \\ \vdots & \vdots \end{array}}{\quad}\ (\mathrm{PB})$$

This means that to infer the formula ψ, it suffices to infer it both under the assumption that φ is the case and under the assumption that $\neg\varphi$ is the case. The square brackets around the formulas φ and $\neg\varphi$ signal that those are pieces of information assumed for the sake of deriving ψ, but not actually held true (they are *discharged*, in natural deduction terminology). We call this type of information *hypothetical*, in contrast to the *actual* information which an agent may hold as her premises.

[3] introduces a logic $\vdash_0$, which does not allow any manipulation of hypothetical information, i.e. any application of PB, and is defined proof-theoretically by a core set of introduction and elmination rules (Intelim Rules [3]), for each connective, both when occurring positively (as the main connective of a formula) and negatively (in the scope of a negation). The family of Depth-Bounded Boolean Logics $\{\vdash_k\}_{k\in\mathbb{N}}$ is then characterized, for $k > 0$, by allowing, in addition to the rules of $\vdash_0$, at most k nested applications of PB.

Results in [3] show that:

- $\vdash_0 \subset \vdash_1 \subset \cdots \subset \vdash_k \subset \cdots$, so the depth-bounded consequence relations form a hierarchy;
- $\lim_{k\to\infty} \vdash_k = \vdash$, i.e. at the limit, the hierarchy of depth-bounded boolean logics coincides with classical logic;
- for each k, $\vdash_k$ has a polynomial decision procedure.

These properties make these logics a suitable starting point for addressing the problems of comparative structures discussed above. In this work, we shall: define a sequence of bounded comparative structures, based on the sequence of depth-bounded boolean logics; identify the conditions under which the bounded comparative structures are asymptotically representable by a probability measure and, conversely, those conditions under which a representable qualitative probability structure can be approximated.

If time allows, we will also discuss current work in progress, in two directions: determining that resulting approximating comparative structures are tractable, along the lines of work done in [1] for the quantitative case, and devising a decision-theoretic framework, on the model of Savage's [10] which grounds our bounded comparative structures on a corresponding notion of bounded preferences between acts.

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Positive (Modal) Logic Beyond Distributivity

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1 Introduction

Duality between modal algebras and modal spaces on the one hand and Heyting algebras and Esakia spaces on the other have been central to the study of modal and intermediate logics [4, 6]. Many important results such as Sahlqvist canonicity and correspondence use duality [13]. In [5], duality between modal algebras and modal spaces is extended to modal distributive lattices (i.e. with distributive lattices taking the role of Boolean algebras) and modal Priestley spaces. Among other things, this led to Sahlqvist theory for positive distributive modal logic.

When the algebraic side of a duality is based on Boolean algebras or distributive lattices, in the spatial side of the duality one works with the space of prime filters of a given lattice. This no longer works for non-distributive lattices. There have been many attempts to extend a duality for Boolean algebras and distributive lattices to the setting of all lattices, e.g. by Urquhart, Hartonas, Gehrke and van Gool, and Goldblatt (we skip the references for lack of space). While this has proven a fruitful and interesting approach, it is quite different from known dualities for propositional logics such as Stone and Priestley duality. As a consequence, it can be difficult to modify existing tools and techniques from other propositional bases for these dualities.

An approach towards duality for non-distributive meet-semilattices was developed by Hofmann, Mislove and Stralka (HMS) [10], along the same lines of the proof of the Van Kampen-Pontryagin duality for locally compact abelian groups. This was later modified to a duality for lattices by Jipsen and Moshier [12]. In HMS duality the dual space is based not on prime filters, but all (proper) filters of a lattice. This is closely related to the possibility semantics of modal logic (Holiday) and to choice-free duality for Boolean algebras (N. Bezhanishvili and Holliday), where again one works with the space of all proper filters. Such an approach was also developed for ortholattices by Goldblatt [9] and later extended by Bimbo [3].

Here we restrict HMS duality to a Stone type duality for lattices, which in turn we extend to modal lattices. As a result we obtain a new Kripke style semantics for non-distributive positive logic, and Sahlqvist correspondence and completeness results for (modal) non-distributive positive logic with their Kripke-style semantics. We also obtain an alternative proof of Bakers and Hales' result [1] that every variety of lattices is closed under ideal completions and extend this result to varieties of modal lattices. This abstract is based on [7, 2].

*Speaker.

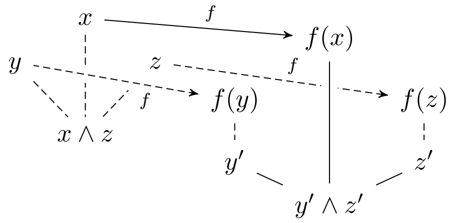
2 Non-distributive positive logic

Let $\mathbf{L}$ be the language of positive logic. We investigate the logic $\mathcal{L}$ consisting of consequence pairs, whose algebraic semantics are (not necessarily distributive) lattices. From a semantic point of view, the move from distributive to non-distributive positive logic is given by:

(Step 1) replace "poset" with "meet-semilattice;"

(Step 2) replace "upset" with "filter."

2.1 Definition. A *lattice Kripke frame* or *L-frame* is a meet-semilattice $(X, \wedge)$. An *L-morphism* from $(X, \wedge)$ to $(X', \wedge')$ is a meet-preserving function $f : (X, \wedge) \to (X', \wedge')$ that satisfies for all $x \in X$ and $y', z' \in X'$: if $y' \wedge z' \leq f(x)$ then there exist $y, z \in X$ such that $y' \leq' f(y)$ and $z' \leq' f(z)$ and $y \wedge z \leq x$ (see figure on the right).



An *L-model* $(X, \wedge, V)$ is an L-frame with a *valuation* that assigns to each proposition letter a filter of $(X, \wedge)$. The interpretation $[\![\phi]\!]$ of $\phi \in \mathbf{L}$ is given by

$$[\![\top]\!] = X \qquad [\![\bot]\!] = \emptyset$$
$$[\![p]\!] = V(p) \qquad [\![\phi \wedge \psi]\!] = [\![\phi]\!] \cap [\![\psi]\!]$$
$$[\![\phi \vee \psi]\!] = [\![\phi]\!] \cup [\![\psi]\!] \cup \uparrow\{x \wedge y \mid x \in [\![\phi]\!], y \in [\![\psi]\!]\}$$

It can be shown that the interpretation of every formula is a filter. We say that a frame $(X, \wedge)$ *validates* the consequence pair $\phi \trianglelefteq \psi$ if $[\![\phi]\!] \subseteq [\![\psi]\!]$ for every model based on it, and write $(X, \wedge) \Vdash \phi \trianglelefteq \psi$. We obtain a duality for the category Lat of lattices by restricting HMS duality.

2.2 Definition. An *HMS space* is a tuple $(X, \wedge, \tau)$ such that $(X, \wedge)$ is a meet-semilattice and (X, τ) is a compact topological space, which additionally satisfies the *HMS separation axiom*:

$$\text{if } x \not\leq y \text{ then there exists a clopen filter } a \text{ such that } x \in a \text{ and } y \notin a.$$

(Here $\leq$ is the order induced by $\wedge$.) An HMS space is called an *L-space* if for every pair of clopen filters a, b, the filter $a \curlyvee b := a \cup b \cup \uparrow\{x \wedge y \mid x \in a, y \in b\}$ is clopen as well.

We write HMS for the category of HMS spaces and continuous meet-semilattice morphisms, and LSpace for the category of L-spaces and continuous L-morphisms.

2.3 Theorem. *We have* $\mathsf{MSL} \equiv^{\mathrm{op}} \mathsf{HMS}$ *[10], and this restricts to* $\mathsf{Lat} \equiv^{\mathrm{op}} \mathsf{LSpace}$.

Here MSL denotes the category of meet-semilattices. Clearly, every L-space $\mathbb{X}$ has an underlying L-frame, denoted by $\kappa\mathbb{X}$. A *clopen valuation* for an L-space is a valuation that assigns to each proposition letter a clopen filter. This gives rise to completeness as usual. Using standard techniques of modal logic (see e.g. [4, Section 3.6]), we obtain the following Sahlqvist results.

2.4 Theorem. *Let* $\phi \trianglelefteq \psi$ *be a consequence pair of* $\mathbf{L}$*-formulae.*

1. $\trianglelefteq \chi$ *locally corresponds to a first-order formula with one free variable.*
2. *For every L-space* $\mathbb{X}$*, if* $\mathbb{X} \Vdash \phi \trianglelefteq \psi$ *then* $\kappa\mathbb{X} \Vdash \phi \trianglelefteq \psi$*.*
3. *If* Γ *is a set of consequence pairs, then* $\mathbf{L}(\Gamma)$ *is sound and complete with respect to the class of L-frames validating all consequence pairs in* Γ*.*

The duality for the Lat gives rise to a new type of lattice completion. We define the F^2-*completion* of a lattice L to be the lattice of all filters of the L-space dual to L. As a consequence of Theorem 3.7 we get the following analogue of [1, Theorem B]:

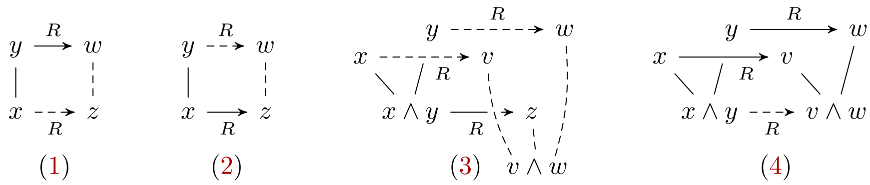
2.5 Theorem. *Every variety of lattices is closed under taking* F^2*-completions.*

3 Modal lattices

We extend the logic from above with modal operators $\Box$ and $\Diamond$. We leave the precise definition of the resulting logic $\mathcal{L}_{\Box\Diamond}$ implicit, and instead give its algebraic semantics in Definition 3.3 below. As a starting point we extend L-frames with an additional relation (used to interpret the modalities), and we stipulate conditions ensuring that every formula is interpreted as a filter.

3.1 Definition. A *modal L-frame* is a tuple $(X, \wedge, R)$ where $(X, \wedge)$ is an L-frame (with induced order $\leq$) and R is a binary relation on X such that:

1. If $x \leq y$ and yRz then there exists a $w \in X$ such that xRw and $w \leq y$;
2. If $x \leq y$ and xRw then there exists a $z \in X$ such that yRz and $w \leq z$;
3. If $(x \wedge y)Rz$ then there exist $v, w \in X$ such that xRv and yRw and $v \wedge w \leq z$;
4. If xRv and yRw then $(x \wedge y)R(v \wedge w)$;
5. For all $x \in X$ there exists an $y \in X$ such that xRy.



A *bounded L-morphism* from $(X, \wedge, R)$ to $(X', \wedge', R')$ is a function $f : X \to X'$ such that $f : (X, \wedge) \to (X', \wedge')$ is an L-morphism and for all $x, y \in X$ and $z' \in X'$:

1. If xRy then $f(x)R'f(y)$;
2. If $f(x)R'z'$ then there exists a $z \in X$ such that xRz and $f(z) \leq z'$;
3. If $f(x)R'z'$ then there exists a $w \in X$ such that xRz and $z' \leq' f(w)$.

A *modal L-model* is a a modal L-frame with a valuation V that assigns to each proposition letter a filter of $(X, \wedge)$. Propositional connectives are interpreted as in Definition 3.1, and

$$[\![\Box\phi]\!] = \{x \in X \mid \forall y \in X, xRy \text{ implies } \mathfrak{M}, y \Vdash \phi\}$$
$$[\![\Diamond\phi]\!] = \{x \in X \mid \exists y \in X \text{ such that } xRy \text{ and } \mathfrak{M}, y \Vdash \phi\}$$

Satisfaction and validity of formulae and consequence pairs are defined as expected.

3.2 Lemma. *The following modal consequence pairs are valid in all modal L-frames:*

$\top \trianglelefteq \Box\top$	$\top \trianglelefteq \Diamond\top$	$\Diamond\bot \trianglelefteq \bot$	*(top and bottom)*
$\Box(p \wedge q) \trianglelefteq \Box p \wedge \Box q$	$\Diamond p \trianglelefteq \Diamond(p \vee q)$		*(monotonicity)*
$\Box p \wedge \Box q \trianglelefteq \Box(p \wedge q)$	$\Diamond p \wedge \Box q \trianglelefteq \Diamond(p \wedge q)$		*(normality and duality)*

3.3 Definition. A *modal lattice* is a tuple $(A, \Box, \Diamond)$ where A is a lattice and $\Box, \Diamond : A \to A$ are maps satisfying the inequalities from Lemma 3.2, with p and q ranging over A and "$\trianglelefteq$" replaced with "$\leq$." With $\Box$- and $\Diamond$-preserving lattice homomorphisms they form the category MLat.

Indeed, $\Diamond$ is not necessarily normal. This resembles the modal extension of intuitionistic logic studied by Kojima [11]. This need not worry us: normality of $\Diamond$ is a Sahlqvist consequence pair, so we can use the results below to restrict to the "fully normal" case. Besides, we have to add seriality ($\top \trianglelefteq \Diamond\top$) because our joins can no longer adequately describe the connection between $\Box$ and $\Diamond$. We obtain a duality for modal lattices by means of L-spaces with relations.

3.4 Definition. A *modal L-space* is a tuple $\mathbb{X} = (X, \wedge, \tau, R)$ such that:

1. $(X, \wedge, \tau)$ is an L-space, R is a binary relation on X, and each $x \in X$ has an R-successor;
2. If a is a clopen filter, then so are $[R]a := \{x \in X \mid R[x] \subseteq a\}$ and $\langle R \rangle a := \{x \in X \mid R[x] \cap a \neq \emptyset\}$;
3. We have xRy iff for all $a \in \mathcal{F}_{clp}\mathbb{X}$, $x \in [R]a$ implies $y \in a$, and $y \in a$ implies $x \in \langle R \rangle a$.

Then it can be shown that $(X, \wedge, R)$ is a modal L-frame. With continuous bounded L-morphisms they form the category $\mathsf{MLSpace}$.

3.5 Theorem. *The duality between* Lat *and* LSpace *lifts to a duality* $\mathsf{MLat} \equiv^{\mathrm{op}} \mathsf{MLSpace}$.

Using standard techniques of modal logic we obtain the following Sahlqvist results.

3.6 Definition. A *boxed atom* is a formula of the form $\Box \cdots \Box p$, with p a proposition letter. A *Sahlqvist antecedent* is a formula made from boxed atoms, $\top$ and $\bot$ by freely using $\wedge$, $\vee$ and $\Diamond$. A *Sahlqvist consequence pair* is a consequence pair $\phi \trianglelefteq \psi$ where ϕ is a Sahlqvist antecedent.

3.7 Theorem. *Let $\phi \trianglelefteq \psi$ be a Sahlqvist consequence pair of $\mathbf{L}_{\Box\Diamond}$-formulae.*

1. *$\trianglelefteq \chi$ locally corresponds to a first-order formula with one free variable.*
2. *For every modal L-space $\mathbb{X}$, if $\mathbb{X} \Vdash \phi \trianglelefteq \psi$ then $\kappa\mathbb{X} \Vdash \phi \trianglelefteq \psi$.*
3. *If Γ is a set of Sahlqvist consequence pairs, then $\mathbf{L}_{\Box\Diamond}(\Gamma)$ is sound and complete with respect to the class of L-frames validating all consequence pairs in Γ.*

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Bi-Intermediate Logics of Trees and Co-trees

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Bi-intuitionistic logic bi-IPC is the conservative extension of intuitionistic logic IPC obtained by adding a new binary connective $\leftarrow$ to the language, called the *co-implication* (or exclusion, or subtraction), which behaves dually to $\rightarrow$. In this way, bi-IPC achieves a symmetry, which IPC lacks, between the connectives $\wedge, \top, \rightarrow$ and $\vee, \bot, \leftarrow$, respectively.

The Kripke semantics of bi-IPC [25] provides a transparent interpretation of co-implication: given a Kripke model $\mathfrak{M}$, a point x in $\mathfrak{M}$, and formulas ϕ, ψ, then

$$\mathfrak{M}, x \models \phi \qquad \iff \exists y \leq x \; (\mathfrak{M}, y \models \phi \text{ and } \mathfrak{M}, y \not\models \psi).$$

Equipped with this new connective, bi-IPC achieves significantly greater expressivity than IPC. For instance, if the points of a Kripke frame are interpreted as states in time, the language of bi-IPC is expressive enough to talk about the past, something that is not possible in IPC. With this example in mind, Wolter extended Gödel's interpretation of IPC into S4 to an interpretation of bi-IPC into tense-S4 [30]. In particular, he proved a version of the Blok-Esakia Theorem [6, 13] stating that the lattice Λ(bi-IPC) *bi-intermediate logics* (i.e., consistent axiomatic[1] extensions of bi-IPC) is isomorphic to that of consistent normal tense logics containing Grz.t, see also [9, 28].

The greater symmetry of bi-IPC with respect to IPC is reflected in the fact that bi-IPC is algebraized in the sense of [7] by the variety bi-HA of *bi-Heyting algebras* [24], i.e., Heyting algebras whose order duals are also Heyting algebras. As a consequence, the lattice Λ(bi-IPC) is dually isomorphic to that of nontrivial varieties of bi-Heyting algebras. The latter, in turn, is amenable to the methods of universal algebra and duality theory because the category of bi-Heyting algebras is dually isomorphic to that of *bi-Esakia spaces* [12], see also [3].

The theory of bi-Heyting algebras was developed in a series of papers by Rauszer and others motivated by the connection with bi-intuitionistic logic. However, bi-Heyting algebras arise naturally in other fields of research as well such as topos theory [20, 21, 26]. Furthermore, the lattice of open sets of an Alexandrov space is always a bi-Heyting algebra, and so is the lattice of subgraphs of an arbitrary graph (see, e.g., [29]) and, similarly, every quantum system can be associated with a complete bi-Heyting algebra [10].

The lattice Λ(IPC) of intermediate logics (i.e., consistent extensions of IPC) has been thoroughly investigated (see, e.g., [8]). On the other hand, the lattice Λ(bi-IPC) of bi-intermediate logics lacks such an in-depth analysis, but for some recent developments see, e.g., [1, 4, 15, 16, 27]. In this paper we shall contribute to fill this gap by studying a simpler, yet nontrivial, sublattice of Λ(bi-IPC): the lattice of consistent extensions of the *bi-intuitionistic linear calculus* (or the bi-Gödel-Dummett's logic),

$$\text{bi-LC} \coloneqq \text{bi-IPC} + (p \rightarrow q) \vee (q \rightarrow p).$$

[1] From now on we will use *extension* as a synonym of *axiomatic extension*.

Notably, the properties of $\Lambda(\mathsf{bi\text{-}IPC})$ and its extensions diverge significantly from those of its intermediate counterpart, i.e., the *intuitionistic linear calculus* (or the Gödel-Dummett's logic) $\mathsf{LC} := \mathsf{IPC} + (p \to q) \vee (q \to p)$ [11, 14].

The choice of bi-LC as a case study was motivated by some of its properties that make it an interesting logic on its own. On the one hand, bi-LC is complete in the sense of Kripke semantics with respect to the class of *co-trees* (i.e., order duals of trees). Moreover, we prove that the bi-intuitionistic logic of linearly ordered Kripke frames is a proper extension of bi-LC. This contrasts with the case of intermediate logics, where LC is both the logic of the class of linearly ordered Kripke frames and of co-trees. Because of this, the language of bi-IPC seems more appropriate to study tree-like structures than that of IPC. Furthermore, because of the symmetric nature of bi-intuitionistic logic, our results on extensions of bi-LC can be extended in a straightforward manner to the extensions of the bi-intermediate logic of trees by replacing in what follows every formula φ by its dual $\neg\varphi^{\partial}$, where φ^{∂} is the formula obtained from φ by replacing each occurrence of $\wedge, \top, \to$ by $\vee, \bot, \leftarrow$ respectively, and every algebra or Kripke frame by its order dual.

On the other hand, the logic bi-LC admits a form of a classical reductio ad absurdum, as we proceed to explain. A deductive system $\vdash$ is said to have a *classical inconsistency lemma* if, for every nonnegative integer n, there exists a finite set of formulas $\Psi_n(p_1, \ldots, p_n)$, which satisfies the equivalence

$$\Gamma \cup \Psi_n(\varphi_1, \ldots, \varphi_n) \text{ is inconsistent in } \vdash \iff \Gamma \vdash \{\varphi_1, \ldots, \varphi_n\}, \tag{1}$$

for all sets of formulas $\Gamma \cup \{\alpha_1, \ldots, \alpha_n\}$ [23] (see also [19, 18]). As expected, the only intermediate logic having a classical inconsistency lemma is CPC (with $\Phi_n := \{\neg(p_1 \wedge \cdots \wedge p_n)\}$). This is far from the case of bi-intermediate logics. For example, we prove that every member of $\Lambda(\mathsf{bi\text{-}LC})$ has a classical inconsistency lemma witnessed by

$$\Phi_n := \{\sim\neg\sim(p_1 \wedge \cdots \wedge p_n)\},$$

where $\neg p$ and $\sim p$ are shorthand for $p \to \bot$ and $\top \leftarrow p$ (see, e.g., [22, Chpt. 4]). Accordingly, logics in $\Lambda(\mathsf{bi\text{-}LC})$ exhibit a certain balance between the classical and intuitionistic behavior of negation connectives.

The logic bi-LC is algebraized by the variety bi-GA of *bi-Gödel algebras*, i.e., the class of bi-Heyting algebras which satisfy Gödel's pre-linearity axiom $(p \to q) \vee (q \to p)$. This is a semi-simple variety of bi-Heyting algebras, hence it follows from [29] that it has a discriminator term, and therefore has EDPC. Moreover, as this variety is axiomatized (relative to bi-HA) by a $\leftarrow$-free formula and has a locally finite Heyting algebra reduct [8], it follows from [22, Chpt. 3] that bi-GA enjoys the finite model property.

As for the geometric models of bi-LC, these take the form of *bi-Esakia co-forests*, i.e., bi-Esakia spaces whose underlying posets are disjoint unions of co-trees. In particular, the dual spaces of the simple bi-Gödel algebras are termed *bi-Esakia co-trees*, and as finite bi-Esakia spaces are equipped with the discrete topology, all finite co-trees can viewed as a bi-Esakia co-trees.

The main contributions of our work can be summarized as follows. We develop a theory of Jankov, subframe and canonical formulas of bi-Gödel algebras. We employ Jankov formulas to obtain a characterization of splittings in $\Lambda(\mathsf{bi\text{-}LC})$ and canonical formulas to uniformly axiomatize all the extensions of bi-LC, cf. [2].

Theorem 1. *If $L \in \Lambda(\mathsf{bi\text{-}LC})$, then:*

1. *L is a splitting logic iff L is the logic of a finite co-tree;*

2. *L is axiomatizable by canonical formulas. Moreover, if L is finitely axiomatized, then L is axiomatizable by finitely many canonical formulas.*

We also use Jankov formulas to show that $\Lambda(\mathsf{bi\text{-}LC})$ has the cardinality of the continuum. This is achieved by means of the construction of an infinite antichain (with respect to the order of being a bi-Esakia morphic image) of finite co-trees[2], and contrasts with the case of $\Lambda(\mathsf{LC})$ which is well known to be a chain of order type $(\omega + 1)^{\partial}$ [8].

Lastly, subframe formulas can be used to describe the fine structure of co-trees, since a bi-Esakia co-tree $\mathcal{X}$ refutes the subframe formula of (the algebraic dual of) a finite co-tree $\mathfrak{F}$ iff $\mathcal{X}$ admits $\mathfrak{F}$ as a subposet. For the present purpose, the interest of subframe formulas is that they help us characterize the locally tabular extensions of bi-LC. This is done in three steps, all relying on the structure of the *finite combs*, a particular class of co-trees depicted in Figure 1.

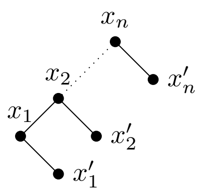

Figure 1: The n-comb $\mathfrak{C}_n$

Firstly, we prove that for all positive integers n, a bi-Esakia co-tree $\mathcal{X}$ admits $\mathfrak{C}_n$ as a subposet iff $\mathfrak{C}_n$ is a bi-Esakia morphic image of $\mathcal{X}$. Secondly, we find a natural bound for the size of m-generated simple bi-Gödel algebras whose bi-Esakia duals do not admit the n-comb $\mathfrak{C}_n$ as a subposet. Finally, by showing that the variety generated by the (algebraic duals of the) finite combs is not locally finite, we derive the following criterion and an immediate corollary:

Theorem 2. *If $L \in \Lambda(\mathsf{bi\text{-}LC})$, then L locally tabular iff $\mathfrak{C}_n$ is not a model of L, for some $n \in \omega$.*

Corollary 3. *A variety V of bi-Gödel algebras is locally finite iff V omits the algebraic dual of a finite comb. Consequently, the variety generated by the duals of the finite combs is the only pre-locally finite variety of bi-Gödel algebras.*

It follows from above that bi-LC is not locally tabular, highlighting yet another contrast with LC, which is well known to be locally tabular [17]. These results are collected in [5].

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Degrees of the finite model property: The antidichotomy theorem

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The most common semantics for modal and superintuitionistic logics is Kripke semantics, which since its inception (in late 1950s and early 1960s), has become one of the main tools in the study of these logics. Logics that are sound and complete with respect to a class of Kripke frames are called *Kripke complete*. A solid body of completeness results for Kripke semantics has been obtained culminating in Sahlqvist canonicity and correspondence results establishing Kripke completeness for a large class of modal logics (see, e.g., [2]; for Sahlqvist theory for superintuitionistic logics see [13]). However, examples of Kripke incomplete logics began to emerge in the 1970s (see, e.g., [6, Ch. 6]).

In order to shed light on the phenomenon of Kripke completeness, Fine [10] associated with each normal modal logic L a cardinal that measures the degree of incompleteness of L. More precisely, let $\mathsf{Fr}(\mathsf{L})$ be the class of Kripke frames validating L. We say that the *degree of incompleteness* of L is the cardinal κ if there are exactly κ normal modal logics L' such that $\mathsf{Fr}(\mathsf{L}') = \mathsf{Fr}(\mathsf{L})$. Notice that all but one of these L' are Kripke incomplete.

Blok [4, 5] gave a very unexpected characterization of degrees of incompleteness, which became known as *Blok's dichotomy theorem*. It states that a normal modal logic L has the degree of incompleteness either 1 or $2^{\aleph_0}$; it is 1 iff L is a join-splitting logic; otherwise it is $2^{\aleph_0}$. We refer to [18] and [16] for a detailed discussion of Blok's dichotomy and its importance in modal logic.

Blok's result implies that some of the most studied normal modal logics, such as $\mathsf{K4}$ (the logic of transitive Kripke frames) and $\mathsf{S4}$ (the logic of reflexive and transitive Kripke frames), have the degree of incompleteness $2^{\aleph_0}$. However, the logics sharing the frames with $\mathsf{K4}$ and $\mathsf{S4}$ are not necessarily normal extensions of $\mathsf{K4}$ or $\mathsf{S4}$. Thus, Blok's result does not automatically transfer to normal extensions of $\mathsf{K4}$ or $\mathsf{S4}$ (or, more generally, to normal extensions of a given normal modal logic L). There have been several attempts to investigate Blok's dichotomy for normal extensions of $\mathsf{K4}$ and $\mathsf{S4}$. However, this remains an outstanding open problem in modal logic [6, Prob. 10.5].

For a logic L, let $\mathsf{Fin}(\mathsf{L})$ be the class of finite Kripke frames validating L. We recall that L has the *finite model property* (*fmp* for short) if L is complete with respect to $\mathsf{Fin}(\mathsf{L})$. Clearly each logic with the fmp is Kripke complete. In addition, every finitely axiomatizable logic with the fmp is decidable by Harrop's theorem (see, e.g., [6, Thm. 16.13]).

Taking inspiration from degrees of incompleteness, it is natural to introduce a similar concept for the fmp. We say that the *degree of fmp* of a logic L is κ provided there exist exactly κ logics L' such that $\mathsf{Fin}(\mathsf{L}') = \mathsf{Fin}(\mathsf{L})$. As with the degree of incompleteness, all but one of such L' lack the fmp. Our main result establishes a complete opposite of Blok's dichotomy theorem for superintuitionistic logics and transitive (normal) modal logics. Namely, we prove that if κ is a nonzero cardinal such that $\kappa \leq \aleph_0$ or $\kappa = 2^{\aleph_0}$, then there exists a superintuitionistic logic (or a transitive modal logic) L whose degree of fmp is κ. Under the Continuum Hypothesis (CH) this implies that each nonzero $\kappa \leq 2^{\aleph_0}$ is realized as the degree of fmp of some superintuitionistic logic (or some transitive modal logic). For this reason, we refer to this result as the *antidichotomy theorem for degrees of fmp.*

In [16, p. 409] Litak asks "if there is any nontrivial completeness notion for which the Blok dichotomy does not hold." Our main result provides such a nontrivial notion for superintuitionistic logics and transitive modal logics. It also provides a solution of a variant of [6, Prob. 10.5] when the degree of incompleteness is replaced with the degree of fmp.

To give more context, we recall that *superintuitionistic logics* are (axiomatic) extensions of the intuitionistic propositional calculus IPC. They have been studied extensively in the literature (see, e.g., [6]). In particular, there is a close connection between superintuitionistic logics and normal extensions of $\mathsf{S4}$. The *Gödel translation* embeds IPC into $\mathsf{S4}$ fully and faithfully [17]. Thus, each superintuitionistic logic L is embedded into a normal extension of $\mathsf{S4}$, called a *modal companion* of L [6, Sec. 9.6]. Each L has many modal companions, but remarkably each L possesses a largest one. By Esakia's theorem [7, 9], the largest modal companion of IPC is the well-known Grzegorczyk logic Grz. Consequently, the largest modal companion of each superintuitionistic logic is a normal extension of Grz, and there exists an isomorphism between the lattice of superintuitionistic logics and the lattice of normal extensions of Grz (the Blok-Esakia theorem) [3, 7].

It is a consequence of Blok's dichotomy theorem that the degree of fmp of a normal extension of the basic modal logic K remains 1 or $2^{\aleph_0}$. Thus, in the lattice of all normal modal logics the dichotomy holds also for the degrees of fmp.

We conclude by discussing how we establish our main results. We first prove the antidichotomy theorem for degrees of fmp of superintuitionistic logics. We heavily rely on Esakia duality for Heyting algebras [8], as well as on Fine's completeness theorem for logics of bounded width [11] and the theory of splittings [6, Sec. 10.5]. Our proof is broken into two parts, depending on whether $\kappa \leq \aleph_0$ or $\kappa = 2^{\aleph_0}$.

When $\kappa \leq \aleph_0$ we work with extensions of the superintuitionistic logic KG, which was introduced by Kuznetsov and Gerčiu [12, 15] and bears their name. The logic KG is the logic of sums of one-generated Heyting algebras, the combinatorics of which allows to construct extensions of KG that lack the fmp [15, 14, 1]. First, we use Fine's completeness theorem to prove that KG is a join-splitting logic over IPC (for a similar result see [14]). Then we develop a method, utilizing a technique of [1], that produces an extension L of KG whose degree of fmp is κ for every nonzero cardinal $\kappa \leq \aleph_0$.

To show that there exist superintuitionistic logics whose degree of fmp is $2^{\aleph_0}$ we work with superintuitionistic logics of finite width. Transitive modal logics of finite width were introduced by Fine [11] who showed that each transitive modal logic of finite width has the fmp. The concept was adapted to superintuitionistic logics by Sobolev [19]. For every positive integer n, let BW_n be the least superintuitionistic logic of width n. We prove that if $n > 2$, the degree of fmp of BW_n is $2^{\aleph_0}$. This is done by a careful analysis of the combinatorics of posets of bounded width.

Under CH our results show that for every nonzero cardinal $\kappa \leq 2^{\aleph_0}$ there exists a superintuitionistic logic L whose degree of fmp is κ, thus yielding the antidichotomy theorem for degrees of fmp of superintuitionistic logics.

Finally, we transfer our results to the setting of modal logics. Following the notation of [6], for a normal modal logic L, let Next L be the lattice of normal extensions of L. We first use the Blok-Esakia theorem to prove our antidichotomy theorem for Next Grz. Next we show that for each normal modal logic $\mathsf{L} \subseteq \mathsf{Grz}$ with the fmp, the antidichotomy theorem holds for Next L provided Grz is a join-splitting logic above L. Since Grz is a join-splitting logic above both $\mathsf{S4}$ and $\mathsf{K4}$ and these logics have the fmp, it follows that the antidichotomy theorem holds for both Next $\mathsf{S4}$ and Next $\mathsf{K4}$.

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Towards a non-integral variant of Łukasiewicz logic

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Łukasiewicz logic is one of the most prominent non-classical logics with very rich metamathematics and with deep connections with many areas of mathematics such as lattice-ordered Abelian groups, continuous model theory, rational polyhedra, Chang MV-algebras, algebraic probability theory, etc. [2,8,10]. One of its defining features is known algebraically as *integrality* (the maximal truth value is the unit of strong conjunction) or proof-theoretically as *weakening* (one can derive $\varphi \to \psi$ from ψ). There are numerous reasons to omit this condition and many of the resulting logics have been studied in the literature under the guise of substructural logics [6]. However none of them can, as of now, boast as deep connections to other areas of logic and mathematics as Łukasiewicz logic.

The existing approaches are arguably either too weak (e.g. the logic of GMV-algebras [7], which drops also commutativity of fusion and semilinearity), or too strong (e.g. Abelian logic [1, 9] which is contraclassical, i.e. proves claims such as $((\varphi \to \psi) \to \psi) \to \varphi)$.

In the recent paper [3], motivated by issues of reasoning with graded predicates, a new logic was proposed based on particular residuated lattice $\boldsymbol{C}$ over the set $\overline{\mathbb{R}}$ of *all* real numbers extended with two limit points $+\infty$ and $-\infty$, in which

- the values in the open interval $(0, 1)$ are still intended as intermediate degrees of truth,
- 1 is intended as the least degree for definitively true statements and will interpret the truth-constant $\mathtt{t}$,
- values above 1 are also intended for definitively true statements and $+\infty$, the largest of these values, will interpret the truth-constant $\top$,
- 0 is intended as the largest degree for definitively false statements and will interpret the truth constant $\mathtt{f}$ (and thus be used to define the negation connective),
- values below 0 are also intended for definitively false statements and $-\infty$, the least of these values, will interpret the truth-constant $\bot$,
- the interpretation of connectives $\to$ and & are defined using the following table; note that for $x, y \in \mathbb{R}$ it is just the untruncated form of the interpretation of these connectives in Łukasiewicz logic:

$x \mathbin{\&^{\boldsymbol{C}}} y$	$y = -\infty$	$y \in \mathbb{R}$	$y = +\infty$
$x = -\infty$	$-\infty$	$-\infty$	$-\infty$
$x \in \mathbb{R}$	$-\infty$	$x + y - 1$	$+\infty$
$x = +\infty$	$-\infty$	$+\infty$	$+\infty$

$x \to^{\boldsymbol{C}} y$	$y = -\infty$	$y \in \mathbb{R}$	$y = +\infty$
$x = -\infty$	$+\infty$	$+\infty$	$+\infty$
$x \in \mathbb{R}$	$-\infty$	$1 - x + y$	$+\infty$
$x = +\infty$	$-\infty$	$-\infty$	$+\infty$

The algebra $\boldsymbol{C} = \langle \overline{\mathbb{R}}, \wedge^{\boldsymbol{C}}, \vee^{\boldsymbol{C}}, \&^{\boldsymbol{C}}, \to^{\boldsymbol{C}}, \mathtt{f}^{\boldsymbol{C}}, \mathtt{t}^{\boldsymbol{C}}, \bot^{\boldsymbol{C}}, \top^{\boldsymbol{C}} \rangle$ is a IUL-*chain* (cf. [5]) and the negation $\neg^{\boldsymbol{C}} x = x \to^{\boldsymbol{C}} \mathtt{f}^{\boldsymbol{C}}$ is the involutive function

$$\neg^{\boldsymbol{C}} x = \begin{cases} 1 - x & \text{for } x \in \mathbb{R} \\ -\infty & \text{for } x = +\infty \\ +\infty & \text{for } x = -\infty \end{cases}$$

The chain $\boldsymbol{C}$ is related to a particular family of standard IUL-chains, denoted as $\mathcal{A}(\circ_{CR}, f)$, which are given by the cross ratio uninorm $\circ_{CR}$:

$$a \circ_{CR} b = \begin{cases} \dfrac{ab}{ab + (1-a)(1-b)}, & \text{if } \{a,b\} \neq \{0,1\}, \\ 0, & \text{otherwise,} \end{cases}$$

and its residuum $\Rightarrow_{CR}$, and by fixing the interpretation of $\mathtt{f}$ as f (clearly $\bot$, $\mathtt{t}$, and $\top$ are interpreted as 0, $\frac{1}{2}$, and 1). We show that $\boldsymbol{C}$ is isomorphic to $\mathcal{A}(\circ_{CR}, f)$ for any $f \in (0, \frac{1}{2})$; e.g. for $f = \frac{1}{3}$ we use the following mapping (for other fs just use a different suitable basis of the logarithm):

$$h\colon [0,1] \to \overline{\mathbb{R}} \qquad \text{defined as} \qquad h(x) = \begin{cases} 1 + \log_2(\frac{x}{1-x}) & \text{if } x \in (0,1) \\ -\infty & \text{if } x = 0 \\ +\infty & \text{if } x = 1 \end{cases}$$

Interestingly enough, up to our knowledge, the logic of $\boldsymbol{C}$ has never been explored (however, the related logic CRL of $\mathcal{A}(\circ_{CR}, \frac{1}{2})$, which conflates the interpretation of $\mathtt{t}$ and $\mathtt{f}$, a clearly an undesired law, has been studied in [5]).

The goal of this contribution is to motivate the logic of $\boldsymbol{C}$ and present its basic mathematical properties in the customary manner of fuzzy logics [4].

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Reasoning with probabilities and belief functions over Belnap–Dunn logic

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Motivation and goal. Probabilities have been developed, mostly in the context of classical logic, to model reasoning based on probabilistic information. Belief functions are a generalisation of probabilities for situations where one is not able to give the exact probability of an event, but an approximation in the terms of an upper/lower bound. They were developed based on classical reasoning to handle situations with incomplete information, but they often produce counter-intuitive results when formalising situations involving contradictory information.

In [8] the authors propose a generalisation of probabilities for reasoning based on Belnap Dunn logic BD. In this paper, we extend their work and propose a generalisation of classical belief functions which is based on BD, and provide two-layered modal logics extending BD for reasoning about probabilities and belief functions. We focus on finite structures, therefore we consider logics over a finite set of atomic propositions and finite algebras.

Representation of uncertainty

Probabilistic reasoning based on incomplete and inconsistent information. The main idea behind Belnap-Dunn logic is to treat positive and negative information independently. A BD model is a tuple $\mathcal{M} = \langle S, v^+, v^- \rangle$ where S is a finite set of states, $v^+, v^- : S \times \text{Prop} \to \{0,1\}$ are valuations encoding respectively the positive and negative information respectively. A *probabilistic model* $\mathcal{M} = \langle S, \mu, v^+, v^- \rangle$ extends a BD model with a probability measure μ on the powerset algebra $\mathcal{P}S$.

Let us call $|\varphi|^+_{\mathcal{M}} = \{s \in \Sigma : v^+(\varphi) = 1\}$ and $|\varphi|^-_{\mathcal{M}} = \{s \in \Sigma : v^-(\varphi) = 1\}$ the positive and negative extensions of φ respectively. They are mutually definable via negation: $|\varphi|^-_{\mathcal{M}} = |\neg\varphi|^+_{\mathcal{M}}$. The *non-standard probability function* based on $\mathcal{M}$ is defined as $\mathsf{p}^+_\mu(\varphi) := \mu(|\varphi|^+_{\mathcal{M}})$ and represents the positive probabilistic evidence for φ. (Positive) non-standard probabilities satisfy the following three axioms:

$$0 \leq \mathsf{p}^+(\varphi) \leq 1 \quad \{\mathsf{p}^+(\varphi) \leq \mathsf{p}^+(\psi) \mid \varphi \vdash_{\mathsf{BD}} \psi\} \quad \mathsf{p}^+(\varphi \wedge \psi) + \mathsf{p}^+(\varphi \vee \psi) = \mathsf{p}^+(\varphi) + \mathsf{p}^+(\psi).$$

We can define negative non-standard probability in a similar manner as $\mathsf{p}^-_\mu(\varphi) = \mu(|\varphi|^-_{\mathcal{M}})$, but from a formal point of view it is sufficient to work with the positive one as $\mathsf{p}^-(\varphi) = \mathsf{p}^+(\neg\varphi)$. Notice that unlike in the classical case, one can no longer prove that $\mathsf{p}^+(\varphi) + \mathsf{p}^+(\neg\varphi) = 1$.

Evidential reasoning via belief functions and Dempster-Shafer combination rule. Here, we generalise the framework introduced in [8] to belief functions. We interpret belief functions on De Morgan algebras and propose a logic to reason with belief function based on BD. Belief functions [9] allow us to reason with the lower approximation of the probability of an event rather than with its exact probability. A *belief function* $\mathsf{bel} : \mathcal{L} \to [0,1]$ on a bounded lattice is a map such that: for every $a, a_1, \ldots a_k, \ldots a_n \in \mathcal{L}$, we have: (1) $\mathsf{bel}(\bot) = 0$ and $\mathsf{bel}(\top) = 1$; (2) for every $a \in \mathcal{L}$, $0 \leq \mathsf{bel}(a) \leq 1$; (3) for every $k \geq 1$, and every $a_1, \ldots, a_k \in \mathcal{L}$,

$$\mathtt{bel}\left(\bigvee_{1 \leq i \leq k} a_i\right) \geq \sum_{\substack{J \subseteq \{1,\ldots,k\} \\ J \neq \varnothing}} (-1)^{|J|+1} \cdot \mathtt{bel}\left(\bigwedge_{j \in J} a_j\right). \tag{1}$$

Recall that a *mass function* $\mathsf{m} : \mathcal{L} \to [0,1]$ on a bounded lattice $\mathcal{L}$ is a map such that: $m(\bot) = 0$ and $\sum_{a \in \mathcal{L}} m(a) = 1$. Every mass function $\mathsf{m} : \mathcal{L} \to [0,1]$ defines a belief function bel_m as follows: for every $a \in \mathcal{L}$, $\mathsf{bel}_\mathsf{m}(a) = \sum_{b \leq a} \mathsf{m}(b)$. Equivalently, for every belief function bel, one can compute its associated mass function m_bel such that the previous equation holds.

Conceptually, mass of a encodes the amount of information provided exactly about a, while the belief of a represents the amount of all the evidence supporting a. Dempster-Shafer combination rule [9] provides a method to aggregate belief functions based on their associated mass functions. Let $m_1, m_2 : \mathcal{L} \to [0,1]$ be two mass functions, their aggregation $m_{1\oplus 2}$ is: $\forall a \in \mathcal{L}$,

$$m_{1\oplus 2}(a) = \frac{1}{1-K} \sum_{b \wedge c = a \neq \bot} m_1(b) m_2(c), \tag{2}$$

where $K = \sum_{b \wedge c = \bot} m_1(b) m_2(c)$. K is a normalisation term that encodes the fact that any fully contradictory information between m_1 and m_2 is ignored. For this reason the combination rule can give very counter intuitive results as demonstrated in the following example.

Example: Two disagreeing doctors. A patient has disease a, b or c and one assumes that he has only one of these diseases. A first expert thinks that the patient has disease a (resp. b and c) with probability 0.9 (resp. 0.1 and 0). This opinion is encoded via the mass function $m_1 : \mathcal{P}(\{a,b,c\}) \to [0,1]$ such that $m_1(a) = 0.9$, $m_1(b) = 0.1$ and $m_1(c) = 0$. A second expert thinks that he has disease a (resp. b and c) with probability 0 (resp. 0.1 and 0.9). This opinion is encoded via the mass function $m_2 : \mathcal{P}(\{a,b,c\}) \to [0,1]$ such that $m_2(a) = 0.9$, $m_2(b) = 0.1$ and $m_2(c) = 0$. Using (2), one gets the following aggregated mass function $\mathsf{m}_{1\oplus 2} : \mathcal{P}(\{a,b,c\}) \to [0,1]$: for every $x \in \mathcal{P}(\{a,b,c\})$, we have $\mathsf{m}_{1\oplus 2}(x) = 1$ if $x = b$, 0 otherwise. This means that $\mathsf{bel}_{1\oplus 2}(b) = 1$ and $\mathsf{bel}_{1\oplus 2}(a) = \mathsf{bel}_{1\oplus 2}(c) = 0$. Therefore while both experts agreed that b was unlikely and that it is highly likely that the patient has an other disease (a or c), one concludes that the patient must have disease b. This results follows from the fact that a, b and c are considered mutually incompatible. Notice that the term K that measure 'contradiction' is equal to 0.99 which means that most of the information given by the experts was ignored.

The same computation over the De Morgan algebra $\mathcal{D}$ generated by $\{a,b,c\}$ leads to a very different conclusion. If one considers the mass functions $m_1 : \mathcal{D} \to [0,1]$ such that $m_1(a \wedge \neg b \wedge \neg c) = 0.9$, $m_1(\neg a \wedge b \wedge \neg c) = 0.1$ and $m_1(\neg a \wedge \neg b \wedge c) = 0$ and $m_2 : \mathcal{D} \to [0,1]$ such that $m_1(a \wedge \neg b \wedge \neg c) = 0$, $m_1(\neg a \wedge b \wedge \neg c) = 0.1$ and $m_1(\neg a \wedge \neg b \wedge c) = 0.9$, one gets the following aggregated mass function $\mathsf{m}_{1\oplus 2}$ (we represent only the elements in $\mathcal{D}$ with non-zero mass):

	$\neg a \wedge b \wedge \neg c$	$a \wedge \neg a \wedge b \wedge \neg b \wedge \neg c$	$a \wedge \neg a \wedge \neg b \wedge c \wedge \neg c$	$\neg a \wedge b \wedge \neg b \wedge c \wedge \neg c$
$\mathsf{m}_{1\oplus 2}$	0.01	0.09	0.81	0.09

Therefore, one reaches the conclusion that one has strong contradictory information regarding a and c and that b is most probably not the case, since $\mathsf{m}_{1\oplus 2}(a \wedge \neg a \wedge \neg b \wedge c \wedge \neg c) = 0.81$. This tells us to search for additional information to figure out whether the patient has disease a or c. This observation leads us to think that in presence of highly conflicting information, it is more relevant to interpret belief functions over De Morgan algebras and therefore to reason with BD rather than with classical logic.

Two-layered Belnapian Logics for probabilities and belief functions

Two-layer logics for reasoning under uncertainty were introduced in [6, 7], and developed further within an abstract algebraic framework by [5] and [2]. Two-layer logics separate two layers of reasoning: the inner layer consists of a logic chosen to reason about events (often classical propositional logic interpreted over sets of possible worlds), the connecting modalities are interpreted by a chosen uncertainty measure on propositions of the inner layer (typically a probability or a belief function), and the outer layer consists of a logical framework to reason about probabilities or beliefs. The modalities apply to inner layer formulas only, to produce outer layer atomic formulas, and they never nest. Logics introduced in [6] use classical propositional logic on the lower layer, and reasoning with linear inequalities on the upper layer. [7] on the other hand uses Łukasiewicz logic on the outer layer, to capture the quantitative, many-valued reasoning about probabilities within a propositional logical language. Building on that idea, and having in mind the two-dimensionality of uncertain information (e.g. positive and negative probabilities), we have introduced a two layer modal logic to reason with non-standard probabilities in [4]. There a two-dimensional extension of Łukasiewicz logic containing an additional De Morgan negation has been proposed. Another two-dimensional extension of Łukasiewicz logic, where De Morgan negation of implication behaves differently, has been introduced in [3], and both logics (which we denote $\text{Ł}^2(\rightarrow)$ and $\text{Ł}^2(\rightarrowtriangle)$) were shown to be coNP complete using constraint tableaux calculi. We provide Hilbert-style axiomatizations for both the logics, which are finitely standard strong complete w.r.t. the twist product of the standard MV algebra $[0,1]_{\text{Ł}}^{\bowtie}$.

In this talk, we consider two-layered logics which use BD as the inner layer, a single unary probability modality P (or a belief modality B) applied to BD formulas, and $\text{Ł}^2(\rightarrow)$ or $\text{Ł}^2(\rightarrowtriangle)$ on the outer layer. The inner formulas are interpreted over a BD model $\mathcal{M} = \langle S, v^+, v^- \rangle$, the atomic modal formulas are interpreted in $[0,1]_{\text{Ł}}^{\bowtie}$ via a given probability (or belief) function on $\mathcal{P}S$ as

$$v^{\mathcal{M}}(P\varphi) = (\mathsf{p}(|\varphi|^+_{\mathcal{M}}), \mathsf{p}(|\varphi|^-_{\mathcal{M}})) \qquad v^{\mathcal{M}}(B\varphi) = (\mathsf{bel}(|\varphi|^+_{\mathcal{M}}), \mathsf{bel}(|\varphi|^-_{\mathcal{M}})),$$

and outer formulas are interpreted in the algebra $[0,1]_{\text{Ł}}^{\bowtie}$ following the semantics of the chosen variant of Ł^2.

We present the resulting two-layer logics via Hilbert-style two-layer axiomatizations of the form $\langle \mathsf{BD}, M_p, \text{Ł}^2 \rangle$, and $\langle \mathsf{BD}, M_b, \text{Ł}^2 \rangle$, and prove their completeness. Here, BD is an axiomatization of the logic BD, and M_p, M_b are sets of modal axioms and rules capturing the behaviour of the P or B modality respectively. Axioms M_p of probability for example look as follows:

$$\vdash_{\text{Ł}^2} P\neg\varphi \leftrightarrow \neg P\varphi \quad \{\vdash_{\text{Ł}^2} P\varphi \rightarrow P\psi \mid \varphi \vdash_{\mathsf{BD}} \psi\} \quad \vdash_{\text{Ł}^2} P(\varphi \vee \psi) \leftrightarrow (P\varphi \ominus P(\varphi \wedge \psi)) \oplus P\psi,$$

where $\oplus, \ominus$ are connectives definable in Ł^2 as in Lukasiewicz logic, corresponding (point-wise) to truncated addition/subtraction on $[0,1]$ respectively.

In the case we deal with belief functions, the first two axiom schemes for B modality stay in place. While expressing the probability axioms in Łukasiewicz logic as above is rather straightforward (see [7, 4]), formulating the belief k-monotonicity axioms is less so. We define a sequence of outer formulas γ_n in propositional letters of the inner language $p_1, \ldots, p_n$ inductively as follows:

$$\gamma_1 \coloneqq Bp_1 \quad \gamma_{n+1} \coloneqq \gamma_n \oplus (Bp_{n+1} \ominus \gamma_n[B\psi : B(\psi \wedge p_{n+1}) \mid B\psi \text{ atoms of } \gamma_n]),$$

where $\gamma_n[B\psi : B(\psi \wedge p_{n+1}) \mid B\psi \text{ modal atoms of } \gamma_n])$ is the result of replacing each modal atom $B\psi$ in γ_n with the modal atom $B(\psi \wedge p_{n+1})$ (semantically, it is a relativisation of the corresponding belief function to the sets $|p_{n+1}|^{+-}$). The n-th belief function axiom (i.e., the n-monotonicity) is expressed by substitution instances (substituting inner formulas for the atomic letters $p_1, \ldots, p_n$) of

$$\alpha_n \coloneqq \gamma_n \to B(\bigvee_{i=1}^{n} p_n).$$

Additionally to Ł^2-based logics, we present a two-layer logic for belief functions based on BD on the lower level, and two-dimensional reasoning about linear inequalities on the upper level. We will relate the two formalism by way of translation, following [1], and we will compare the resulting logic to the one introduced in [10].

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Embeddings of metric Boolean algebras in $\mathbb{R}^N$

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A metric Boolean algebra (see e.g. [1, 2, 3]) consists of a Boolean algebra $\mathbf{A}$, equipped with a strictly positive (finitely-additive) probability measure[1] $m\colon \mathbf{A} \to [0,1]$, which makes $(\mathbf{A}, d_m)$ a metric space, where the distance between any two points $a, b \in A$ is defined as:

$$d_m(a,b) \coloneqq m((a \wedge b') \vee (a' \wedge b)).$$

From a geometrical point of view, it is natural to wonder under which conditions a metric Boolean algebra $(\mathbf{A}, d_m)$, or some of its relevant subspaces, can be isometrically embedded in $\mathbb{R}^N$ (equipped with the Euclidean distance), for a given positive integer N. Actually, for $|A| > 2$, there is no such embedding. However, under the assumption that $\mathbf{A}$ is finite (or, more generally, atomic), it makes sense to restrict the question to the subspace $\mathsf{At}(\mathbf{A})$ of its atoms.

A classical result by Morgan [5] states that a metric space (X, d) embeds in $\mathbb{R}^N$ if and only if it is flat and has dimension less or equal to N, where (X, d) is flat if the determinant of the matrix $M(\vec{x}_n)$, whose generic entry is $M_{ij} = \frac{1}{2}(d(x_0, x_i)^2 + d(x_0, x_j)^2 - d(x_i, x_j)^2)$, is non-negative for every n-simplex (namely every choice of $n+1$ points $\vec{x}_n = \{x_0, \ldots, x_n\}$ in X) and the dimension of (X, d) is the greatest N (if exists) such that there exists a N-simplex with positive determinant.

Given a finite metric Boolean algebra $\mathbf{A}$ with $\mathsf{At}(\mathbf{A}) = \{a_0, a_1, \ldots, a_k\}$, it is easily checked that the matrix $M(\vec{x}_n) = \{M_{ij}\}$, $2 \leq n \leq k$ (introduced in Morgan's theorem) has generic entry

$$M_{ij} = (x_0 + x_i)^2 \delta_{ij} + (x_0^2 + x_0 x_1 + x_0 x_j - x_i x_j)(1 - \delta_{ij}),$$

where $x_\alpha = m(a_\alpha)$ (thus $x_\alpha > 0$, for every $\alpha \in \{0, 1, \ldots, k\}$). Therefore the form of the determinant can be simplified according to the following.

Lemma 1. *Let $M(\vec{x}_n)$, $2 \leq n \leq k$ be the matrix associated to a finite metric atomic Boolean algebra $\mathbf{A}$ with $k+1$ atoms. Then*

$$\det(M(\vec{x}_n)) = 2^{n-1} \left[\left(\sum_{\alpha=0}^{n} x_0 \cdot \dots \cdot \hat{x}_\alpha \cdot \dots \cdot x_n \right)^2 - (n-1) \quad \sum_{\alpha=0}^{n} x_0^2 \cdot \dots \cdot \hat{x}_\alpha^2 \cdots x_n^2 \right) \right],$$

where $\hat{x}_i$ means that x_i has to be omitted.

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*Speaker.

[1] Recall that a *strictly positive* (finitely additive) probability measure over a Boolean algebra $\mathbf{A}$ is a map $m\colon \mathbf{A} \to [0,1]$ such that:

1. $m(\bot) = 1$,
2. $m(a \vee b) = m(a) + m(b)$, for every $a, b \in A$ such that $a \wedge b = \bot$,
3. $m(a) > 0$, for every $a \in A$, $a \neq \bot$.

It follows, for instance, that the space $(\mathsf{At}(\mathbf{A}), d_m)$ of the $k+1$ atoms of a finite metric Boolean algebra such that $m(a_i) = \frac{1}{k+1}$ (for every $a_i \in \mathrm{At}(\mathbf{A})$) embeds in $\mathbb{R}^k$ with the Euclidean metric and that $\det(M(\vec{x}_2)) > 0$.

Upon indicating by $\mathcal{M}_{ind}(\mathrm{At}(\mathbf{A}))$ the space of the (finitely additive) probability measures m such that $(\mathrm{At}(\mathbf{A}), d_m)$ admits an isometric embedding into some Euclidean space $\mathbb{R}^N$, in virtue of Morgan's theorem one has

$$\mathcal{M}_{ind}(\mathrm{At}(\mathbf{A})) = \bigcap_{n=3}^{k} C_n \cap \Pi_k,$$

where $C_n = \{\vec{x} \in \mathbb{R}_+^{k+1} \mid \det M(\vec{x}_n) \geq 0\}$, with $3 \leq n \leq k$ and Π_k is the interior of the standard k-simplex (or probability simplex) of $\mathbb{R}^{k+1}$, namely

$$\Pi_k = \{\vec{x} \in (0,1)^{k+1} \mid \sum_{\alpha=0}^{k} x_\alpha = 1\}.$$

We are interesting in solving the following.

Problem. Study the topology of $\mathcal{M}_{ind}(\mathrm{At}(\mathbf{A}))$ with the topology induced by $(0,1)^{k+1} \subset \mathbb{R}_+^{k+1}$.

In order to get a solution, we first analyze the topology of C_n.

Lemma 2. *For each $3 \leq n \leq k$, the space $C_n \cong H_n \times \mathbb{R}_+^{k-n}$ where H_n is a solid half-hypercone in $\mathbb{R}_+^{n+1}$.*

The solution to the above presented problem is given by the following.

Theorem 3. *Let $k \geq 3$. Then:*

1. *$\mathcal{M}_{ind}(\mathrm{At}(\mathbf{A}))$ is contractible.*
2. *$\mathcal{M}(\mathrm{At}(\mathbf{A})) \setminus \mathcal{M}_{ind}(\mathrm{At}(\mathbf{A}))$ is simply-connected (not contractible).*

In the final part of the talk, we will draw some considerations on the significance of our results for probability theory and on their possibile extensions to metric MV-algebras (MV-algebras equipped with a faithful state [6, 4]).

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Kites and pseudo MV-algebras

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Abstract

We investigate the structure of perfect residuated lattices, focussing especially on perfect pseudo MV-algebras. We show that perfect pseudo MV-algebras can be represented as a generalised version of kites from [8]. We characterise varieties generated by kites and describe the lattice of these varieties as a complete sublattice of the lattice of perfectly generated varieties of perfect pseudo MV-algebras.

1 Introduction

We work in the framework of *residuated lattices*, that is, algebras $\mathbf{A} = (A; \wedge, \vee, \cdot, \backslash, /, 1)$ such that $(A; \wedge, \vee)$ is a lattice, $(A; \cdot, 1)$ is a monoid, and the equivalences

$$y \leq x \backslash z \quad \Leftrightarrow \quad xy \leq z \quad \Leftrightarrow \quad x \leq z / y$$

hold for all $x, y, z \in A$, where the ordering relation $\leq$ is the natural lattice order on A, and multiplication is written as juxtaposition. A residuated lattice expanded by an additional constant 0 is an *FL-algebra* (for **F**ull **L**ambek calculus), and an FL-algebra satisfying $0 \leq x \leq 1$ is an FL_w-algebra.

Our general terminology and notation is that of universal algebra, with a minimum of category theory. For the theory of residuated lattices and all concepts not defined below, we refer the reader to [9], from where we also adopt the convention of using calligraphic letters as variables for arbitrary classes of algebras, and sans-serif for the acronyms of named classes. The acronyms themselves also come from [9], with the exception of the variety of pseudo MV-algebras which we call $\mathsf{\Psi MV}$, and not psMV as in [9].

The present work grew out of an attempt at answering Question 8.4 from [8], concerning a construction of certain algebras called *kites*, most naturally associated with a noncommutative generalisation of BL-algebras known as *pseudo BL-algebras* (see also [6]). The construction has also been used in a broader context of residuated lattices (e.g., [2]) and algebras related to quantum computation (e.g., [7], [1] and [5]). Here we narrow the focus to *pseudo MV-algebras*, and for the most part indeed to *perfect pseudo MV-algebras*. This narrowing of view bears fruit: we obtain several structural results that we believe would be much more difficult to discover (or do not hold at all) in a broader context. We begin however in broad strokes, by establishing a few facts about *perfect residuated lattices*.

Definition 1. *An FL_w-algebra* $\mathbf{A}$ *is* perfect *if there is a homomorphism* $h_{\mathbf{A}} \colon \mathbf{A} \to \mathbf{2}$ *such that for any* $x \in h_{\mathbf{A}}^{-1}(0)$ *and any* $y \in h_{\mathbf{A}}^{-1}(1)$ *the inequality* $x \leq y$ *holds.*

*Speaker.

We say that a variety $\mathcal{V}$ of FL_w-algebras is *perfectly generated* if it is generated by its perfect members. Let $\mathbf{A}$ be an FL-algebra, and $a, b \in A$. The *left conjugate* of $a \in A$ by $b \in A$ is the element $\lambda_b(a) := (b \setminus ab) \wedge 1$ and the *right conjugate* is $\rho_b(a) := (ba \,/\, b) \wedge 1$. A *conjugation polynomial* $\boldsymbol{\alpha}$ over $\mathbf{A}$ is any unary polynomial $(\gamma_{a_1} \circ \gamma_{a_2} \circ \cdots \circ \gamma_{a_n})(x)$ where $\gamma \in \{\lambda, \rho\}$ and $a_i \in A$ for $1 \leq i \leq n$. We write $\mathrm{cPol}(\mathbf{A})$ for the set of all conjugation polynomials over $\mathbf{A}$. For an element $u \in A$, an *iterated conjugate* of u is $\boldsymbol{\alpha}(u)$ for some $\boldsymbol{\alpha} \in \mathrm{cPol}(\mathbf{A})$.

Theorem 1. *A subvariety $\mathcal{V}$ of $\mathsf{FL_w}$ is perfectly generated if and only if $\mathcal{V}$ is nontrivial and satisfies the following identities:*

$$\boldsymbol{\alpha}(x \,/\, x^-) \vee \boldsymbol{\beta}(x^- \,/\, x) = 1, \tag{1}$$

$$\boldsymbol{\alpha}((x \vee x^-) \cdot (y \vee y^-))^- \leq \boldsymbol{\alpha}((x \vee x^-) \cdot (y \vee y^-)), \tag{2}$$

$$x \wedge x^- \leq y \vee y^- \tag{3}$$

for every $\mathbf{A} \in \mathcal{V}$ and all $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathrm{cPol}(\mathbf{A})$.

2 Kites and perfect pseudo MV-algebras

As we already mentioned, ΨMV will stand for the variety of *pseudo MV-algebras*. We will write pfΨMV for the class of perfect members of ΨMV, and PΨMV for the variety generated by pfΨMV. Now we define a generalised version of a kite.

Definition 2. *Let $\mathbf{L}$ be an ℓ-group and $\lambda\colon \mathbf{L} \to \mathbf{L}$ be an automorphism. We define the algebra*

$$\mathcal{K}(\mathbf{L}, \lambda) := (L^- \uplus L^+; \wedge, \vee, \odot, \setminus, /, 0, 1)$$

where $L^- \uplus L^+$ is a disjoint union, $0 := e \in L^+$, $1 := e \in L^-$, and the other operations are given by

$$x \wedge y := \begin{cases} x \wedge y & \text{if } x, y \in L^-, \\ x & \text{if } x \in L^+, y \in L^- \\ y & \text{if } x \in L^-, y \in L^+, \\ x \wedge y & \text{if } x, y \in L^+, \end{cases} \qquad x \vee y := \begin{cases} x \vee y & \text{if } x, y \in L^-, \\ y & \text{if } x \in L^+, y \in L^- \\ x & \text{if } x \in L^-, y \in L^+, \\ x \vee y & \text{if } x, y \in L^+, \end{cases}$$

$$x \odot y := \begin{cases} x \cdot y & \text{if } x, y \in L^-, \\ \lambda(x) \cdot y \vee e & \text{if } x \in L^-, y \in L^+ \\ x \cdot y \vee e & \text{if } x \in L^+, y \in L^-, \\ e & \text{if } x, y \in L^+, \end{cases}$$

$$x \setminus y := \begin{cases} x^{-1} \cdot y \wedge e & \text{if } x, y \in L^-, \\ e & \text{if } x \in L^+, y \in L^- \\ \lambda(x)^{-1} \cdot y \vee e & \text{if } x \in L^-, y \in L^+, \\ x^{-1} \cdot y \wedge e & \text{if } x, y \in L^+, \end{cases} \qquad y \,/\, x := \begin{cases} y \cdot x^{-1} \wedge e & \text{if } x, y \in L^-, \\ e & \text{if } x \in L^+, y \in L^- \\ y \cdot x^{-1} \vee e & \text{if } x \in L^-, y \in L^+, \\ \lambda^{-1}(y \cdot x^{-1}) \wedge e & \text{if } x, y \in L^+, \end{cases}$$

Remark 1. *The negations $x^- := 0 \,/\, x$ and $x^\sim := x \setminus 0$ in $\mathcal{K}(\mathbf{L}, \lambda)$ are given by*

$$x^- = \begin{cases} x^{-1} & \text{if } x \in L^-, \\ \lambda^{-1}(x)^{-1} & \text{if } x \in L^+. \end{cases} \qquad x^\sim = \begin{cases} \lambda(x)^{-1} \in L^+ & \text{if } x \in L^-, \\ x^{-1} \in L^- & \text{if } x \in L^+. \end{cases}$$

In any perfect pseudo MV-algebra $\mathbf{A}$ the normal filter $F_{\mathbf{A}}$ is the universe of a cancellative IGMV-algebra $\mathbf{F_A}$. It is well known that $\mathbf{F_A}$ uniquely determines an ℓ-group $\ell(\mathbf{F_A})$; indeed ℓ is a functor from the category CanIGMV of cancellative IGMV-algebras to the category LG of ℓ-groups. Since pseudo MV-algebras satisfy the identities

$$(x \star y)^{\sim\sim} = x^{\sim\sim} \star y^{\sim\sim} \qquad x^{-\sim\sim} = x^{\sim\sim-}$$

where $\star \in \{\wedge, \vee, \cdot\}$, the map $-^{\sim\sim}$ is an automorphism of $\mathbf{F_A}$. Applying the functor ℓ we lift $-^{\sim\sim}$ to an automorphism

$$\ell^{\approx} \colon \ell(\mathbf{F_A}) \to \ell(\mathbf{F_A})$$

defined as $\ell^{\approx}(-) := \ell(-^{\sim\sim})$.

Theorem 2. *Let $\mathbf{A}$ be a perfect pseudo MV-algebra. Then $\mathbf{A} \cong \mathcal{K}(\ell(\mathbf{F_A}), \ell^{\approx})$.*

It was shown in [4] that perfect MV-algebras are categorically equivalent to Abelian ℓ-groups, and in [3] the result was generalised to a categorical equivalence between *symmetric* perfect pseudo MV-algebras and ℓ-groups. We generalise both results below.

Theorem 3. *The categories of perfect pseudo MV-algebras, and of ℓ-groups with a distinguished automorphism, are equivalent. If the distingushed automorphism is the identity, the equivalent category is that of symmetric perfect pseudo MV-algebras.*

3 Varieties generated by kites

For any ℓ-group $\mathbf{L}$, and any bijection $\beta \colon B \to B$, a very natural automorphism $\lambda \colon \mathbf{L}^B \to \mathbf{L}^B$ is induced by taking $\lambda(x(i)) := x(\beta(i))$ for each $i \in B$. Then $\mathcal{K}(\mathbf{L}^B, \lambda)$ is a perfect pseudo MV-algebra.

Definition 3. *A monounary algebra $\mathbf{B} = (B; \beta)$ where β is a bijection on B will be called a* B-cycle*. Homomorphisms of B-cycles are maps $f \colon \mathbf{B} \to \mathbf{C}$ satisfying $f \circ \lambda^{\mathbf{B}} = \lambda^{\mathbf{C}} \circ f$. Objects of the category* BC *are B-cycles and arrows are homomorphisms.*

The definition below is equivalent to the original definition of a kite from [8].

Definition 4. *Let $\mathbf{B} = (B; \beta)$ be a B-cycle and $\mathbf{L}$ and ℓ-group. A* kite *over $\mathbf{B}$ and $\mathbf{L}$ is the algebra*

$$\mathcal{K}_{\mathbf{B}}(\mathbf{L}) := \mathcal{K}(\mathbf{L}, \lambda)$$

where $\lambda \colon \mathbf{L}^B \to \mathbf{L}^B$ is the automorphism given by $\lambda(x(i)) = x(\beta(i))$ for any $i \in B$.

We write $\Lambda(\mathcal{V})$ for the lattice of subvarieties of a variety $\mathcal{V}$, and $\Lambda^+(\mathcal{V})$ for the poset of nontrivial subvarieties of $\mathcal{V}$. Since the variety BA of Boolean algebras is the unique atom of $\Lambda(\mathsf{P\Psi MV})$, we have that $\Lambda^+(\mathsf{P\Psi MV})$ is a (complete algebraic) sublattice of $\Lambda(\mathsf{P\Psi MV})$.

For any pseudo MV-algebra $\mathbf{A}$, the operation $-^{\approx}$ is a bijection on A, so for any $\mathbf{A}$ we define the dimension of $\mathbf{A}$ to be $dim(-^{\approx})$. From now on, $\mathbb{D}$ will stand for the lattice $(\mathbb{N}; |)$ of natural numbers under the divisibility ordering.

Definition 5. *Let $\mathbf{A} \in \mathsf{PMV}$ and $\mathcal{V} \in \mathsf{\Psi MV}$. Then*

1. $dim(\mathbf{A}) := dim(-^{\approx})$,
2. $dim(\mathcal{V}) := \min^{\mathbb{D}}\{dim(\mathbf{A}) \mid n : \text{ for all } \mathbf{A} \in \mathcal{V}\}$,

3. $\mathsf{P\Psi MV}_n := \mathsf{P\Psi MV} \cap \mathrm{Mod}\{\lambda^n(x) = x\}$*, for any* $n \in \mathbb{D}$*.*

It is immediate that $\mathsf{P\Psi MV}_n$ defined in (3) is the largest subvariety of $\mathsf{P\Psi MV}$ of dimension n. Moreover, for all $n, m \in \mathbb{N}$ we have

$$\mathsf{P\Psi MV}_n \subseteq \mathsf{P\Psi MV}_m \text{ if and only if } n \mid m$$

so in particular $\mathsf{P\Psi MV}_0 = \mathsf{P\Psi MV}$.

Definition 6. *We define two pairs of maps*

$$\psi\colon \Lambda(\mathsf{P\Psi MV}) \to \Lambda(\mathsf{CanIGMV}), \text{ where } \psi(\mathcal{V}) = V\{\mathbf{F_A} : \mathbf{A} \in \mathcal{V}_{pf}\},$$
$$\Psi\colon \Lambda(\mathsf{P\Psi MV}) \to \Lambda(\mathsf{CanIGMV}) \times \mathbb{D}, \text{ where } \Psi(\mathcal{V}) = (\psi(\mathcal{V}),\ dim(\mathcal{V})),$$

for any $\mathcal{V} \in \Lambda(\mathsf{P\Psi MV})$ *and*

$$\delta\colon \Lambda(\mathsf{CanIGMV}) \to \Lambda(\mathsf{P\Psi MV}), \text{ where } \delta(\mathcal{V}) = V\{\mathbf{A} \in \mathsf{pf\Psi MV} : \mathbf{F_A} \in \mathcal{V}\},$$
$$\Delta\colon \Lambda(\mathsf{CanIGMV}) \times \mathbb{D} \to \Lambda(\mathsf{P\Psi MV}), \text{ where } \Delta(\mathcal{V}, n) = \delta(\mathcal{V}) \cap \mathsf{P\Psi MV}_n,$$

for any $\mathcal{V} \in \Lambda(\mathsf{CanIGMV})$ *and* $n \in \mathbb{D}$*.*

Theorem 4. *Let* $\mathcal{V} \in \Lambda(\mathsf{P\Psi MV})$*. The following are equivalent.*

1. $\mathcal{V}$ *is generated by kites.*
2. $\mathcal{V} = \Delta\Psi(\mathcal{V})$*.*
3. $\mathcal{V} = \Delta(\mathcal{W}, n)$ *for some* $\mathcal{W} \in \Lambda(\mathsf{CanIGMV})$ *and some* $n \in \mathbb{D}$*.*

We end by a more detailed description of the lattice of varieties generated by kites. Incidentally, it answers Questions 8.1 and 8.2 from [8] insofar as they apply in this context.

Theorem 5. *Let* $\mathbb{K}$ *be the lattice of subvarieties of* $\mathsf{P\Psi MV}$ *generated by kites.*

$$\mathbb{K} \cong \mathbf{1} \oplus \big(\Lambda^+(\mathsf{CanIGMV}) \times \mathbb{D}\big) \cong \mathbf{1} \oplus \big(\Lambda^+(\mathsf{LG}) \times \mathbb{D}\big)$$

where $\mathbf{1}$ *is the trivial lattice and* $\oplus$ *is the operation of ordinal sum.*

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Hereditary Structural Completeness over K4

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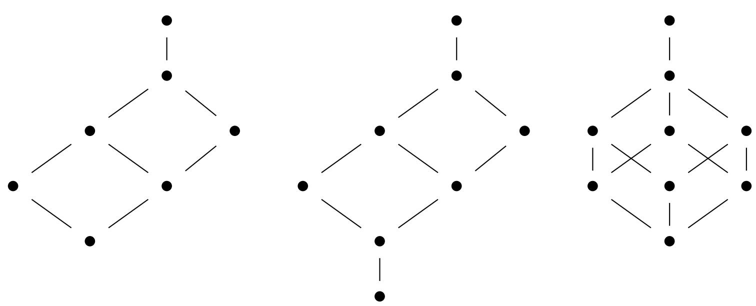
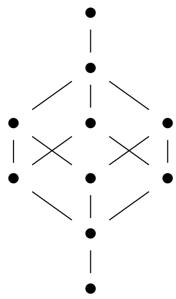
In deductive systems a rule is said to be *admissible* if the tautologies of the system are closed under its applications and *derivable* if the rule itself holds in the system [5]. Whilst every derivable rule for a system is admissible whether the converse holds, that is whether every admissible rule is derivable, varies between deductive systems. As one might expect in the classical propositional calculus (CPC) this converse holds, but it fails for many non-classical systems including the intuitionistic propositional calculus (IPC) [1]. A classical problem is to determine which deductive systems are *structurally complete*, that is share with CPC the property that all admissible rules are derivable. Early investigations suggestion that even if a full characterisation of the structurally complete modal and intuitionistic logics was out of reach it might be possible to precisely characterise the *hereditarily structurally complete* (HSC) systems, those which are not only themselves structurally complete but whose finitary extensions are too. This proved a fruitful question, Citkin [3] produced a characterisation for intermediate logics and Rybakov [6, 7] did so for transitive modal logics. Both these characterisations take a similar form.

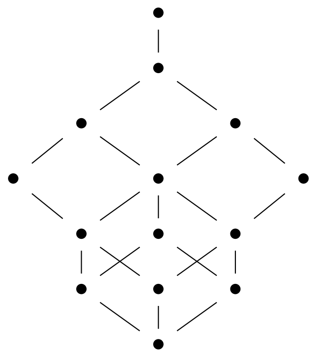
Citkin's Theorem An intermediate logic is HSC iff the variety of Heyting algebras associated with it omits the following five finite algebras [3].



C_1 C_2 C_3

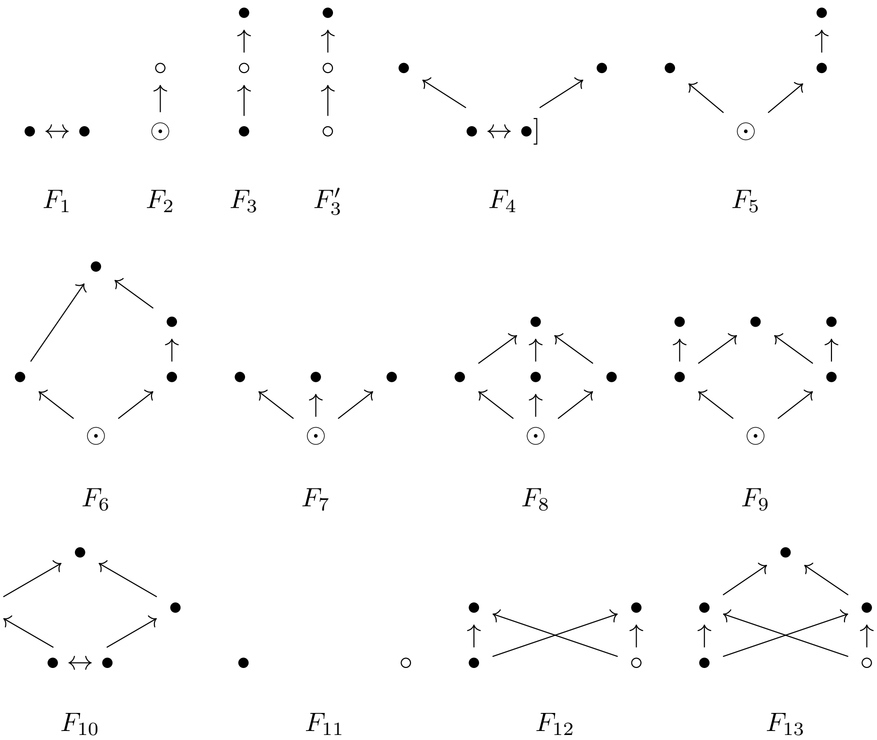


C_4



C_5

Rybakov's Theorem A transitive modal logic is HSC iff it is not included in the logic of a list of 20 frames [7, pg 274].



In the above diagrams • represents a reflexive point, ∘ an irreflexive point and ⊙ a point that may be either reflexive or irreflexive.

Recently, Bezhanishvili and Moraschini [1] gave a new proof of Citkin's theorem. Their approach draws upon both abstract algebraic logic and duality theory. Techniques from abstract algebraic logic allow one to establish that an algebrizable logic is HSC iff its associated variety of algebras is primitive [1, Section 2], that is every all its sub quasi-varieties are in fact varieties. IPC is algebrizable by the variety of Heyting algebra and consequently the task of characterising hereditary structurally complete intermediate logics is equivalent to that of characterising primitive subvarieties of Heyting algebras[1, Section 2]. Results from universal algebra further reduce the problem to centre around the notion of weak projectivity. An algebra A is *weakly projective* in a variety V iff for every $B \in V$ if A is a homomorphic image of B then A is isomorphic to a subalgebra of B.

Lemma 1 Let V be a locally finite variety, that is all its finitely generated members are finite. Then V is primitive iff its finite, non-trivial, finitely subdirectly irreducible (FSI) members are weakly projective in V.

The investigation is further aided through the Esakia duality between Heyting algebras and Esakia spaces[1, Section 3]. This allows the reduced algebraic question to be investigated with topological methods.

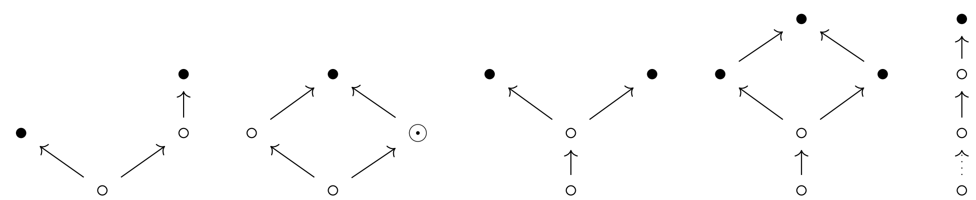
Notably a similar framework exists for transitive modal logics; they are algebrizable by the variety of K4-algebras [4] which are linked by Jónnson-Tarski duality to the class of transitive modal spaces. This allows us to do for Rybakov's result what Bezhanishvili and Moraschini did for Citkin's and investigate HSC modal logics through K4-algebras and transitive modal spaces.

However, more than simply provide a new proof of Rybakov's theorem our investigation illuminates a mistake in Rybakov's characterisation. The list of frames given by Rybakov is too restrictive due to the following.

Theorem 2 The variety generated by the algebra dual to F_3' is primitive. Note that as a finite transitive space the topology on F_3' is the discrete topology.

Accordingly, the characterisation of HSC transitive modal logics is revised.

Revised Theorem A transitive modal logic is HSC iff the variety of K4-algebras associated with it omits the algebras $(F_i)^* : 1 \leq i \leq 17$ and omit the algebra $(G_n)^*$ for some $n \in \omega$.



F_{14} F_{15} F_{16} F_{17} G_n

The frame G_n is a reflexive point receeded by a chain of n irreflexive points.

The proof strategy for the new revised system is the same. However, varieties of K4-algebras are not necessarily locally finite and so an alternative necessary and sufficient condition for being primitive is needed.

Lemma 3 Let V be a variety of K4-algebras. If V is primitive then the finite, non-trivial FSI members of V are weakly projective in V. Moreover, suppose all sub-varieties of V have the finite model property (FMP). Then if the finite, non-trivial FSI members of V are weakly projective in V then V is primitive.

Consequently, the poof strategy for the revised theorem has four components. The first is to establish the easier direction of the revised theorem.

Lemma 4 Primitive varieties of K4-algebras omit the algebras $(F_i)^* : 1 \leq i \leq 17$ and $(G_n)^*$ for some $n \in \omega$.

The second harder direction is much more involved. A crucial step is to give a precise description of the finitely generated, non-trivial, subdirectly irreducible (SI) members of varieties of K4-algebras omitting the given algebras. This description then drives the proofs of the final two key results.

Lemma 5 Let V be a variety of K4-algebras omitting $(F_i)^* : 1 \leq i \leq 17$ and $(G_n)^*$ for some $n \in \omega$. Then V has the FMP.

Lemma 6 Let V be a variety of K4-algebras omitting $(F_i)^* : 1 \leq i \leq 17$ and $(G_n)^*$ for some $n \in \omega$. Every finite, non-trivial FSI member of V is weakly projective in V.

Combing lemmas 3, 4 and 5 then yields a proof of the new revised theorem.

This work is a summary of a master's thesis undertaken at the Institute for Logic, Language and Computation [2].

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Game semantics for constructive modal logic.

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1 Abstract

Semantics is the area of logic concerned with specifying the meaning of the logical constructs. We distinguish between two main kind of semantic approach to logic. The first, the model-theoretic approach, is concerned with specifying the meaning of formulas in terms of truth in some model. The second, the denotational semantic approach, is concerned with specifying the meaning of proofs of the logic under a compositional point of view. Proofs are interpreted as mathematical objects called denotation, and the meaning of composed proofs is obtained by composing denotations. One of the desired feature of denotational models is full completeness: in a fully complete model, every morphism is the interpretation of some proof. Reasoning about the property of full complete models allows one to have a syntax-free characterization of the property of proofs. We say that a denotational model is *concrete* if its elements are not obtained by the quotient on proofs induced by cut-elimination. Game semantics [6, 5, 1, 2] is a form of denotational semantics in which proofs are interpreted as winning strategies for two player games.

In this presentation, we focus on denotational semantics for modal logics. Modal logics are, traditionally, an extension of *classical logic* making use of unary connectives, called *modalities*, that qualify the truth of a judgement. More precisely, modal logics are obtained by extending classical logic with a modality operator $\Box$ (together with its dual operator $\Diamond$), which are usually interpreted as *necessity* (respectively *possibility*).

Beginning with Simpson's work [10], intuitionistic and constructive modal logics have aroused growing interest. In particular, during the last three decades the proof theory of constructive modal logics has been developed considerably providing proof systems by means of sequent calculi [8, 11], natural deduction and λ-calculus [9, 7, 4].

The subject of our talk will be the basic constructive modal logic: the constructive version of the modal logic K (called CK) [4]. The formulas of CK are written using the connectives $\supset$ and $\wedge$ and the modalities $\Box$ and $\Diamond$. A complete sequent calculus system for this logic is obtained by adding the following two rules to a standard sequent calculus system for minimal logic

$$\frac{A_1, \ldots, A_n \vdash C}{\Box A_1, \ldots, \Box A_n \vdash \Box C}\ \mathsf{K}^{\Box} \qquad\qquad \frac{A_1, \ldots, A_n, B \vdash C}{\Box A_1, \ldots, \Box A_n, \Diamond B \vdash \Diamond C}\ \mathsf{K}^{\Diamond}$$

In particular, we present a concrete denotational semantics for CK (introduced in [3]). Our semantics is a game semantics. We present winning strategies that correspond to proofs of CK, we show that our winning strategies can be composed, and that —furthermore— our semantics is fully complete: each modal winning strategy is the interpretation of some sequent calculus proof.

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Degrees of FMP in extensions of bi-intuitionistic logic

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Abstract

We investigate degrees of FMP in extensions of bi-intuitionistic logic. Motivated by the proof of the intuitionistic case, we define a bi-intuitionistic version of the logic KG and restrict attention to its extensions. There we find useful properties of simple algebras and give a description of extensions with the FMP. Consequently, we provide a full characterisation of degrees of FMP, stating that the only existing degrees are 1 and $2^{\aleph_0}$, which is in stark contrast with the intuitionistic case.

1 Introduction

The notion of a degree of incompleteness, introduced in [5] (see also [4, Chapter 10.5]), is an important property in the theory of modal logic. It measures the cardinality of logics that have the same frames as a given logic. The study of degrees of incompleteness culminates in the result known as Blok's dichotomy [2], [3]. It states that every normal modal logic has degree of incompleteness 1 or $2^{\aleph_0}$.

This inspired the inception of another notion, called degree of the finite model property (degree of FMP for short) [1]. It counts the cardinality of logics that share the same finite frames as a given logic. Although this yields a similar definition, a striking difference appears when looking at the existing degrees of FMP of extensions of S4. As proven in [1], it turns out that in addition to 1 and $2^{\aleph_0}$, for every cardinal between 2 and $\aleph_0$ inclusive, there exists a logic of that degree. Moreover, the same characterisation holds for degrees of FMP of superintuitionistic logics. This gives motivation for the open problem of finding all existing degrees in extensions of bi-intuitionistic logic – a conservative extension of intuitionistic logic with an additional co-implication connective. Our work aims to shed light on this question by adapting the techniques used in the intuitionistic setting to the bi-intuitionistic setting.

2 The Kuznetsov-Gerĉiu logic

The characterisation of degrees of FMP of superintuitionistic logics follows from the explicit construction of logics with desired degrees. This construction takes advantage of the Kuznetsov-Gerĉiu logic KG [6] and its rich combinatorial properties. In order to define KG, one can make use of the following operation.

Definition 1. Let A and B be Heyting algebras. By the sum of A and B we mean the Heyting algebra obtained by placing A below B and identifying the top of A with the bottom of B.

Recall that the free Heyting algebra on one generator is the Rieger-Nishimura lattice [8], [7]. Hence all 1-generated Heyting algebras are obtained as homomorphic images of the Rieger-Nishimura lattice.

Definition 2. The logic KG is defined as the logic of all finite sums of 1-generated Heyting algebras.

Crucially, for every cardinal κ between 1 and $\aleph_0$, there exists an extension of KG of degree κ. Therefore, we are interested in defining a suitable bi-intuitionistic analogue of KG and characterising possible degrees of its extensions.

3 The logic bi-KG

We look more closely at the definition of KG with the aim of adapting it to a bi-intuitionistic version. Notice that every 1-generated Heyting algebra can be uniquely equipped with a co-implication, thus becoming a bi-Heyting algebra. This allows us to introduce the following logic.

Definition 3. Let $\mathscr{G}$ be the class of all finite sums of 1-generated Heyting algebras, viewed as bi-Heyting algebras. The logic bi-KG is defined as the logic of $\mathscr{G}$.

It turns out that a very useful step towards understanding bi-KG is finding the universal class $\mathbb{U}(\mathscr{G}) = \mathbb{S}\mathbb{P}_U(\mathscr{G})$. We accomplish this by describing precisely the local structure of algebras in $\mathscr{G}$ via universal formulas. Two algebras in $\mathbb{U}(\mathscr{G})$ are of particular interest – a Rieger-Nishimura variant that goes downward instead of upward and a Rieger-Nishimura variant that goes both upward and downward. Importantly, we have the following.

Theorem 4. Let $\mathscr{G}'$ be the collection $\mathscr{G}$ together with the two new Rieger-Nishimura variants. Then $\mathbb{U}(\mathscr{G})$ consists of all sums of algebras in $\mathscr{G}'$.

Since we know that these sums are always simple algebras[1], we deduce the following strong property.

Corollary 5. bi-KG is semi-simple and it is generated by $\mathbb{U}(\mathscr{G})$.

4 The FMP in extensions of bi-KG

The remainder of our work is dedicated to giving a characterisation of the FMP and degrees of FMP in extensions of bi-KG. Both of these make heavy use of the understanding of $\mathbb{U}(\mathscr{G})$.

We begin with the description of extensions of bi-KG with the FMP. In general, a logic enjoys the FMP if it is generated by its finite algebras. In our specific case, this requirement narrows down to verifying that the simple algebras validating the logic can be generated by the finite simple algebras. Furthermore, given a simple algebra A, we know the shape of the finite simple algebras that can generate A – these are what we call m-compressions of A, where m is a natural number. While the precise definition of an m-compression of A is quite technical, a quick intuition is that it is the result of replacing infinite segments of A with finite parts of size at least m.

These observations lead to the following result.

Theorem 6. An extension L of bi-KG has the FMP if and only if for each of its simple algebras A and each natural number m, there exists an m-compression of A satisfying L.

By applying this result to particular extensions, we get the the following corollaries.

Corollary 7. The logic bi-KG has the FMP.

Corollary 8. The logic generated by the bi-Heyting Rieger-Nishimura lattice lacks the FMP.

The latter is a notable difference with the intuitionistic case, where the logic generated by the Rieger-Nishimura lattice enjoys the FMP.

[1] There are actually two exceptions, but they play no important role.

5 Degrees of FMP in extensions of bi-KG

Using the work from the previous sections, we reach a full characterisation of degrees of FMP in extensions of bi-KG. Interestingly, we obtain a dichotomy-style theorem, in contrast with the KG case.

Theorem 9. In extensions of bi-KG, all possible degrees of FMP are 1 and $2^{\aleph_0}$.

In order to prove this statement, we follow the following strategy. Firstly, we observe that we already have witnesses of the degrees of FMP 1 and $2^{\aleph_0}$ – these are bi-KG and the logic generated by the bi-Heyting Rieger-Nishimura lattice respectively. This can be seen with the help of Corollary 7 and 8.

Secondly, we prove that if a given logic extending bi-KG has degree of FMP greater than 1, then its degree of FMP is $2^{\aleph_0}$. This is achieved through the explicit construction of continuum many logics with the same finite algebras. In particular, we build countably many algebras and generate continuum many logics by taking subsets of these algebras. The algebras are carefully selected in order to ensure that every subset of algebras generates a unique logic.

6 Directions for future work

A natural continuation of our work would be a characterisation of degrees of FMP for all extensions of bi-intuitionistic logic. We showed that the only know technique to construct finite degrees in intuitionistic logic does not work bi-intuitionistically and we believe that this hints at a possible dichotomy theorem for extensions of bi-intuitionistic logic.

Moreover, we find it interesting whether our ideas for bi-intuitionistic logic can be applied to other similar logical systems. For instance, we see temporal logic and intuitionistic modal logic as potential candidates.

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The general algebraic framework for Mathematical Fuzzy Logic

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Originating as an attempt to provide solid logical foundations for fuzzy set theory [19], and motivated also by philosophical and computational problems of vagueness and imprecision [16], Mathematical Fuzzy Logic (MFL) has become a significant subfield of mathematical logic [17]. Throughout the years many particular many-valued logics and families of logics have been proposed and investigated by MFL and numerous deep mathematical results have been proven about them (see the three volumes of handbook of MFL [5]). In the early years, the necessay exploratory work of the pioneers resulted naturally in a certain amount of repetition in the papers published on this topic; it was common to encounter articles that studied slightly different logics by repeating the same definitions and essentially obtaining the same results by means of analogous proofs. Therefore, MFL was an area of science screaming for systematization through the development and application of uniform, general, and abstract methods.

Abstract algebraic logic presented itself as the ideal toolbox to rely on; indeed, this general theory is applicable to all non-classical logics and provides an abstract insight into the fundamental (meta)logical properties at play. However, the existing works in that area (summarized in excellent monographs [2,14,15]) did not readily give the desired answers. Despite their many merits, these texts live at a level of abstraction a little too far detached from the intended field of application in MFL. They are indeed great sources of knowledge and inspiration, but there is still a lot of work to be done in order to bring the theory closer to the characteristic particularities of MFL, in particular in first-order logics.

These considerations led us, the authors of this contribution, to writing an extensive series of papers (e.g., [1,3,4,6–8,10–12,18] to name the most important ones) in which we have developed various aspects of the general theory of MFL at different levels of generality and abstraction.

Our first attempt at systematizing this bulk of research was a chapter published in 2011 in the Handbook of Mathematical Fuzzy Logic [9] where we provided rudiments of a well rounded theory constituting solid foundations sufficient (and necessary!) for a rapid development of new particular fuzzy logics demanded by emerging applications. The goal of this talk is to summarize the subsequent 10 years of development and refinements of this theory and present its now matured state of the art as described in our recent monograph [13].

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Axiomatization of Logics with Two-Layered Modal Syntax: The Protoalgebraic Case

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Two-layered modal syntax is given by three propositional languages (collections of connectives together with their arities): the *inner* one (used, in the common applications, to speak about events), the *modal* one (whose connectives are actually called *modalities*), and the *outer* one (used to speak about measures of events). Using these three languages and a fixed set of inner variables, we construct three disjoint sets of formulas:

- *inner* formulas are built from event variables using the inner language,
- *atomic outer* formulas are built by applying the modalities to inner formulas, and
- *complex* outer formulas are built from the atomic ones using the outer language.

Early examples of logics with two-layered syntax were modal logics of uncertainty stemming from Hamblin's seminal idea of reading the atomic outer formulas $P\varphi$ as 'probably φ' [16] and semantically interpreting it (in a given Kripke frame equipped with a finitely additive probability measure) as *true* iff the probability of the set of worlds where φ is true is bigger than a given threshold. This idea was later elaborated and extended by Fagin, Halpern and many others; see e.g. [5, 15].

These initial examples used classical logic to govern the behavior of formulas on both the inner and outer layers. A departure from this classical paradigm was proposed by Hájek and Harmancová in [13] and later developed by them in collaboration with Godo and Esteva in [12]. They kept classical logic as the interpretation of the inner syntactical layer of events, but proposed Łukasiewicz logic to govern the outer layer of statements on probabilities of these events, so that the truth degree of the atomic outer formula $P\varphi$ could be directly identified with the probability of the set of worlds where φ is true. Later, numerous other authors changed even the logic governing the inner layer (e.g., another fuzzy logic in order to allow for the treatment of uncertainty of vague events) or considered additional possibly non-unary modalities (e.g. for conditional probability), see e.g. [6–11, 14, 17].

This research thus gave rise to an interesting way of combining logics which allows to use one logic to reason about formulas (or rules) of another one with numerous examples described and developed in the literature. The existing bulk of literature constitutes an area of logic screaming for systematization through the development and application of uniform, general, and abstract methods. In our previous work [3] we took the first steps towards such a theory by providing an abstract notion of two-layered syntax and logic, a general semantics of *measured* Kripke frames and proved, in a rather general setting, two forms of completeness theorem most commonly appearing in the literature. Although the level of generality seemed quite sufficient back then (*finitary weakly implicative logics with unit and lattice conjunction*, see [4]), the recent development in the field shows the need for more: e.g., the inner logic in [2] and the outer logic in [1] are not weakly implicative, and in the former case they are not even equivalential.

The aim of this talk is to overcome the restrictions of [3] and present the completeness proof for an arbitrary inner logic and an arbitrary protoalgebraic outer logic.

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One-Variable Lattice-Valued Logics

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The *one-variable fragment* of any first-order logic yields an "S5-like" modal logic, obtained by replacing each occurrence of an atom $P(x)$ with a propositional variable p, and $(\forall x)$ and $(\exists x)$ with $\Box$ and $\Diamond$, respectively. The first-order semantics typically induces a relational semantics for this modal logic, but finding an axiomatization for its algebraic semantics is hindered by the fact that an axiomatization of the one-variable fragment cannot be directly extracted from an axiomatization of the full logic. Nevertheless, axiomatizations have been obtained in certain well-known cases. Monadic Boolean algebras [12] and monadic Heyting algebras [3, 14] correspond to the one-variable fragments of first-order classical logic and intuitionistic logic, respectively. More generally, varieties of monadic Heyting algebras corresponding to one-variable fragments of first-order intermediate logics have been investigated in [1, 2, 4–6, 15, 17, 18]. One-variable fragments of some first-order many-valued logics have also been studied in some depth; notably, monadic MV-algebras [7, 10, 16] and monadic Abelian ℓ-groups [13] correspond to the one-variable fragments of first-order Łukasiewicz logic and Abelian logic, respectively.

In [9], we initiate a general approach to addressing this axiomatization problem. Let $\mathcal{L}$ be an algebraic signature containing binary operations $\wedge$ and $\vee$, and consider the sets $\mathrm{Fm}^1_\forall(\mathcal{L})$ of *(first-order) one-variable $\mathcal{L}$-formulas* (with quantifiers $\forall$ and $\exists$) and $\mathrm{Fm}_\Box(\mathcal{L})$ of *propositional modal formulas* (with modalities $\Box$ and $\Diamond$), denoting by $(-)^*$ the standard translation function from $\mathrm{Fm}^1_\forall(\mathcal{L})$ to $\mathrm{Fm}_\Box(\mathcal{L})$. Members of both $\mathrm{Fm}^1_\forall(\mathcal{L})$ and $\mathrm{Fm}_\Box(\mathcal{L})$ are interpreted using semantics based on algebraic structures for the signature $\mathcal{L}$ with a lattice reduct, called *$\mathcal{L}$-lattices*. For $\mathrm{Fm}^1_\forall(\mathcal{L})$, we define structures over complete $\mathcal{L}$-lattices and interpret the quantifiers $\forall$ and $\exists$ as infima and suprema. For $\mathrm{Fm}_\Box(\mathcal{L})$, we call an algebraic structure $\langle \mathbf{A}, \Box, \Diamond\rangle$ an *m-$\mathcal{L}$-lattice* if $\mathbf{A}$ is an $\mathcal{L}$-lattice and $\Box, \Diamond$ are unary operations satisfying

$(\mathrm{L1}_\Box)$	$\Box x \wedge x \approx \Box x$	$(\mathrm{L1}_\Diamond)$	$\Diamond x \vee x \approx \Diamond x$
$(\mathrm{L2}_\Box)$	$\Box(x \wedge y) \approx \Box x \wedge \Box y$	$(\mathrm{L2}_\Diamond)$	$\Diamond(x \vee y) \approx \Diamond x \vee \Diamond y$
$(\mathrm{L3}_\Box)$	$\Box\Diamond x \approx \Diamond x$	$(\mathrm{L3}_\Diamond)$	$\Diamond\Box x \approx \Box x$,

and for each n-ary operation symbol $\star$ of $\mathcal{L}$,

$$(\star_\Box) \quad \Box(\star(\Box x_1, \dots, \Box x_n)) \approx \star(\Box x_1, \dots, \Box x_n).$$

For any class $\mathcal{K}$ of complete $\mathcal{L}$-lattices, semantical sentential consequence $\vDash^\forall_\mathcal{K}$ is defined over $\mathrm{Fm}^1_\forall(\mathcal{L})$-equations, i.e., formal expressions of the form $\varphi \approx \psi$ where $\varphi, \psi \in \mathrm{Fm}^1_\forall(\mathcal{L})$. Similarly, for any class $\mathcal{M}$ of m-$\mathcal{L}$-lattices, semantical consequence $\vDash_\mathcal{M}$ is defined over $\mathrm{Fm}_\Box(\mathcal{L})$-equations.

Observe now that any complete $\mathcal{L}$-lattice $\mathbf{A}$ and set W yields an m-$\mathcal{L}$-lattice $\langle \mathbf{A}^W, \Box, \Diamond\rangle$, that we call *full functional*, where the operations of $\mathbf{A}^W$ are defined pointwise and for each $f \in A^W$ and $u \in W$,

$$\Box f(u) = \bigwedge_{v \in W} f(v) \quad \text{and} \quad \Diamond f(u) = \bigvee_{v \in W} f(v).$$

We also call an m-$\mathcal{L}$-lattice *functional* if it embeds into a full functional m-$\mathcal{L}$-lattice.

Given any class $\mathcal{K}$ of complete $\mathcal{L}$-lattices, let $\mathcal{K}_f$ denote the class of all full functional m-$\mathcal{L}$-lattices $\langle \mathbf{A}^W, \Box, \Diamond \rangle$ with $\mathbf{A} \in \mathcal{K}$. It follows easily that for any set of $\mathrm{Fm}^1_\forall(\mathcal{L})$-equations $T \cup \{\varphi \approx \psi\}$ (lifting the translation * to sets of $\mathrm{Fm}^1_\forall(\mathcal{L})$-equations in the obvious way),

$$T \models^\forall_\mathcal{K} \varphi \approx \qquad \Longleftrightarrow \qquad T^* \models_{\mathcal{K}_f} \varphi^* \approx \psi^*.$$

The general problem addressed here is to provide an (elegant) axiomatization of the generalized quasivariety of m-$\mathcal{L}$-lattices generated by $\mathcal{K}_f$: that is, the class of *all* m-$\mathcal{L}$-lattices $\mathbf{M}$ satisfying $T^* \models_\mathbf{M} \varphi^* \approx \psi^*$ whenever $T \models^\forall_\mathcal{K} \varphi \approx \psi$ for a set of $\mathrm{Fm}^1_\forall(\mathcal{L})$-equations $T \cup \{\varphi \approx \psi\}$. In this work, we solve this problem for the case where $\mathcal{K}$ is the class of complete members of a variety that satisfies two natural algebraic properties.

Given any class $\mathcal{K}$ of $\mathcal{L}$-lattices, let $\overline{\mathcal{K}}$ denote the class of complete members of $\mathcal{K}$ and let $m\mathcal{K}$ denote the class of m-$\mathcal{L}$-lattices $\langle \mathbf{A}, \Box, \Diamond \rangle$ with $\mathbf{A} \in \mathcal{K}$. Following closely the proof of the same result for monadic Heyting algebras given in [2], we obtain a general *functional representation theorem* that gives sufficient conditions on $\mathcal{K}$ for all algebras in $m\mathcal{K}$ to be functional. Recall that a class $\mathcal{K}$ of $\mathcal{L}$-lattices

(i) admits *regular completions* if for any $\mathbf{A} \in \mathcal{K}$, there exist a $\mathbf{B} \in \overline{\mathcal{K}}$ and an embedding $f\colon \mathbf{A} \to \mathbf{B}$ that preserves all existing meets and joins of $\mathbf{A}$;

(ii) has the *superamalgamation property* if for any $\mathbf{A}, \mathbf{B}_1, \mathbf{B}_2 \in \mathcal{K}$ and embeddings $f_1\colon \mathbf{A} \to \mathbf{B}_1$, $f_2\colon \mathbf{A} \to \mathbf{B}_2$, there exist a $\mathbf{C} \in \mathcal{K}$ and embeddings $g_1\colon \mathbf{B}_1 \to \mathbf{C}$, $g_2\colon \mathbf{B}_2 \to \mathbf{C}$ such that $g_1 \circ f_1 = g_2 \circ f_2$ and for any $b_1 \in B_1$, $b_2 \in B_2$ and distinct $i, j \in \{1, 2\}$ such that $g_i(b_i) \leq g_j(b_j)$, there exists an $a \in A$ satisfying $g_i(b_i) \leq g_i \circ f_i(a) = g_j \circ f_j(a) \leq g_j(b_j)$.

Theorem 1. *Let $\mathcal{K}$ be a class of $\mathcal{L}$-lattices that is closed under subalgebras and direct limits, admits regular completions, and has the superamalgamation property. Then every member of $m\mathcal{K}$ is functional.*

Combing this functional representation theorem with our previous observation regarding the relationship between consequence in a class of complete $\mathcal{L}$-lattices and the corresponding class of full functional m-$\mathcal{L}$-lattices, we obtain the following result:

Corollary 1. *Let $\mathcal{V}$ be a variety of $\mathcal{L}$-lattices that admits regular completions and has the superamalgamation property. Then for any set $T \cup \{\varphi \approx \psi\}$ of $\mathrm{Fm}^1_\forall(\mathcal{L})$-equations,*

$$T \models^\forall_{\overline{\mathcal{V}}} \varphi \approx \psi \quad \Longleftrightarrow \quad T^* \models_{m\mathcal{V}} \varphi^* \approx \psi^*.$$

In particular, when $\mathcal{V}$ is the variety of Boolean algebras or Heyting algebras, both of which admit regular completions and have the superamalgamation property, $m\mathcal{V}$ is the variety of monadic Boolean algebras [12] or monadic Heyting algebras [14], respectively, and Corollary 1 yields well-known completeness results for the one-variable fragments of first-order classical logic and intuitionistic logic.

Further examples can be taken from the class of substructural logics (see, e.g., [11]). In particular, letting $\mathcal{L}_s$ be a signature with binary connectives $\vee$, $\wedge$, $\cdot$, and $\to$, and constant symbols f and e, an $\mathrm{FL_e}$-*algebra* is an $\mathcal{L}_s$-lattice $\mathbf{A} = \langle A, \vee, \wedge, \cdot, \to, \mathrm{f}, \mathrm{e} \rangle$ such that $\langle A, \cdot, \mathrm{e} \rangle$ is a commutative monoid and $\to$ is the residuum of $\cdot$, i.e., $a \cdot b \leq c \iff a \leq b \to c$ for all $a, b, c \in A$. Let us denote by $\mathcal{FL}_e$ the variety of $\mathrm{FL_e}$-algebras and by $\mathcal{FL}_{ew}$ and $\mathcal{FL}_{ec}$ the subvarieties of $\mathrm{FL_e}$-algebras satisfying $\mathrm{f} \leq x \leq \mathrm{e}$ and $x \leq x \cdot x$, respectively, noting that $\mathcal{FL}_{ew} \cap \mathcal{FL}_{ec}$ is term-equivalent to the variety of Heyting algebras. Since these varieties are closed under MacNeille completions and have the superamalgamation property (see, e.g., [11]), Theorem 1 and Corollary 1 yield the following result:

Theorem 2. *Let $\mathcal{V} \in \{\mathcal{FL}_e, \mathcal{FL}_{ew}, \mathcal{FL}_{ec}\}$. Then any member of $m\mathcal{V}$ is functional and for any set $T \cup \{\varphi, \psi\}$ of $\mathrm{Fm}^1_\forall(\mathcal{L}_s)$-equations,*

$$T \models^\forall_\mathcal{V} \varphi \approx \psi \quad \iff \quad T \models_{m\mathcal{V}} \varphi^* \approx \psi^*.$$

Note also that it was proved in [8] that a variety of $\mathrm{FL_e}$-algebras axiomatized relative to $\mathcal{FL}_e$ by "$\mathcal{N}_2$-equations" (i.e., equations of a certain simple syntactic form) is closed under MacNeille completions if and only if it has an analytic sequent calculus of a certain form. It is also known that a variety of $\mathrm{FL_e}$-algebras has the superamalgamation property if and only if it has the Craig interpolation property (see, e.g., [11]); however, a precise characterization of which varieties of $\mathrm{FL_e}$-algebras (even those with an analytic sequent calculus) have these properties is not known.

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Truthmaker Semantics for Degreeism of Vagueness *

No name for blind review[1]

No institute for blind review

Philosophers have been discussing vagueness and tackling its related paradox known as the sorites for many reasons, mostly linguistic and sometimes metaphysical and more (cf. [3]). When philosophers talk about vagueness, they often end up talking about semantics. In fact, most solutions towards the paradox revise semantics and/or logic: supervaluationism renovates semantics with supervalues, degreeism suggests many-valued logic; epistemicism is a traditionalist who keeps classical logic and semantics. In the market of semantic builders, truthmaker is a rising star with its expressive power powerful enough for hyperintensionality. Still, few have adopted truthmakers for vagueness. An exceptional case [5] suggests an argument appealing to truthmaker gaps but only for his version of epistemicism. Is there any other application of truthmaker semantics in the study of vagueness?

The goal of this paper is to offer an affirmative answer to these questions, by designing truthmaker semantics for a different position on vagueness. Among many positions at our hands, this paper works on a popular one: degreeism (degree theory). As its name tells, degreeism renews the semantic concept of truth value from binary (truth 1 and false 0 and nothing else) to many-valued (often infinite). However, importing truthmakers into degreeism is not straightforward. While truthmakers are about quality and use mereology when formalizing, degreeism is based on a quantity idea, namely a segment of real numbers [0,1]. How can we convert mereological structures of truthmaking into degreeists' real numbers?

The key idea of this transition is to import measure theory. A measure is, roughly put, a mathematical generalization of geometrical measures such as distance, length, area, and volume. This formal notion is applied to many things including physical mass and probability of events. Given degreeism is often associated with probability theory as they both feature the real fragment $[0, 1]$ as a central part of their formalization, this already seems a good match. We see an evaluation function μ that assigns a truth value to given truthmakers as a measure function, which satisfies the standard axioms of measure theory. The definition tells us how naturally these concepts fit degreeism. For one thing, an axiom says that the measure of the null set is zero,

$$\mu(\emptyset) = 0.$$

This corresponds to our intuitive idea that if a sentence has no truthmaker at all its truth value should be zero. Also, (countable) additivity confirms our idea on the relationship between truthmakers and truth values — the more truthmakers (e.g. evidence) a truth has, the more certain it is.

Having introduced truthmaker semantics for degreeism, this paper discusses the benefits of this semantics to further support how truthmakers are useful for the discussion of vagueness, at least for degreeism. This resulted semantics can resolve two formal issues of degreeism. One is about triviality [4]. Some may want to characterize vague predicates (from non-vague) by the formal concept of continuity. More technically speaking, one may want to characterize vague terms by whether its evaluation function from (a subset of) N (the number of hair) to truth values [0,1]. Unfortunately, this does not work because the domain (the number of hair, with the most natural topology) is discrete, hence any function from there is trivially continuous. In our renewed framework, such a worry disappears. Our domain is not the natural number but a

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set of truthmakers, whose topology is not necessarily discrete. The other is called the problem of "penumbral connection" [1]. This problem is about how to calculate truth values of two vague clauses connected by logical connectives. What happens if two indefinite clauses (i.e. borderline cases) are connected with a conjunction, say, "This ball is purple and this ball is red"? The truth value of this sentence should be zero, i.e. definitely false because one ball cannot have different colors at the same time. But typical degreeists say it is also indefinite. Truthmakers prepare an easy way out. Just suppose that a truthmaker for being red and another truthmaker for being purple are not compatible, formally speaking, they have no overlap on each other.

The motivation originally comes from philosophical debates. Nevertheless, this work offers an insight to rather formal studies. For a broader picture, this work can bridge two different approaches towards truth — qualitative (truthmakers) and quantitative (degree theory and probability theories). Also, since truthmaker semantics has been working as a good candidate tool for relevant logic (see [2]), it may highlight the connection between degree and relevance.

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Connexive implication in substructural logics

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Connexive Logic is a stream of research devoted to formalize conditionals expressing coherence/connection requirements between their antecedent and consequent. The current interest in these logics relies on their capability of formalizing indicative natural language conditionals (see [1]), counterfactuals (see [4]), and some species of physical and "causal" implications (see [3]).

We say a logic $\mathcal{L}$ is connexive provided that it has a negation $\neg$ and an implication $\to$ satisfying *Aristotle's Theses*:

$$\neg(\alpha \to \neg\alpha) \tag{AT1}$$

$$\neg(\neg\alpha \to \alpha) \tag{AT2}$$

e.g., that no formula implies or is implied by its own negation; *Boethius' Theses*:

$$(\alpha \to \beta) \to \neg(\alpha \to \neg\beta) \tag{BT1}$$

$$(\alpha \to \neg\beta) \to \neg(\alpha \to \beta) \tag{BT2}$$

e.g., that if α implies β (respectively, $\neg\beta$), then it is not the case that α implies β (respectively, $\neg\beta$) as well; and lastly, and crucially, the stipulation that $\to$ be non-symmetric, as to properly distinguish it from bi-implication. Apparently, these theses are falsified by classical logic whenever implications with false antecedents are considered.

Over the past years the research on connexive logic has been focused on defining new deductive systems satisfying connexive principles. However, to the best of our knowledge, the literature does not offer a systematic attempt to verify to what extent familiar systems of non-classical logic, e.g. substructural logics, admit (definable) connexive implications. At least not until recently where, in the work of Fazio, Ledda, and Paoli, it is shown that intuitionistic logic is deductively equivalent to their so-called *Connexive Heyting logic*. From the semanitcal perspective, they show that the variety HA of Heyting algebras is term-equivalent to a class of *Connexive Heyting algebras*. In particular, they show that in HA, the operation $\Rightarrow$ defined via

$$x \Rightarrow y := (x \to y) \wedge (y \to \neg\neg x),$$

where $\to$ is Heyting implication and $\neg x := x \to 0$ is Heyting negation, is generally non-symmetric and, in conjunction with Heyting negation, satisfies (the equational renderings of) laws for a connexive implication, i.e., Aristotle's and Boethius' theses.

Contributing to this line of research, we consider a broader class of substructural logics *vis-à-vis* their semantic lens in residuated structures. That is, we investigate those (sub)classes of commutative pointed residuated lattices, i.e., $\mathsf{FL_e}$-algebras, for which $\Rightarrow$, and similarly related operations, satisfy such connexive principles. We demonstrate that these properties are intimately related-to, and in many cases equivalent-to, having the equational Glivenko property hold relative to Boolean algebras (see [2] for more on the Glivenko property).

*Speaker.

In particular, given an $\mathsf{FL_e}$-algebra $\mathbf{A}$ and a operation $\Rightarrow : A \times A \to A$, we say $(\mathbf{A}, \Rightarrow)$ is *weakly connexive* if the following identities are satisfied:

$$1 \leq \neg(x \Rightarrow \neg x) \tag{at1}$$
$$1 \leq \neg(\neg x \Rightarrow x) \tag{at2}$$
$$1 \leq (x \Rightarrow y) \Rightarrow \neg(x \Rightarrow \neg y) \tag{bt1}$$
$$1 \leq (x \Rightarrow \neg y) \Rightarrow \neg(x \Rightarrow y) \tag{bt2}$$

and we say $\mathbf{A}$ is *connexive* if furthermore $\Rightarrow$ is non-symmetric. We prove the following:

Theorem 1. *Let C be the class of $\mathsf{FL_e}$-algebras satisfying the equation (bt1) (Boethius' thesis) for the connective $x \Rightarrow y := (x \to y) \wedge (y \to \neg\neg x)$, where $\neg x := x \to 0$. Then the following hold:*

1. *C is connexive, i.e., $(\mathbf{A}, \Rightarrow)$ is weakly connexive for every member $\mathbf{A}$ of C and $\Rightarrow$ is not generally symmetric in C;*
2. *C is exactly $\mathbf{G}_{\mathsf{FL_e}}(\mathsf{BA})$, the largest variety of $\mathsf{FL_e}$-algebras for which the equational Glivenko property holds relative to Boolean algebras.*

We also investigate those subvarieties of $\mathsf{FL_e}$ that are integral and/or where 0 is the least element along with a broader class of candidate connexive arrows. In particular, for the class $\mathsf{FL_{ew}}$ of 0-bounded integral $\mathsf{FL_e}$-algebras, we obtain the following:

Theorem 2. *Let $\mathbf{A}$ be an $\mathsf{FL_{ew}}$-algebra and define the operations $\Rightarrow_\circ$ and $\Rightarrow_\wedge$ on $\mathbf{A}$ via:*

$$x \Rightarrow_\circ y := (x \to y) \cdot (y \to \neg\neg x)$$
$$x \Rightarrow_\wedge y := (x \to y) \wedge (y \to \neg\neg x)$$

and note that the interval $[\Rightarrow_\circ, \Rightarrow_\wedge]$ of binary operation (under the usual ordering) is non-empty. Then the following are equivalent:

1. *$\mathbf{A}$ is a member of $\mathbf{G}_{\mathsf{FL_{ew}}}(\mathsf{BA})$, the largest variety of integral 0-bounded $\mathsf{FL_e}$-algebras for which the equational Glivenko property holds relative to Boolean algebras.*
2. *For all $\Rightarrow \in [\Rightarrow_\circ, \Rightarrow_\wedge]$, $(\mathbf{A}, \Rightarrow)$ is weakly connexive.*
3. *There exists $\Rightarrow \in [\Rightarrow_\circ, \Rightarrow_\wedge]$ such that $(\mathbf{A}, \Rightarrow) \models$ (at1) (Aristotle's thesis).*

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Quantified Relevant logic **RQ** with Constant Domains!? A Perspective from Quantified Modal Logics

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Abstract

This work explicates and aims to solve the problems of (general frame) constant domain semantics for relevant logics, and presents some recent results concerning the Barcan formula(s) in relevant (substructural) logics with neighbourhood semantics.

The relevant and substructural logic **R** — the logic of De Morgan monoids, with an implication that rejects syntactic irrelevance — was first given a ternary relational semantics by Sylvan (né Routley) and Meyer in a series of papers [8, 9]. This semantics enjoys many of the philosophical and interpretative benefits of Kripke-style, relational frame-based semantics. To the relevantist's disappointment, the most straightforward way of generalizing this semantics to model first-order extensions of **R**, namely by adding one universal domain and interpreting the quantifiers using generalized (infinite) intersection and union, produces a semantics for which quantified **R** (**RQ**) is incomplete. The incompleteness, shown by Fine [4], was remedied (again by Fine [3]) by a genius but complicated variable domain semantics (and some additional machinery). More recently, Mares and Goldblatt [7] have developed an alternative semantics for **RQ** which employs (i) general frames, and (ii) a non-Tarskian interpretation of the quantifiers. General frames are frames built on a set of points (worlds, situations, etc) K such that it need not be that every set of points (worlds, situations, etc) can express a proposition, and so an admissible subset of $\wp(K)$, called the *admissible propositions*, is given. The Tarskian interpretation of the quantifiers uses the generalized intersection and unions, such that, given a point $a \in K$ and a variable assignment f, $a, f \vDash \forall x\mathcal{A}$ iff $a, f' \vDash \mathcal{A}$, for each f' that differs from f in at most the assignment of the variable x. The non-Tarskian interpretation of the universal quantified $\forall x\mathcal{A}$ in the Mares-Goldblatt semantics is the strongest admissible proposition that entails every instance $\mathcal{A}[\tau/x]$, where the generalized intersection of the truth sets of the instances need not be an admissible proposition.

The Mares-Goldblatt approach has been extended and employed to model a wide range of quantified modal relevant logics (Ferenz [1], Ferenz and Tedder [11, 2]), identity in relevant logics (Ferenz [1], Standefer [10]), and quantified modal classical logic (Goldblatt [5], Goldblatt and Mares [6]).

Of particular interest to the author is a handful of results in Goldblatt and Mares [6] and Goldbatt [5], which show that certain quantified modal logics are (1) incomplete with respect to the constant domain, non-general-frame semantics with a non-Tarskian interpretation of the quantifiers, (2) complete with respect to the Mares-Goldblatt semantics, but (3) complete with respect to constant domain, non-general-frame semantics with a *Tarskian* interpretation of the quantifiers. That is, we can obtain completeness for these logics without using the full power of the Mares-Goldblatt semantics.

The case for **RQ** is similar in some respects. First, the incompleteness shown by Fine shows that a constant domain, non-general semantics with Tarskian truth does not characterize **RQ**. Second, the Mares-Goldblatt semantics does in fact characterize **RQ**. What is left to show is whether or not **RQ** can be characterized by employing non-general frames with Tarskian truth conditions. The present paper aims at solving this problem.

In the classical setting, the canonical model is constructed from ω-complete theories, where *omega*-complete theories are those theories of a logic which do not contain every

instance $\mathcal{A}[\tau/x]$ of formula without also containing the universally quantified $\forall x\mathcal{A}$. The Barcan formula plays a critical role in the completeness proof, where Thomason [12] initially demonstrated that the formula ensures that certain theories obtained from collections of modal formulas are ω-complete. In this regard, the Barcan formula essentially "repairs" the completeness of some logics with respect to universal domain semantics.

For **RQ**, it is an open question whether or not any additional axioms are sufficient to repair completeness with respect to Tarskian, non-general frames. A solution to the problem aimed here — completeness for Tarskian general frames — may provide an avenue to repairing completeness in the non-general case. In particular, there may be formulas that provide a service analogous to the Barcan formula. That is, showing that a set of formulas generated from implicational formulas — by taking just the right set of antecedents or just the right set of consequents of a theory — must be ω-complete.

As both implication and modalities are treated intensionally — that is, modelled using relations between points in a frame — there are relevant questions as to the relations that hold between properties of the ternary and binary relations, the interpretation of the quantifiers, and formulas which 'mix' the various intensional operators, such as the Barcan formula. I will give independence results of several properties and Barcan formulas for neighborhood ternary relational semantics, and discuss some implications for stronger logics and philosophical perspectives.

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Intuitionistic Sahlqvist correspondence for deductive systems

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In this talk we present a Sahlqvist Correspondence Theorem [9] for finitary protoalgebraic logics. Our proof is based on the extension of Sahlqvist theory to some fragments of IPC provided in the previous talk [4]. A formula in the language

$$\mathcal{L} \ ::= \ x \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \varphi \to \psi \mid \neg\varphi \mid 0 \mid 1$$

is said to be

(i) a *Sahlqvist antecedent* if it is constructed from variables, negative formulas, and the constants 0 and 1 using only $\wedge$ and $\vee$;

(ii) a *Sahlqvist implication* if either it is positive, or it has the form $\neg\varphi$ for a Sahlqvist antecedent φ, or it has the form $\varphi \to \psi$ for a Sahlqvist antecedent φ and a positive formula ψ.

Moreover, a *Sahlqvist quasiequation* is a universal sentence of the form

$$\forall \vec{x}, y, z((\varphi_1 \wedge y \leqslant z \,\&\, \ldots \,\&\, \varphi_n \wedge y \leqslant z) \Longrightarrow y \leqslant z),$$

where y, z are distinct variables that do not occur in $\varphi_1, \ldots, \varphi_n$ and each φ_i is constructed from Sahlqvist implications using only $\wedge$ and $\vee$.

Remark 1. The focus on quasiequations (as opposed to formulas or equations) is necessary as we deal with fragments where equations have a very limited expressive power. ⊠

Let PSL, (b)ISL, PDL, IL and HA be, respectively, the varieties of pseudocomplemented semilattices, (bounded) implicative semilattices, pseudocomplemented distributive lattices, implicative lattices, and Heyting algebras. Furthermore, given a poset $\mathbb{X}$, let $\mathsf{Up}(\mathbb{X})$ be the Heyting algebra of its upsets. The Sahlqvist theorem for fragments of IPC presented in [4] takes the following form:

Theorem 2. *The following holds for every variety* K *between* PSL, (b)ISL, PDL, IL *and* HA *and every Sahlqvist quasiequation* Φ *in the language of* K:

(i) Canonicity*: For every* $\boldsymbol{A} \in \mathsf{K}$*, if* $\boldsymbol{A}$ *validates* Φ*, then also* $\mathsf{Up}(\boldsymbol{A}_*)$ *validates* Φ*, where* $\boldsymbol{A}_*$ *is the poset of the meet irreducible filters of* $\boldsymbol{A}$*;*

(ii) Correspondence*: There exists an effectively computable sentence* $\mathsf{tr}(\Phi)$ *in the language of posets such that* $\mathsf{Up}(\mathbb{X}) \vDash \Phi$ *iff* $\mathbb{X} \vDash \mathsf{tr}(\Phi)$*, for every poset* $\mathbb{X}$*.*

*Speaker.

A *logic* $\vdash$ is a finitary substitution invariant consequence relation on the set of formulas of some algebraic language. Let $\vdash$ be a logic and $\boldsymbol{A}$ an algebra. A subset F of $\boldsymbol{A}$ is said to be a *deductive filter* of $\vdash$ on $\boldsymbol{A}$ if it is closed under the interpretation of the rules valid in $\vdash$. When ordered under the inclusion relation, the set of deductive filters of $\vdash$ on $\boldsymbol{A}$ forms an algebraic lattice $\mathsf{Fi}_{\vdash}(\boldsymbol{A})$ with semilattice of compact elements $\mathsf{Fi}^{\omega}_{\vdash}(\boldsymbol{A})$. Lastly, the poset of meet irreducible elements of $\mathsf{Fi}_{\vdash}(\boldsymbol{A})$ will be denoted by $\mathsf{Spec}_{\vdash}(\boldsymbol{A})$.

In order to extend Sahlqvist Correspondence to arbitrary logics, recall that a logic $\vdash$ is said to have

(i) The *inconsistency lemma* (IL) [8] if for every $n \in \mathbb{Z}^+$ there is a finite set of formulas ${\sim_n}(x_1, \dots, x_n)$ such that for every set of formulas $\Gamma \cup \{\varphi_1, \dots, \varphi_n\}$,

$$\Gamma \cup \{\varphi_1, \dots, \varphi_n\} \text{ is inconsistent iff } \Gamma \vdash {\sim_n}(\varphi_1, \dots, \varphi_n);$$

(ii) The *deduction theorem* (DT) [1] if for every $n, m \in \mathbb{Z}^+$ there is a finite set $(x_1, \dots, x_n) \Rightarrow_{nm} (y_1, \dots, y_m)$[1] of formulas such that for every set of formulas $\Gamma \cup \{\psi_1, \dots, \psi_n, \varphi_1, \dots, \varphi_m\}$,

$$\Gamma, \psi_1, \dots, \psi_n \vdash \varphi_1, \dots, \varphi_m \text{ iff } \Gamma \vdash (\psi_1, \dots, \psi_n) \Rightarrow_{nm} (\varphi_1, \dots, \varphi_m);$$

(iii) The *proof by cases* (PC) [2, 3] if for every $n, m \in \mathbb{Z}^+$ there is a finite set of formulas $(x_1, \dots, x_n) \curlyvee_{nm} (y_1, \dots, y_m)$ such that for every set of formulas $\Gamma \cup \{\psi_1, \dots, \psi_n, \varphi_1, \dots, \varphi_m, \gamma\}$,

$$\Gamma, \psi_1, \dots, \psi_n \vdash \gamma \text{ and } \Gamma, \varphi_1, \dots, \varphi_m \vdash \gamma \text{ iff } \Gamma, (\psi_1, \dots, \psi_n) \bigcurlyvee_{nm} (\varphi_1, \dots, \varphi_m) \vdash \gamma.$$

A formula φ in $\mathcal{L}$ is *compatible* with a logic $\vdash$ when

(i) If 0 (resp. 1) occurs in φ, then $\vdash$ has the IL (resp. the IL or the DT);

(ii) If $\neg$ (resp. $\to, \vee$) occurs in φ, then $\vdash$ has the IL (resp. DT, PC).

In this case, for every $k \in \mathbb{Z}^+$ we associate a finite set $\boldsymbol{\varphi}^k(\vec{x}_1, \dots, \vec{x}_n)$ of formulas $\vdash$ (where each $\vec{x}_i$ is a sequence of variables of length k) with φ as follows:

(i) If $\varphi = x_i$, then $\boldsymbol{\varphi}^k := \{\vec{x}_i\}$;

(ii) If $\varphi = \psi \wedge \gamma$, then $\boldsymbol{\varphi}^k := \;\; ^k \cup \boldsymbol{\gamma}^k$;

(iii) If $\varphi = \neg\psi$, then $\vdash$ has the IL and, therefore, we set $\boldsymbol{\varphi}^k := {\sim_m}(\gamma_1, \dots, \gamma_m)$ where $\boldsymbol{\psi}^k = \{\gamma_1, \dots, \gamma_m\}$;

(iv) The cases where φ has the form $\psi \to \gamma$ or $\psi \vee \gamma$ are handled similarly to the previous one.

A Sahlqvist quasiequation

$$\Phi = \forall \vec{x}, y, z((\varphi_1(x_1, \dots, x_m) \wedge y \leqslant z \,\&\, \dots \,\&\, \varphi_n(x_1, \dots, x_m) \wedge y \leqslant z) \Longrightarrow y \leqslant z),$$

is said to be *compatible with a logic* $\vdash$ if so are $\varphi_1, \dots, \varphi_n$. With it, we associate the set $\mathsf{R}_{\vdash}(\Phi)$ of metarules for $\vdash$ of the form

$$\frac{\Gamma, \boldsymbol{\varphi}_1^k(\vec{\gamma}_1, \dots, \vec{\gamma}_m) \vdash \psi, \dots, \Gamma, \boldsymbol{\varphi}_n^k(\vec{\gamma}_1, \dots, \vec{\gamma}_m) \vdash}{\Gamma \vdash \psi.}$$

[1]We signify that $\Rightarrow_{nm}$ is a set of formulas in the variables $x_1, \dots, x_n, y_1, \dots, y_m$ by the more suggestive notation $(x_1, \dots, x_n) \Rightarrow_{nm} (y_1, \dots, y_m)$. A similar convention applies to Condition (iii).

where $k \in \mathbb{Z}^+$, $\Gamma \cup \{\psi\}$ is a finite set of formulas, and $\vec{\gamma}_1, \ldots, \vec{\gamma}_m$ are sequences of formulas of length k.

A logic is *protoalgebraic* if there exists a set of formulas $\Delta(x, y)$ such that $\emptyset \vdash \Delta(x, x)$ and $x, \Delta(x, y) \vdash y$. Our general Sahlqvist Correspondence Theorem takes the following form:

Sahlqvist Correspondence. *Let Φ be a Sahlqvist quasiequation compatible with a protoalgebraic logic $\vdash$. Then*

$$\vdash \textit{ validates the metarules in } \mathsf{R}_{\vdash}(\Phi) \quad \textit{iff} \quad \mathsf{Spec}_{\vdash}(\boldsymbol{A}) \vDash \mathsf{tr}(\Phi), \textit{ for every algebra } \boldsymbol{A}.$$

As a consequence, we obtain for instance that a protoalgebraic logic with the IL satisfies a generalization of the excluded middle law (resp. of the bounded top width n formula) iff it is semisimple (resp. principal upsets in $\mathsf{Spec}_{\vdash}(\boldsymbol{A})$ have at most n maximal elements, for every algebra $\boldsymbol{A}$) [6, 7]. The results of this talk are collected in [5].

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Sahlqvist theory for fragments of Intuitionistic Logic

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The aim of this talk is to present an extension of Sahlqvist theory [9] to the fragments of intuitionistic logic IPC associated with the varieties PSL, (b)ISL, PDL, IL and HA, of pseudocomplemented semilattices, (bounded) implicative semilatices, pseudocomplemented distributive lattices and Heyting algebras, respectively. This result will serves as a basis for another talk in this conference, namely [4].

Consider the modal language

$$\mathcal{L}_{\Box} \quad ::= \quad x \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \varphi \to \psi \mid \neg\varphi \mid \Box\varphi \mid \Diamond\varphi \mid 0 \mid 1.$$

Formulas of $\mathcal{L}_{\Box}$ will be assumed to have variables in a denumerable set $Var = \{x_n : n \in \mathbb{Z}^+\}$ and arbitrary elements of Var will often be denoted by x, y, and z.

Definition 1. Let φ be a formula of $\mathcal{L}_{\Box}$ and x a variable. An occurrence of x in φ is said to be *positive* (resp. *negative*) if the sum of negations and antecedents of implications within whose scopes it appears is even (resp. odd). Moreover, we say a x is *positive* (resp. *negative*) in φ if every occurrence of x in φ is positive (resp. negative). Lastly, φ is said to be *positive* (resp. *negative*) if every variable is positive (resp. negative) in φ.

Formulas of the form $\Box^n x$ with $x \in Var$ and $n \in \mathbb{N}$ will be called *boxed atoms*.

Definition 2. A formula of $\mathcal{L}_{\Box}$ is said to be

(i) a *Sahlqvist antecedent* if it is constructed from boxed atoms, negative formulas and the constants 0 and 1 using only $\wedge$, $\vee$ and $\Diamond$;

(ii) a *Sahlqvist implication* if either it is positive, or it is of the form $\neg\varphi$ for a Sahlqvist antecedent φ, or it is of the form $\varphi \to \psi$ for a Sahlqvist antecedent φ and a positive formula ψ.

Remark 3. When applied to modal logic, our definition of a Sahlqvist implication is intentionally redundant. For if φ is positive and ψ a Sahlqvist antecedent, then φ is equivalent to $1 \to \varphi$ and $\neg\psi$ is equivalent to $\psi \to 0$. ⊠

Definition 4. A *Sahlqvist quasiequation* is a universal sentence of the form

$$\forall \vec{x}, y, z((\varphi_1(\vec{x}) \wedge y \leqslant z \,\&\, \ldots \,\&\, \varphi_n(\vec{x}) \wedge y \leqslant z) \Longrightarrow y \leqslant z),$$

where y and z are distinct variables that do not occur in $\varphi_1, \ldots, \varphi_n$ and each φ_i is constructed from Sahlqvist implications using only $\wedge$, $\vee$, and $\Box$.

Remark 5. The role of Sahlqvist quasiequations is usually played by the so-called *Sahlqvist formulas*, i.e., formulas that can be constructed from Sahlqvist implications using only $\wedge, \vee$, and $\Box$. To clarify the relation between Sahlqvist quasiequations and formulas, recall that a *modal algebra* is a structure $\langle A; \wedge, \vee, \neg, \Box, 0, 1\rangle$ where $\langle A; \wedge, \vee, \neg, 0, 1\rangle$ is a Boolean algebra and for every $a, b \in A$,

$$\Box(a \wedge b) = \Box a \wedge \Box b \text{ and } \Box 1 = 1.$$

Then, a Sahlqvist quasiequation $\Phi = \forall \vec{x}, y, z((\varphi_1(\vec{x}) \wedge y \leqslant z \,\&\ldots\&\, \varphi_n(\vec{x}) \wedge y \leqslant z) \Longrightarrow y \leqslant z)$ is valid in a modal algebra $\boldsymbol{A}$ if and only if $\boldsymbol{A} \vDash \varphi_1 \vee \cdots \vee \varphi_n$. The focus on Sahlqvist quasiequations (as opposed to formulas) is motivated by the fact that we deal with fragments where equations have a very limited expressive power. For instance, in PSL there are only three nonequivalent equations [8], while there are infinitely many nonequivalent Sahlqvist quasiequations. ⊠

With every Kripke frame $\mathbb{X} = \langle X, R\rangle$ we can associate a modal algebra

$$\mathscr{P}_{\mathsf{M}}(\mathbb{X}) := \langle \mathscr{P}(X); \cap, \cup, \neg, \Box, \emptyset, X\rangle,$$

where $\neg$ and $\Box$ are defined for every $Y \subseteq X$ as

$$\neg Y := X \smallsetminus Y \text{ and } \Box Y := \{x \in X : \text{if } \langle x, y\rangle \in R, \text{ then } y \in Y\}.$$

Conversely, with a modal algebra $\boldsymbol{A}$ we can associate a Kripke frame $\boldsymbol{A}_+ := \langle X, R\rangle$, where X is the set of ultrafilters of $\boldsymbol{A}$ and

$$R := \{\langle F, G\rangle \in X \times X : \text{for every } a \in A, \text{ if } \Box a \in F, \text{ then } a \in G\}.$$

Our aim is to extend the next classical version of Sahlqvist Theorem to the above-mentioned fragments of IPC.

Theorem 6. *The following conditions hold for a Sahlqvist quasiequation* Φ*:*

(i) Canonicity*: If a modal algebra* $\boldsymbol{A}$ *validates* Φ*, then also* $\mathscr{P}_{\mathsf{M}}(\boldsymbol{A}_+)$ *validates* Φ*;*

(ii) Correspondence*: There is an effectively computable first order sentence* $\mathsf{tr}(\Phi)$ *in the language of Kripke frames such that* $\mathscr{P}_{\mathsf{M}}(\mathbb{X}) \vDash \Phi$ *iff* $\mathbb{X} \vDash \mathsf{tr}(\Phi)$*, for every Kripke frame* $\mathbb{X}$*.*

In order to do so, first we extend Sahlqvist Theorem to IPC using Gödel translation of IPC into S4 [7] and its duality theoretic interpretation (see, e.g., [2]). Then, we individuate a correspondence between homomorphisms in the varieties PSL, (b)ISL, PDL, IL, and HA and appropriate partial functions between (possibly empty) posets that generalize the notion of a p-morphism typical of *Esakia duality* for Heyting algebras [5, 6]. Our approach is inspired by [1].

For a poset $\mathbb{X}$ and $Y \subseteq X$, let

$$\uparrow^{\mathbb{X}} Y := \{x \in X : \text{there exists } y \in Y \text{ s.t. } y \leqslant x\};$$
$$\downarrow^{\mathbb{X}} Y := \{x \in X : \text{there exists } y \in Y \text{ s.t. } x \leqslant y\}.$$

Definition 7. An order preserving partial function $p \colon \mathbb{X} \rightharpoonup \mathbb{Y}$ between posets is

(i) a *partial negative p-morphism* if

$$X = \downarrow^{\mathbb{X}} \{x \in X : \uparrow^{\mathbb{X}} x \subseteq \mathsf{dom}(p)\}$$

and for every $x \in \mathsf{dom}(p)$ and $y \in Y$,

$$\text{if } p(x) \leqslant^{\mathbb{Y}} y, \text{ there exists } z \in \mathsf{dom}(p) \text{ s.t. } x \leqslant^{\mathbb{X}} z \text{ and } y \leqslant^{\mathbb{Y}} p(z);$$

(ii) a *partial positive p-morphism* if for every $x \in \mathsf{dom}(p)$ and $y \in Y$,

$$\text{if } p(x) \leqslant^{\mathbb{Y}} y, \text{ there exists } z \in \mathsf{dom}(p) \text{ s.t. } x \leqslant^{\mathbb{X}} z \text{ and } y = p(z);$$

(iii) a *partial p-morphism* if it is both a partial negative p-morphism and a partial positive p-morphism.

When p is a total function, we drop the adjective *partial* in the above definitions.

With every variety K among PSL, (b)ISL, PDL, IL, and HA we associate a collection K^{∂} consisting of the class of all posets with suitable partial functions between them as follows:[1]

$\mathsf{PSL}^{\partial} :=$ the collection of posets with partial negative p-morphisms;
$\mathsf{ISL}^{\partial} :=$ the collection of posets with partial positive p-morphisms;
$\mathsf{bISL}^{\partial} :=$ the collection of posets with partial p-morphisms;
$\mathsf{PDL}^{\partial} :=$ the collection of posets with negative p-morphisms;
$\mathsf{IL}^{\partial} :=$ the collection of posets with almost total partial positive p-morphisms;
$\mathsf{HA}^{\partial} :=$ the collection of posets with p-morphisms.

We will refer to the partial functions in K^{∂} as to the *arrows* of K^{∂}. Given $\boldsymbol{A}, \boldsymbol{B} \in \mathsf{K}$ and a homomorphism $f \colon \boldsymbol{A} \to \boldsymbol{B}$, let $f_* \colon \boldsymbol{B}_* \rightharpoonup \boldsymbol{A}_*$ be the partial function between the posets of meet irreducible filters of $\boldsymbol{B}$ and of $\boldsymbol{A}$ respectively, with

$$\mathsf{dom}(f_*) := \{F \in \boldsymbol{B}_* : f^{-1}[F] \in \boldsymbol{A}_*\}$$

defined as $f_*(F) := f^{-1}[F]$ for every $F \in \mathsf{dom}(f_*)$. Conversely, given a poset $\mathbb{X}$, let $\mathsf{Up}_{\mathsf{K}}(\mathbb{X})$ be the reduct in the language of K of the Heyting algebra

$$\langle \mathsf{Up}(\mathbb{X}); \cap, \cup, \to, \emptyset, X \rangle,$$

where $\mathsf{Up}(\mathbb{X})$ is the set of upsets of $\mathbb{X}$ and $\to$ is defined by

$$U \to V := X \smallsetminus {\downarrow}(U \smallsetminus V).$$

Lastly, given an arrow $p \colon \mathbb{X} \rightharpoonup \mathbb{Y}$ in K^{∂}, let $\mathsf{Up}_{\mathsf{K}}(p) \colon \mathsf{Up}_{\mathsf{K}}(\mathbb{Y}) \to \mathsf{Up}_{\mathsf{K}}(\mathbb{X})$ be the map defined for every $U \in \mathsf{Up}_{\mathsf{K}}(\mathbb{Y})$ as $\mathsf{Up}_{\mathsf{K}}(p)(U) := X \smallsetminus {\downarrow}^{\mathbb{X}} p^{-1}[Y \smallsetminus U]$.

Remark 8. In the case of HA, the applications $(-)_*$ and $\mathsf{Up}(-)$ are the contravariant functors underlying *Esakia duality* [5, 6]. ⊠

Proposition 9. *Let* K *be a variety among* PSL, (b)ISL, PDL, IL, *and* HA. *The following conditions hold for every* $\boldsymbol{A}, \boldsymbol{B} \in \mathsf{K}$ *and every pair* $\mathbb{X}$, $\mathbb{Y}$ *of posets:*

(i) *If* $f \colon \boldsymbol{A} \to \boldsymbol{B}$ *is a homomorphism, then* $f_* \colon \boldsymbol{B}_* \rightharpoonup \boldsymbol{A}_*$ *is an arrow in* K^{∂};

(ii) *If* $p \colon \mathbb{X} \rightharpoonup \mathbb{Y}$ *is an arrow in* K^{∂}, *then* $\mathsf{Up}_{\mathsf{K}}(p) \colon \mathsf{Up}_{\mathsf{K}}(\mathbb{Y}) \to \mathsf{Up}_{\mathsf{K}}(\mathbb{X})$ *is a homomorphism.*

Furthermore, if f *is injective* (*resp.* p *is surjective*), *then* f_* *is surjective* (*resp.* $\mathsf{Up}_{\mathsf{K}}(p)$ *is injective*).

[1] The collection K^{∂} need not be a category in general.

By making use of Proposition 9, one can extend Sahlqvist Theorem 6 as announced, in the following way:

Theorem 10. *The following conditions hold for every variety* K *between* PSL, $(\mathsf{b})\mathsf{ISL}$, PDL, IL *and* HA *and every Sahlqvist quasiequation* Φ *in the language of* K:

(i) Canonicity*: For every* $\boldsymbol{A} \in \mathsf{K}$*, if* $\boldsymbol{A}$ *validates* Φ*, then also* $\mathsf{Up}_{\mathsf{K}}(\boldsymbol{A}_*)$ *validates* Φ*;*

(ii) Correspondence*: There exists an effectively computable sentence* $\mathtt{tr}(\Phi)$ *in the language of posets such that* $\mathsf{Up}_{\mathsf{K}}(\mathbb{X}) \vDash \Phi$ *iff* $\mathbb{X} \vDash \mathtt{tr}(\Phi)$*, for every poset* $\mathbb{X}$*.*

These results are collected in the manuscript [3].

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Lifting Properties from Finitely Subdirectly Irreducible Algebras

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When a deductive system $\vdash$ is algebraized by a variety $\mathcal{V}$, we may often establish that $\vdash$ enjoys a given metalogical property by proving that $\mathcal{V}$ has a corresponding algebraic property. For instance, $\vdash$ has a local deduction theorem if and only if $\mathcal{V}$ has the congruence extension property [3], and in this setting $\vdash$ has the deductive interpolation property if and only if $\mathcal{V}$ has the amalgamation property [5]. 'Bridge theorems' of this type give a powerful technique for obtaining metalogical results, at least when there is a suitable path for establishing that $\mathcal{V}$ has the corresponding properties. Against this backdrop, the present study gives an array of results for transferring several well-known and logically-relevant algebraic properties to a variety $\mathcal{V}$ from its class of finitely subdirectly irreducible members $\mathcal{V}_{\text{FSI}}$. This yields a potent strategy for establishing these algebraic properties for varieties algebraizing deduction systems, for which the class of finitely subdirectly irreducibles is often simpler and easier to work with than the class of subdirectly irreducibles.[1]

To express our results in detail, first recall that an algebra $\mathbf{B}$ has the *congruence extension property* (or CEP) if for every subalgebra $\mathbf{A}$ of $\mathbf{B}$ and every congruence Θ of $\mathbf{A}$, there exists a congruence Ψ of $\mathbf{B}$ such that $\Psi \cap A^2 = \Theta$. A class $\mathcal{K}$ of similar algebras is said to have the CEP if each $\mathbf{A} \in \mathcal{K}$ does. The following generalizes [6, Theorem 3.3].

Theorem 1. *Let $\mathcal{V}$ be a congruence-distributive variety. Then $\mathcal{V}$ has the* CEP *if and only if $\mathcal{V}_{FSI}$ has the* CEP.

Several of our results may be expressed succinctly by referencing commutative diagrams. Given a class $\mathcal{K}$ of similar algebras, a *span* in $\mathcal{K}$ is a 5-tuple $\langle \mathbf{A}, \mathbf{B}, \mathbf{C}, \varphi_B, \varphi_C \rangle$ such that $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{K}$ and $\varphi_B \colon \mathbf{A} \to \mathbf{B}$, $\varphi_C \colon \mathbf{A} \to \mathbf{C}$ are homomorphisms. We say that a span in $\mathcal{K}$ is *injective* if φ_B is an embedding, *doubly injective* if both φ_B and φ_C are embeddings, and *injective-surjective* if φ_B is an embedding and φ_C is surjective. The class $\mathcal{K}$ is said to have the *extension property* (or EP) if for any injective-surjective span $\langle \mathbf{A}, \mathbf{B}, \mathbf{C}, \varphi_B, \varphi_C \rangle$ in $\mathcal{K}$, there exist an algebra $\mathbf{D} \in \mathcal{K}$, a homomorphism $\psi_B \colon \mathbf{B} \to \mathbf{D}$, and an embedding $\psi_C \colon \mathbf{C} \to \mathbf{D}$ such that $\psi_B \varphi_B = \psi_C \varphi_C$ (i.e., the diagram in Figure 1(i) commutes). It is well-known that a variety $\mathcal{V}$ has the EP if and only if $\mathcal{V}$ has the CEP [1], but this does not hold for arbitrary classes of algebras. However, under the assumption that $\mathcal{V}_{\text{FSI}}$ is closed under subalgebras, we may strengthen Theorem 1.

Theorem 2. *Let $\mathcal{V}$ be a congruence-distributive variety such that $\mathcal{V}_{FSI}$ is closed under subalgebras. The following are equivalent:*

1. *$\mathcal{V}$ has the congruence extension property.*

*Speaker.

[1] For example, if $\mathcal{V}$ has equationally definable principal meets (a common property for varieties algebraizing logics), then $\mathcal{V}_{\text{FSI}}$ is a universal class [2, Theorem 1.5]. For another prominent example, if $\mathcal{V}$ is a class of semilinear residuated lattices, then $\mathcal{V}_{\text{FSI}}$ consists of exactly the totally ordered algebras in $\mathcal{V}$ [4].

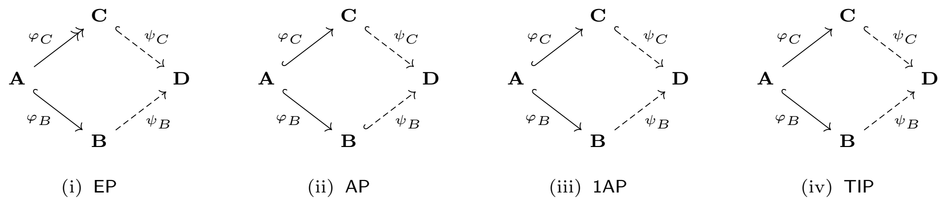

Figure 1: Commutative diagrams for algebraic properties

2. *$\mathcal{V}$ has the extension property.*

3. *$\mathcal{V}_{FSI}$ has the congruence extension property.*

4. *$\mathcal{V}_{FSI}$ has the extension property.*

If $\mathcal{K}$ and $\mathcal{K}'$ are two classes of algebras in a common signature and $\langle \mathbf{A}, \mathbf{B}, \mathbf{C}, \varphi_B, \varphi_C \rangle$ is a doubly injective span in $\mathcal{K}$, an *amalgam in* $\mathcal{K}'$ of this span is a triple $\langle \mathbf{D}, \psi_B, \psi_C \rangle$ where $\mathbf{D} \in \mathcal{K}'$ and $\psi_B \colon \mathbf{B} \to \mathbf{D}$ and $\psi_C \colon \mathbf{C} \to \mathbf{D}$ are embeddings such that $\psi_B\varphi_B = \psi_C\varphi_C$ (i.e., the diagram in Figure 1(ii) commutes). The class $\mathcal{K}$ has the *amalgamation property* (or AP) if every doubly injective span in $\mathcal{K}$ has an amalgam in $\mathcal{K}$. The class $\mathcal{K}$ is said to have the *one-sided amalgamation property* (or 1AP) if for any doubly injective span $\langle \mathbf{A}, \mathbf{B}, \mathbf{C}, \varphi_B, \varphi_C \rangle$ in $\mathcal{K}$, there exist a $\mathbf{D} \in \mathcal{K}$, a homomorphism $\psi_B \colon \mathbf{B} \to \mathbf{D}$, and an embedding $\psi_C \colon \mathbf{C} \to \mathbf{D}$ such that $\psi_B\varphi_B = \psi_C\varphi_C$ (i.e., the diagram in Figure 1(iii) commutes). For varieties, the AP and 1AP coincide, but this does not hold for arbitrary classes of algebras. We obtain the following, generalizing [9, Theorem 9].

Theorem 3. *Let $\mathcal{V}$ be a variety with the congruence extension property such that $\mathcal{V}_{FSI}$ is closed under subalgebras. The following are equivalent:*

1. *$\mathcal{V}$ has the amalgamation property.*

2. *$\mathcal{V}$ has the one-sided amalgamation property.*

3. *$\mathcal{V}_{FSI}$ has the one-sided amalgamation property.*

4. *Every doubly injective span of finitely generated algebras from $\mathcal{V}_{FSI}$ has an amalgam in $\mathcal{V}_{FSI} \times \mathcal{V}_{FSI}$.*

A class $\mathcal{K}$ of similar algebras is said to have the *transferable injections property* (or TIP) if for any injective span $\langle \mathbf{A}, \mathbf{B}, \mathbf{C}, \varphi_B, \varphi_C \rangle$ in $\mathcal{K}$, there exist an algebra $\mathbf{D} \in \mathcal{K}$, a homomorphism $\psi_B \colon \mathbf{B} \to \mathbf{D}$, and an embedding $\psi_C \colon \mathbf{C} \to \mathbf{D}$ such that $\psi_B\varphi_B = \psi_C\varphi_C$ (i.e., the diagram in Figure 1(iv) commutes). From [1, Lemma 1.7], a variety has the TIP if and only if it has both the CEP and AP. Building on the previously-announced results, we obtain the following.

Theorem 4. *Let $\mathcal{V}$ be a congruence-distributive variety such that $\mathcal{V}_{FSI}$ is closed under subalgebras. Then $\mathcal{V}$ has the transferable injections property if and only if $\mathcal{V}_{FSI}$ has the transferable injections property.*

Under appropriate hypotheses, the transfer theorems we have announced may be used in conjunction with Jónsson's Lemma [8] to obtain decidability results for the algebraic properties we have discussed. In particular, we prove the following.

Theorem 5. *Let $\mathcal{V}$ be a finitely generated congruence-distributive variety such that $\mathcal{V}_{FSI}$ is closed under subalgebras. Then there exist effective algorithms to decide if $\mathcal{V}$ has the congruence extension property, amalgamation property, or transferable injections property.*

As previously discussed, the class $\mathcal{V}_{\mathrm{FSI}}$ is often well-behaved when $\mathcal{V}$ is the algebraic counterpart of a deductive system $\vdash$. As an illustration of the transfer theorems we have articulated, we conclude with a case study concerning several subvarieties of BL-algebras, which give equivalent algebraic semantics for certain axiomatic extensions of Hájek's basic fuzzy logic [7]. We use the results articulated previously to determine which of these has the AP, thereby obtaining the deductive interpolation property for the corresponding logics.

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A proof-theoretic approach to ignorance

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1 Introduction

Several recent works in epistemic logic focus on finding a way to model the notion of ignorance (see, e.g., [10], [8], [1], [3]). One of the difficulties in achieving this task is that there is no agreement on which notion of ignorance to model. Indeed, van der Hoek & Lomuscio [10] take ignorance to be 'not knowing whether'; Steinsvold [8] considers ignorance as 'unknown truth'; finally, Kubyshkina and Petrolo [3] introduce a primitive ignorance operator relying on the factive nature of ignorance. We argue that these three different approaches should not be considered as exclusive alternatives, but as representing different aspects of the polysemic notion of ignorance. From this perspective, these three types of ignorance should coexist in the same formal framework. On the basis of this pluralist view, our main objective is to provide a unified framework expressing all the aforementioned types of ignorance, in order to analyse their behaviour and interactions.

We introduce a class of Kripke models, *ignorance models*, which interpret the three types of ignorance. We then define a labelled sequent calculus called labWUDI, and prove its soundness and completeness with respect to ignorance models. Completeness is proved by constructing a countermodel from a failed and finite proof search tree. In future work we plan to define a Hilbert-style axiomatization for ignorance models, to prove admissibility of cut for labWUDI, and to investigate alternative non-labelled calculi to treat ignorance. Furthermore, to study the interactions between ignorance and knowledge modalities, we intend to strengthen our models by imposing (combinations of) reflexivity, transitivity and symmetry on the accessibility relation, and to define sequent calculi formalising these frameworks.

2 Ignorance models

Given a countable set of propositional variables $Atm = \{p, q, \dots\}$, formulas of our language are constructed using the following grammar: $\phi ::= p \mid \bot \mid \phi \to \phi \mid \Box\phi \mid I^w\phi \mid I^u\phi \mid I^d\phi$. Negation is set to be $\neg\phi := \phi \to \bot$, and the other propositional connectives can be standardly defined. Operator I^w, for *ignorance whether*, was introduced by van der Hoek & Lomuscio [10]; I^u, for *unknown truth*, by Steinsvold [8, 9], and I^d by Kubyshkina & Petrolo [3]. Differently from [3], we intuitively interpret I^d as representing a specific type of ignorance, namely, *disbelieving ignorance*, which is characterized by Peels [7] as follows: "[a subject] S is disbelievingly ignorant that p iff (i) it is true that p, and (ii) S disbelieves that p."

For each ignorance operator there exists a complete Hilbert-style system. However, no unified framework for all the three ignorance operators is present in the literature.

Definition 2.1. An *ignorance model* is a triple $\mathcal{M} = \langle W, R, v\rangle$, where W is a set of possible worlds, $R \subseteq W \times W$ and $v : Atm \to 2^W$ is a valuation of propositional variables. We assume R to satisfy the *two-worlds property*, that is: for all $x \in W$, there is a $y \in W$ such that xRy and $x \neq y$. The satisfiability relation of formulas in a world x of a model $\mathcal{M}$ is defined as:

$\mathcal{M}, x \models p$	*iff*	$x \in v(p)$ and $\mathcal{M}, x \not\models \bot$;
$\mathcal{M}, x \models \phi \to \psi$	*iff*	$\mathcal{M}, x \not\models \phi$ or $\mathcal{M}, x \models \psi$;
$\mathcal{M}, x \models \Box\phi$	*iff*	for all $y \in W$, if xRy then $\mathcal{M}, y \models \phi$;
$\mathcal{M}, x \models I^w\phi$	*iff*	there are $y, z \in W$ s.t. xRy, xRz, $\mathcal{M}, y \not\models \phi$ and $\mathcal{M}, z \models \phi$;
$\mathcal{M}, x \models I^u\phi$	*iff*	$\mathcal{M}, x \models \phi$ and there is $y \in W$ s.t. xRy and $\mathcal{M}, y \not\models \phi$;
$\mathcal{M}, x \models I^d\phi$	*iff*	$\mathcal{M}, x \models \phi$ and for all $y \in W$, if xRy and $y \neq x$ then $\mathcal{M}, y \not\models \phi$.

We say that ϕ *is valid in* $\mathcal{M}$ and write $\mathcal{M} \models \phi$ if $\mathcal{M}, x \models \phi$ for all x in W. If for all $\mathcal{M}$ we have $\mathcal{M} \models \phi$, we say that ϕ *is valid*, and write $\models \phi$.

Ignorance whether and unknown truth can be defined in terms of the $\Box$ operator as follows: $I^w\phi := \neg\Box\phi \wedge \neg\Box\neg\phi$ and $I^u\phi = \neg\Box\phi \wedge \phi$. Interestingly, disbelieving ignorance is not definable in terms of $\Box$ in none of the standard frames (K, T, S4, and S5), see [2]. Since we here focus on ignorance operator, we take I^w and I^u as primitive in our language.

The two-worlds property ensures that all worlds in a model have access to some world other than themselves. This allows one to avoid some counterintuitive consequences: for instance, when evaluating formulas at a one-world model $\mathcal{M}$, we get that $\mathcal{M} \models I^d\top$, $\mathcal{M} \models \neg I^w\top$, and $\mathcal{M} \models \neg I^u\top$. Indeed, it seems implausible that an agent is disbelievingly ignorant of a tautology, but she is not ignorant of its truth (neither in the sense of I^w, nor of I^u). By assuming the two-worlds property we obtain validity of formula $\neg I^d\top$.

3 Labelled sequent calculus

In this section, we shall introduce a labelled calculus *lab*WUDI, following the methodology from [4]. We enrich our language by an infinite set of variables, called *labels*: x, y, z, etc. Then, *relational atoms* have the form xRy or $x \neq y$, and *labelled formulas* have the form $x : \phi$. A *labelled sequent* has the form $\Gamma \Rightarrow \Delta$, where Γ is a multiset of relational atoms and labelled formulas and Δ is a multiset of labelled formulas.

The rules of *lab*WUDI are illustrated in Figure 1. The calculus features only one structural rule, 2w, expressing the two-worlds property. Propositional rules and the rules for $\Box$ are standard. The rules for ignorance operators have been defined based on the truth condition of the operators at ignorance models. The condition for I^d on the left is captured by a pair of rules, one of which only introducing formulas within the label of the principal formula, x[1]. Rules I^u_{R} and I^w_{R} introduce $\Box$-formulas in its premisses, needed to express the universal conditions in the negated truth conditions for I^u and I^w respectively.

Labelled sequents do not have a direct formula interpretation, and thus we need to interpret them over ignorance models to prove soundness of the calculus, which is straightforward.

Definition 3.1. Given a labelled sequent $\Gamma \Rightarrow \Delta$ and an ignorance model $\mathcal{M} = \langle W, R, v\rangle$, let $S = \{x \mid x \in \Gamma \cup \Delta\}$ and $\rho : S \to W$. We define the following relation: $\mathcal{M}, \rho \models xRy$ iff $\rho(x)R\rho(y)$; $\mathcal{M}, \rho \models x \neq y$ iff $\rho(x) \neq \rho(y)$; and $\mathcal{M}, \rho \models x : \phi$ iff $\rho(x) \models \phi$. A sequent $\Gamma \Rightarrow \Delta$ *is satisfied at* $\mathcal{M}$ *under* ρ if, if for all formulas $\phi \in \Gamma$ it holds that $\mathcal{M}, \rho \models \phi$, then there exists a

[1] In presence of 2w, rules I^d_{L1} and I^d_{L2} can be formulated as a single rule. Our choice is motivated by modularity: the rules from Figure 1 *without* 2w are adequate w.r.t. ignorance models without the two-worlds property.

$$\textsf{init}\ \frac{}{x:p,\Gamma\Rightarrow\Delta,x:p} \qquad \bot\ \frac{}{x:\bot,\Gamma\Rightarrow\Delta} \qquad \textsf{2w}\ \frac{xRy,x\neq y,\Gamma\Rightarrow\Delta}{\Gamma\Rightarrow\Delta}\ *$$

$$\to_{\mathsf{L}}\ \frac{\Gamma\Rightarrow\Delta,x:\phi \quad x:\psi,\Gamma\Rightarrow\Delta}{x:\phi\to\psi,\Gamma\Rightarrow\Delta} \qquad \to_{\mathsf{R}}\ \frac{x:\phi,\Gamma\Rightarrow\Delta,x:}{\Gamma\Rightarrow\Delta,x:\phi\to\psi}$$

$$\Box_{\mathsf{L}}\ \frac{xRy,x:\Box\phi,y:\phi,\Gamma\Rightarrow\Delta}{xRy,x:\Box\phi,\Gamma\Rightarrow\Delta} \qquad \Box_{\mathsf{R}}\ \frac{xRy,\Gamma\Rightarrow\Delta,y:\phi}{\Gamma\Rightarrow\Delta,x:\Box\phi}\ * \qquad I^d_{\mathsf{L1}}\ \frac{x:I^d\phi,x:\phi,\Gamma\Rightarrow\Delta}{x:I^d\phi,\Gamma\Rightarrow\Delta}$$

$$I^d_{\mathsf{L2}}\ \frac{xRy,x\neq y,x:I^d\phi,\Gamma\Rightarrow\Delta,y:\phi}{xRy,x\neq y,x:I^d\phi,\Gamma\Rightarrow\Delta} \qquad I^d_{\mathsf{R}}\ \frac{\Gamma\Rightarrow\Delta,x:\phi \quad xRy,x\neq y,y:\phi,\Gamma\Rightarrow\Delta}{\Gamma\Rightarrow\Delta,x:I^d\phi}\ *$$

$$I^u_{\mathsf{L}}\ \frac{xRy,x:\phi,\Gamma\Rightarrow\Delta,y:\phi}{x:I^u\phi,\Gamma\Rightarrow\Delta}\ * \qquad I^u_{\mathsf{R}}\ \frac{\Gamma\Rightarrow\Delta,x:\phi \quad x:\Box\phi,\Gamma\Rightarrow\Delta}{\Gamma\Rightarrow\Delta,x:I^u\phi}$$

$$I^w_{\mathsf{L}}\ \frac{xRy,xRz,y:\phi,\Gamma\Rightarrow\Delta,z:\phi}{x:I^w\phi,\Gamma\Rightarrow\Delta}\ * \qquad I^w_{\mathsf{R}}\ \frac{x:\Box\neg\phi,\Gamma\Rightarrow\Delta \quad x:\Box\phi,\Gamma\Rightarrow\Delta}{\Gamma\Rightarrow\Delta,x:I^w\phi}$$

$*$: y (and z, if present) is fresh, i.e., it does not occur in $\Gamma\cup\Delta$.

Figure 1: Sequent calculus *lab*WUDI

formula $\psi\in\Delta$ such that $\mathcal{M},\rho\models\psi$. Then, $\Gamma\Rightarrow\Delta$ *is valid in* $\mathcal{M}$ if the sequent is satisfied at $\mathcal{M}$ for all ρ. Finally, $\Gamma\Rightarrow\Delta$ is valid if the sequent is valid in all models.

Theorem 3.2 (Soundness)**.** *If* $\Rightarrow x:\phi$ *is provable in lab*WUDI*, then* ϕ *is valid.*

To prove completeness of *lab*WUDI with respect to ignorance models, we show how to construct a finite countermodel from a failed and finite proof search tree, adapting to our setting the proof-or-countermodel approach to completeness for labelled calculi introduced in [5] (refer also to [6]). Thus, we first show termination of *lab*WUDI, the main difficulty being that rule 2w may lead to non-termination of root-first proof search. We shall introduce a *proof search strategy* restricting the application of rule 2w. We first define a measure for formulas.

Definition 3.3. We define the *weight of a labelled formula* as follows: $w(xRy)=w(x\neq y)$ $=0$ and, for a labelled formula $x:\phi$, we set $w(x:\chi)=w(\chi)$, where $w(\chi)$ is inductively defined as follows: $w(p)=w(\bot)=1$; $w(\phi\to\psi)=w(\phi)+w(\psi)+1$; $w(K\phi)=w(\phi)+2$; and $w(I^d\phi)=w(I^u\phi)=w(I^w\phi)=w(\phi)+3$.

Next, we define the notion of *saturated sequent*. Intuitively, a sequent is saturated if it is not an initial sequent and if all the rules have been non-redundantly applied to it. More formally, given a branch $\mathcal{B}=\{\Gamma_i\Rightarrow\Delta_i\}_{i>0}$ in a proof search tree and a sequent $\Gamma_n\Rightarrow\Delta_n$ in $\mathcal{B}$, let $\downarrow\Gamma_n=\bigcup_{i=1}^n\Gamma_i$ and $\downarrow\Delta_n=\bigcup_{i=1}^n\Delta_i$. Moreover, given two labels z and x occurring in a sequent $\Gamma_n\Rightarrow\Delta_n$, we write $\mathsf{For}(z)=\mathsf{For}(x)$ meaning that the set of formulas labelled by z and occurring in $\downarrow\Gamma_n$ coincides with the set of formulas labelled with x and occurring in $\downarrow\Gamma_n$, and similarly for $\downarrow\Delta_n$. We then associate to each rule a saturation condition. We explicitly show the one for 2w and, by means of example, the one for I^d_{R}:

(I^d_{R}) If $x:I^d\phi\in\downarrow\Delta_n$, then either $x:\phi\in\downarrow\Delta_n$ or for some y, $xRy\in\Gamma_n$, $x\neq y\in\Gamma_n$ and $y:\phi\in\downarrow\Gamma_n$.

(2w) For all x in $\downarrow \Gamma_n \cup \downarrow \Delta_n$, either $xRy \in \Gamma_n$ and $x \neq y \in \Gamma_n$ for some y, or $zRx \in \Gamma_n$ and $z \neq x \in \Gamma_n$, for some z such that $\mathsf{For}(z) = \mathsf{For}(x)$.

A labelled sequent is *saturated* if it meets the saturation conditions for all the rules, and if it not an instance of $\perp$ or init. Next, we define our *proof search strategy* as follows: given a sequent, we first apply to it rules that do not introduce bottom-up new labels, and rules that do introduce new labels, except for 2w. Once all the other rules have been applied, we apply 2w, taking care of not applying the rule to a label x if one of the two conditions described in the saturation condition is met. The saturation condition (2w) allows to prove the following:

Theorem 3.4 (Termination). *Root-first proof search for a sequent $\Rightarrow x : \phi$ built in accordance with the strategy comes to an end in a finite number of steps, and each leaf of the proof-search tree contains either an initial sequent or a saturated sequent.*

To conclude, we sketch the proof of completeness:

Theorem 3.5 (Completeness). *If ϕ is valid, there is a derivation of $\Rightarrow x : \phi$.*

Proof sketch. We prove the counterpositive. Suppose that $\Rightarrow x : \phi$ is *not* derivable in *lab*WUDI. By termination, if ϕ is not derivable then there is a proof search branch $\mathcal{B}$ whose upper node is occupied by a saturated sequent, $\Gamma_n \Rightarrow \Delta_n$. We construct a model $\mathcal{M}^{\mathcal{B}} = \langle \mathcal{W}^{\mathcal{B}}, \mathcal{R}^{\mathcal{B}}, \mathcal{V}^{\mathcal{B}} \rangle$ that satisfies all formulas in $\downarrow \Gamma_n$ and falsifies all formulas in $\downarrow \Delta_n$ as follows: $\mathcal{W}^{\mathcal{B}} = \{x \mid x \in \downarrow \Gamma_n \cup \downarrow \Delta_n\}$, $\mathcal{R}^{\mathcal{B}} = \{(x, y) \mid xRy \in \Gamma_n\}$ and $\mathcal{V}^{\mathcal{B}}(p) = \{x \in \mathcal{W}^{\mathcal{B}} \mid x : p \in \Gamma_n\}$. Note that distinct variables in $\downarrow \Gamma_n \cup \downarrow \Delta_n$ get mapped to distinct elements in $\mathcal{W}^{\mathcal{B}}$. As it is, $\mathcal{M}^{\mathcal{B}}$ does not satisfy the two-worlds condition. We modify the model as follows. Whenever we have a world x that has no access to worlds other than itself, by the saturation condition (2w) there needs to be a world z such that zRx and $z \neq x$ occur in Γ_n. We add $(x, z) \in \mathcal{R}^{\mathcal{B}}$, and conclude that x satisfies the two-worlds condition. To conclude the proof, one needs to show that $\mathcal{M}^{\mathcal{B}}$ satisfies formulas in $\downarrow \Gamma_n$ and falsifies all formulas in $\downarrow \Delta_n$. This is proved by induction on the weight of formulas, and by taking $\rho(x) = x$, for all $x \in \downarrow \Gamma_n \cup \downarrow \Delta_n$. The crucial case is proving that if $x : I^d \phi$ occurs in $\downarrow \Gamma_n$, then $\mathcal{M}^{\mathcal{B}}, \rho \models x : I^d \phi$. □

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Structural completeness and lattice of extensions in many-valued logics with rational constants

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1 Introduction

The logics **RŁ**, **RP**, and **RG** are obtained by expanding Łukasiewicz logic **Ł**, product logic **P**, and Gödel logic **G** with rational constants $\{\mathbf{c}_q : q \in [0,1] \cap \mathbb{Q}\}$ and adding the bookkeeping axioms: For every $p, q \in [0,1] \cap \mathbb{Q}$,

$$\mathbf{c}_p \cdot \mathbf{c}_q \leftrightarrow \mathbf{c}_{p*q} \qquad (\mathbf{c}_p \to \mathbf{c}_q) \leftrightarrow \mathbf{c}_{p\Rightarrow q} \qquad \mathbf{c}_0 \leftrightarrow \bot \qquad \mathbf{c}_1$$

where $*$ is the Łukasiewicz, Product, and Gödel (minimum) t-norm and $\Rightarrow$ is the Łukasiewicz, Product, and Gödel standard residuated implication in each case.
The history of these logics goes back to the pioneering works of Goguen [13] and Pavelka [20, 21, 22]. Expanding the language with constants can be viewed as taking advantage of the rich algebraic setting to gain more expressivity; see, e.g., [2, 4, 9, 10, 15, 23, 24]. In this talk, we study the lattices of extensions and structural completeness of these three expansions, obtaining results that stand in contrast to the known situation in **Ł**, **P**, and **G**.
A *rule* is an expression of the form $\Gamma \rhd \varphi$, where $\Gamma \cup \{\varphi\} \subseteq Fm$ is a finite set. A rule $\Gamma \rhd \varphi$ is said to be *derivable* in a logic $\vdash$ when $\Gamma \vdash \varphi$. It is *admissible* in $\vdash$ when for every substitution σ on Fm,

$$\text{if } \emptyset \vdash \sigma(\gamma) \text{ for all } \gamma \in \Gamma, \text{ then } \emptyset \vdash \sigma(\varphi).$$

In other words, a rule is admissible in $\vdash$ when its addition to $\vdash$ does not produce any new theorem. Clearly, every rule that is derivable in $\vdash$ is also admissible in $\vdash$. If the converse holds, $\vdash$ is said to be *structurally complete* (SC). Logics whose extensions are all structurally complete have been called *hereditarily structurally complete* (HSC).
During the last two decades, research in structural completeness has turned also to the family of fuzzy logics. While **G** and **P** are hereditarily structurally complete [8, 5], **Ł** is structurally incomplete [7] and a base for its admissible rules was exhibited by Jeřábek [17], see also [16, 18]. Admissibility in extensions of **Ł** was investigated in [11, 12].

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Łukasiewicz logic **Ł**, product logic **P**, and Gödel logic **G** can be obtained as axiomatic extensions of Hájek's basic logic **BL** [14], even if they were defined independently prior to the definition of **BL**. All logics in the **BL** family are algebraizable in the sense of Blok and Pigozzi [3]: the equivalent algebraic semantics of the three logics are the varieties of MV-algebras, product algebras and Gödel algebras, respectively. In fact, from algebraizablity we obtain a dual lattice isomorphism from the lattice of finitary extensions of each logic $L_E(\vdash)$ into the lattice of quasivarieties of each equivalent variety semantics $L_Q(\mathbb{V})$. Moreover, if we restrict this isomorphism to the lattice of axiomatic extensions $L_{AE}(\vdash)$ we get an isomorphism $L_{AE}(\vdash) \cong L_V(\mathbb{V})$, where $L_V(\mathbb{V})$ denotes the lattice of all subvarieties of $\mathbb{V}$.
Komori in [19] characterizes $L_{AE}(\text{Ł})$ which forms an infinite non totally ordered denumerable pseudo Boolean algebra. The lattice of all extensions $L_E(\text{Ł})$ is as complicated as it can be, since the class of all MV-algebras $\mathbb{MV}$ is Q-universal [1]. That is, for every quasivariety $\mathbb{K}$ of finite type $L_Q(\mathbb{K})$ is a homomorphic image of a sublattice of $L_Q(\mathbb{MV})$.
The lattice of all axiomatic extensions of **P** is just the three element chain where the only consistent proper axiomatic extension of **P** is classical logic. Since **P** is hereditary structurally complete $L_E(\mathbf{P}) = L_{AE}(\mathbf{P})$ [6].
Finally, every axiomatic consistent extension of **G** is a finite valued Gödel logic and $L_{AE}(\mathbf{G}) \cong \omega + 1$. Since **G** is hereditary structurally complete $L_E(\mathbf{G}) \cong \omega + 1$ (see [8]).
Structural completeness and the structure of the lattice of axiomatic extensions and the lattice of extensions need not to be preserved when expanding with rational constants, while algebraizability is preserved:

(i) **RŁ** is an algebraizabe conservative expansion of **Ł** and the variety of all rational MV-algebras $\mathbb{RMV}$ is its equivalent variety semantics.

(ii) **RP** is an algebraizable conservative expansion of **P** and the variety of all rational product algebras $\mathbb{RP}$ is its equivalent variety semantics.

(iii) **RG** is an algebraizable conservative expansion of **G** and the variety of all rational Gödel algebras $\mathbb{RG}$ is its equivalent variety semantics.

We recall that a *rational MV-algebra* , *rational product algebra* and *rational Gödel algebra* is an algebra **A** in the language $\mathcal{L} = \{\wedge, \vee, \cdot, \rightarrow, \bot, \top\} \cup \{\mathbf{c}_q : q \in [0,1] \cap \mathbb{Q}\}$ such that the $\{\wedge, \vee, \cdot, \rightarrow, \bot, \top\}$-reduct is an MV-algebra, Product algebra and Gödel algebra respectively and it satisfies the following bookkeeping equations: For every $p, q \in [0,1] \cap \mathbb{Q}$,

$$\mathbf{c}_p \cdot \mathbf{c}_q \approx \mathbf{c}_{p*q} \qquad (\mathbf{c}_p \rightarrow \mathbf{c}_q) \approx \mathbf{c}_{p \Rightarrow q} \qquad \mathbf{c}_0 \approx \bot \qquad \mathbf{c}_0 \approx \top$$

2 Main results

2.1 Rational Łukasiewicz logic

For the case of Łukasiewicz adding rational constants trivializes the lattice of extensions:

Theorem 2.1. ***RŁ*** *has no proper consistent extensions, hence* ***RŁ*** *is hereditary structurally complete.*

$$L_E(\boldsymbol{R\text{Ł}}) = L_{AE}(\boldsymbol{R\text{Ł}}) \cong 2$$

2.2 Rational Product logic

In the case of product logic adding rational constants does not have a significant change in the lattice of axiomatic extensions

Theorem 2.2. **RP** *has two proper consistent axiomatic extensions: namely PL and CL.*

- *PL is axiomatized by c_q for each (some) $q \in (0,1] \cap \mathbb{Q}$*
- *CL is axiomatized by c_q for each (some) $q \in (0,1] \cap \mathbb{Q}$ plus $\varphi \vee (\varphi \to \bot)$*

Corollary 2.3. *$L_{AE}(\mathbf{RP})$ is a four element chain.*

Notice that PL is equivalent to the original **P** and CL is equivalent to classical logic, hence when studying admissible rules we will only need to study admissble rules for **RP**.

Theorem 2.4. *Every proper extension of* **RP** *is structurally complete, but* **RP** *is not structurally complete. A base for the admissible rules of* **RP** *is given by the set of rules of the form*

$$\mathbf{c}_q \vee z \rhd z \qquad (\mathbf{c}_p \leftrightarrow x^n) \vee z \rhd z,$$

for each (equiv. some) $q \in (0,1) \cap \mathbb{Q}$ and each $p \in [0,1] \cap \mathbb{Q}$, $n \in \omega$ such that $\sqrt[n]{p}$ is irrational.

Finally, the biggest contrast expanding with rational constants is in the lattice of extensions. We can not obtain a nice description of the lattice $L_E(\mathbf{RP})$ because of the following result:

Theorem 2.5. *The variety $\mathbb{RP}$ is Q-universal.*

2.3 Rational Gödel Logic

Expanding Gödel logic with rational constants have a significant efect in the lattice of axiomatic extensions. In fact next result shows that we go from a numerable chain to an uncountable chain.

Theorem 2.6. *Every consistent axiomatic extension of* **RG** *is of the form*
$\mathbf{RG}_r := \mathbf{RG} + \{\mathbf{c}_q : q \in [r,1] \cap \mathbb{Q}\}$ *for some* $r \in (0,1]$,
$\mathbf{RG}_p^\omega := \mathbf{RG} + \{\mathbf{c}_q : q \in (p,1] \cap \mathbb{Q}\}$ *for some rational* $p \in [0,1)$ *or*
$\mathbf{RG}_p^n := \mathbf{RG}_p^\omega + \bigvee_{0 \leqslant i < j \leqslant n+2} (\boldsymbol{c}_p \vee x_i) \leftrightarrow (\boldsymbol{c}_p \vee x_j)$ *for some rational* $p \in [0,1)$ *and* $n \in \omega$.
Moreover, $L_{AE}(\mathbf{RG})$ is an uncountable chain dually isomorphic to the poset obtained adding a new bottom element to the Dedekind–MacNeille completion of the lexicographic order of $([0,1) \cap \mathbb{Q}) \times (\omega + 1)$.

Observe that $\mathbf{RG}_1 = \mathbf{RG}$ and that $\mathbf{RG}_0^\omega$ is equivalent to **G** and $\mathbf{RG}_0^n$ is equivalent to the $(n+2)$-valued Gödel logic. Next result shows that none of the other extensions of **RG** is structurally complete.

Theorem 2.7. *The only consistent axiomatic extensions of* **RG** *structurally complete are $\mathbf{RG}_0^\omega$ and $\mathbf{RG}_0^n$ for each $n \in \omega$. Moreover, for all $r \in (0,1]$, $p \in [0,1) \cap \mathbb{Q}$, and $\gamma \in \omega + 1$:*

- *A base for the admissible rules of $\mathbf{RG}_r$ is given by the rules of the form $\boldsymbol{c}_q \vee z \rhd z$, for all $q \in [0,r) \cap \mathbb{Q}$;*
- *A base for the admissible rules of $\mathbf{RG}_p^\gamma$ is given by the rule $\boldsymbol{c}_p \vee z \rhd z$.*

If we denote by $\overline{\mathbf{RG}_r}$ the structural completion of $\mathbf{RG}_r$, then $\{\overline{\mathbf{RG}_r} : r \in (0,1]\}$ is an uncountable antichain in $L_E(\mathbf{RG})$. Conseqüently, $L_E(\mathbf{RG})$ seems not easy to describe since it contains an uncountable antichain and, by Theorem 2.6, it contains an uncountable chain. The question whether $\mathbb{RG}$ is Q-universal remains open.

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From implicative reducts to Mundici's functor

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The connection between substructural logics and residuated lattices is one of the most relevant results of algebraic logic. Indeed, it establishes a framework where different systems, or equivalently, classes of structures, can be both compared and studied uniformly.

Among the most well-known connections among different structures in this framework surely stands Mundici's theorem, which establishes a categorical equivalence between the algebraic category of MV-algebras and lattice-ordered abelian groups (abelian ℓ-groups in what follows) with strong order unit (an archimedean element with respect to the lattice order), with unit preserving homomorphisms. This equivalence, connecting the equivalent algebraic semantics of infinite-valued Łukasiewicz logic (i.e., MV-algebras) with ordered groups, has been deeply investigated and also extended to more general structures. In particular, Dvurečenskij first extended Mundici's approach to the case where the monoidal operation involved is not commutative, showing a categorical equivalence between ℓ-groups with strong unit and *pseudo* MV-algebras.

Alternative algebraic approaches to Mundici's functor have been proposed by other authors. In particular, Galatos and Tsinakis in [5] extended both Mundici and Dvurčenskij's result to the unbounded and non-commutative setting of *generalized* MV-algebras, using a truncation construction based on the work of Bosbach on *cone algebras* [2, 3]. The connection between ℓ-groups and other relevant structures in algebraic logic is also deeply explored in [1].

In the present contribution we re-elaborate Rump's work [6], which is again inspired by Bosbach's ideas [2],[3],[4], but focuses on structures with only one implication and a constant (whereas Bosbach's cone algebras, used also in [5], have two implications). The key idea is to characterize which structures in this reduced signature embed in an ℓ-group. We find conditions that are different (albeit equivalent) to the ones found by Rump, and moreover we extend some of Rump's constructions to categorical equivalences of the algebraic categories involved.

Let us now give more details. The construction starts from structures having only one binary operation, which interprets some form of implication. We call *unital magma* a structure $\langle M, \to, 1\rangle$, satisfying the following equations and quasi-equation:

(M1) $x \to x \approx 1$;

(M2) $1 \to x \approx x$;

(M3) $x \to 1 \approx 1$;

(M4) $(x \to y \approx y \to x) \Rightarrow x \approx y$.

In particular, we call a unital magma a *H-algebra* if it satisfies two more equations:

(H) $(x \to y) \to (x \to z) \approx (y \to x) \to (y \to z)$;

(K) $x \to (x \to y) \approx 1$.

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With the properties (H) and (K) one can define a partial order on the structures in the usual way: $x \leq y$ iff $x \to y = 1$.

H-algebras turn out to be the appropriate framework to identify the fundamental properties that are satisfied by an implicative reduct of an ℓ-group, and thus they are the starting point for the construction. In particular, given any H-algebra $\mathbf{A}$, one first constructs the free generated monoid from $\mathbf{A}$, and then suitably considers a particular quotient in order to get a right-cancellative left-complemented (i.e., left-residuated) monoid, let us call it $\mathbf{L_A}$. The starting algebra $\mathbf{A}$ embeds (with respect to its reduced signature) in $\mathbf{L_A}$. The idea is now to embed $\mathbf{L_A}$ into the negative cone of an ℓ-group. Thus, the next step is to construct a group $\mathbf{G_{L_A}}$ from the previously obtained left-complemented monoid, with a construction that is similar to Øre's group of fractions. That is, considering $\mathbf{L_A}$, one can define the following equivalence relation on $L_A \times L_A$ to construct $G_{\mathbf{L_A}}$:

$$(a, b) \equiv (c, d) \text{ if and only if there exists } u, v \in L_A \text{ such that } ua = vc \text{ and } ub = vd.$$

Given particular properties of $\mathbf{L_A}$, $G_{\mathbf{L_A}}$ can be equipped with the operations of a partially ordered group. Then it follows from well-known results that cancellativity of $\mathbf{L_A}$ is a sufficient and necessary condition for $\mathbf{L_A}$ to be embedded as a partially ordered monoid into the negative cone of an ℓ-group. In order to obtain en embedding with respect to $\to$ (where in the ℓ-group, $x \to y = yx^{-1} \wedge 1$), one needs to require a condition called *regularity*, that can be expressed in terms of the implication $\to$. Thanks to regularity, the generated group $\mathbf{G_{L_A}}$ is in particular an ℓ-group. If one also requires the starting algebra $\mathbf{A}$ to be *full*, that is, for all $b, c \in A$, if $b \leq c$, there exists $a \in A$ such that $a \to b = c$, one gets an isomorphism between $\mathbf{L_A}$ and the negative cone of $\mathbf{G_{L_A}}$ in the reduced signature. Fullness and regularity turn out to be necessary and sufficient conditions for a cancellative, left complemented monoid to be isomorphic with the negative cone of an ℓ-group (seen as a left complemented monoid). Moreover, this characterization can be extended to a categorical equivalence. Indeed we can show that the algebraic categories of negative cones of ℓ-groups and of full, regular, cancellative, left-complemented monoids are equivalent.

In order to recover Mundici's theorem, we focus our attention on bounded H-algebras. In order to characterize embeddings for bounded H-algebras, we show the following lemma:

Lemma 1. *A bounded H-algebra* $\mathbf{A}$ *is* $\to$*-isomorphic with the interval* $[u, 1]$ *of a given* ℓ*-group with strong unit* u*, if and only if*

- *for all* $a, b \in A$*, if there exists* $v \in \mathbf{L_A}$*,* $v \leq a, b$*, such that* $a \to v = b \to v$*, then* $a = b$*;*
- A *is full and regular.*

As Rump's observes in [6], if an H-algebra satisfies Tanaka's equation, then $\mathbf{L_A}$ is commutative as a monoid, it has a lattice order, and moreover, we show that it satisfies the conditions of the previous lemma. Thanks to this observation, we gain that given any $\mathbf{A}$ full, regular and bounded H-algebra satisfying Tanaka's equation, then $\mathbf{A}$ is $\to$-isomorphic to the interval $[u, 1]$ of the negative cone of an ℓ-group. The latter extends to Mundici's result.
In order to deal with the non-commutative case, a little more work is required to show that the two implications can be recovered with this construction. Nonetheless, we can prove that, with some further technicalities, the same construction works. In the process we get another categorical equivalence of some interest and as another particular case, Dvurečenskij's result.

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Multi-type modal extensions of the Lambek calculus for structural control

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In the tradition of 'parsing as deduction', various logical calculi have been considered for applications in formal linguistics. A lively strand of research focuses on the analysis of logical systems specifically designed to model a controlled linguistic resource management [20, 16, 21, 22, 11, 1, 28]. Research on the so-called *structural control* (in combination with various modal and substructural logics) is also motivated by applications in other domains and has given rise to a rich literature in logic (see [7, 10, 12, 5, 27]).

Lambek's Syntactic Calculus [17, 18] is an early representative of substructural logic. The original Lambek calculus lacks the required expressivity to serve as a tool for realistic grammar development. The extended Lambek calculi introduced in the 1990ies enrich the type language with modalities for structural control. These modalities have found two distinct uses [16]. On the one hand, modalities can act as *licenses*, granting the applicability of so-called *structural rules* that by default would not be permitted. On the other hand, modalities can be used to *block* structural rules that otherwise would be available.

Examples of modalities as licensors relate to various aspects of grammatical resource management: multiplicity, order and structure. As for multiplicity, under the control of modalities limited forms of copying can be introduced in grammar logics that overall are resource-sensitive systems, see [26, 25, 13, 19] for some recent examples. As for order and structure, modalities may be used to license changes of word order and/or constituent structure that leave the form-meaning correspondence intact, as illustrated e.g. in [24, 3].

An example of the complementary use of modalities as blocking devices can be found in [14, 15]. The authors propose a modally-extended type language designed to simultaneously account for function-argument structure and *dependency structure*. For function-argument structure the key opposition is between a function type A/B (or $B\backslash A$) that combines with its argument B to produce an A. Dependency structures [4] on the other hand are based on the opposition between a *head* and its *dependents*; these dependents can either be *complements* selected by the head, or *adjuncts* modifying the head. In the phrase "Alice left unexpectedly", for example, the verb "left" is the head selecting for "Alice" as a complement with the *subject* role; "unexpectedly" is an adjunct modifying the head. To capture these dependency relations, [14, 15] refine the Lambek types $NP\backslash S$ and $S\backslash S$ for "left" and "unexpectedly" to $(\Diamond^{\mathrm{su}}NP)\backslash S$ and $\Box^{\mathrm{adv}}(S\backslash S)$. In general, $(\Diamond^{\mathrm{c}}A)\backslash B$ is the type for a head selecting an A complement with dependency role c, and $\Box^{\mathrm{m}}(A\backslash B)$ for an adjunct with dependency role m modifying a head A. The dependency modalities do not come with structural rules, but they have the effect of sealing off a structure consisting of a head together with its dependents as a *domain of locality*.

In this talk, we reconsider the licensing of structural rules in the light of the locality domains induced by the dependency-enhanced type language. To put the discussion in perspective, we introduce the class of multi-type logics for explicit structural control management together with their algebraic semantics, and provide proper display calculi for the basic logics and their extensions via axioms of a specific syntactic shape (the so-called *analytic-inductive axioms* [9]) in a modular fashion (e.g. preserving completeness, subformula property and cut elimination) according to the general methodology emerged in the field of structural and algebraic proof theory [2, 6, 8, 9]; in particular, all the logics considered in

[23, 14, 15] and related work, when recast as mSCLs, can profit from the pleasant proof-theoretic and model theoretic benefits that the multi-type approach brings with it.

For each $i \in I$, a *heterogeneous structural control algebra* is a structure

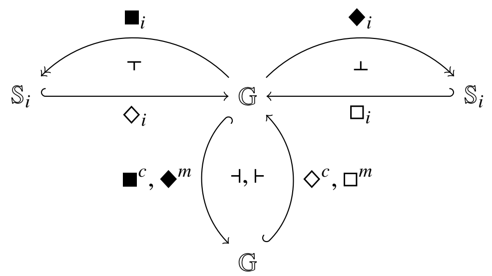
$$\mathbb{H} := (G, S_i, \Diamond_i, \blacksquare_i, \Box_i, \blacklozenge_i, \mathcal{F}, \mathcal{G}, \leq_G, \leq_{S_i})$$

such that

- $\mathbb{G} := (G, \leq_G, \mathcal{F}, \mathcal{G})$ is a partially ordered algebra, $\mathcal{F}$ (resp. $\mathcal{G}$) is a set of maps from $\mathbb{G}^n$ to $\mathbb{G}$ for some natural number n, and for each map in $\mathcal{F} \cup \mathcal{G}$ the corresponding adjoint/residual is also in $\mathcal{F} \cup \mathcal{G}$ (where the maps in $\mathcal{F}$ are left adjoints/residuals and the maps in $\mathcal{G}$ are right adjoints/residuals);

- $(S_i, \leq_{S_i})$ is a partial order; we refer to $\Diamond_i, \blacksquare_i, \Box_i, \blacklozenge_i$ as *structural control modalities*, they are unary *heterogeneous* (given their source and target do not coincide) modalities, namely such that $\blacksquare_i : G \twoheadrightarrow S_i$, $\Diamond_i : S_i \hookrightarrow G$ (where $\Diamond_i \dashv \blacksquare_i$), and $\blacklozenge_i : G \twoheadrightarrow S_i$, $\Box_i : S_i \hookrightarrow G$ (where $\blacklozenge_i \dashv \Box_i$);

- the composition $\blacksquare^c\Diamond^c : G \to G$ (resp. $\blacklozenge^m\Box^m : G \to G$) defines a closure operator, (resp. an interior operator), and the compositions $\blacksquare_i\Diamond_i : S_i \to S_i$ and $\blacklozenge_i\Box_i : S_i \to S_i$ define identity on S_i.



G is the set of *general* elements. The sort $(S_i, \leq_{S_i})$ is a set of *special* elements that witness the (controlled) licence of structural rules (that by default would not be permitted). The structural control modalities identify special elements in the general regime/type modulo the composition of adjoint pairs. For instance, in the expanded signature of the Lambek calculus the postulate $(x \otimes y) \otimes \Diamond_a \alpha \leq_G x \otimes (y \otimes \Diamond_a \alpha)$ represents a controlled form of left-to-right associativity. The x, y here are general elements, $\otimes$ is the binary fusion operator of the Lambek calculus, and $\Diamond_a \alpha$ is the image of a special element α which then *licenses* the restructuring.

In this talk the dependency modalities are *homogeneous* (as opposed to heterogeneous and given that their source and target coincide) primitive modalities defined in the general type $\mathbb{G}$ (so, they are unary maps in $\mathcal{F} \cup \mathcal{G}$). We will focus on the use of dependency modalities as means to block structural rules. Nonetheless, other design choices are also conceivable. We will also briefly expand on a few alternative design options, and we will discuss their pros and cons from the perspective of their use in linguistics.

Each and every design option falls under the scope of a general methodology that allow us to introduce multi-type proper display calculi enjoining canonical cut elimination. In particular, we observe that all the logical introduction rules are standard and reflect the minimal order-theoretic properties of the primitive operators, while the controlled linguistic resource management is explicitly captured by structural rules, so maintaining a neat division of labour that guarantees a modular treatment. At last, all the structural rules are automatically generated via the algorithm ALBA [9] (here generalized to a multi-type environment).

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Probability via Łukasiewicz logic: a multi-type semantic and proof theoretical account

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Providing good proof systems for probabilistic logics is a long standing problem in proof theory and logics for uncertainty. This work is a part of larger research project aimed at providing good proof systems for probabilistic logics and other logics of uncertainty in a uniform and modular way. In this project, we use a generalization of display calculi introduced by Belnap. This choice is motivated by the following two reasons. Firstly, display calculi are by design modular, insofar they implement a neat division of labour between logical rules (introducing the connectives and relying on their minimal order-theoretic properties) and so-called structural rules (capturing the specific features of the logic under consideration). Secondly, they provide a framework in which cut-elimination, a crucial property of proof systems, can be proved in a principled way as an application of a general meta-theorem.

Logics for reasoning about probability have been extensively studied. In 1990, [4] introduces a logic to reason about probabilities and its Hilbert style calculus that contains three types of axioms and rules: the ones that govern the arithmetical part, i.e., the reasoning about inequalities; the ones that axiomatise probabilities; and the rules and axioms of classical propositional logic. In [13] and later in [12], probabilities are axiomatized via fuzzy logics, in a language with two sorts: a sort for expressing Boolean statements and a sort for statements about probabilities. This two-layer approach for probabilistic logics is further developed in [6, 5, 7]. Finally, in 2020, [1] utilises a two-layered modal logic to formalise reasoning about probabilities. The proposed calculus consists of three parts: the rules and axioms of the logic of events (i.e. classical logic) or 'inner logic'; the 'outer logic' that formalises reasoning with probabilities; and finally, the modalities that transform events into probabilistic statements.

The main difficulties in applying the theory of display calculi to the probability logics lies in the handling of the operators $+$ and $-$ (i.e. the truncated sum and difference, respectively) and their interaction with the probability operator P in well-known axiomatization of probability. Here we rely on an ongoing work, where we introduce a generalization of standard display calculi to capture Łukasewicz logic and, in particular, to deal with the axiom

$$((A \to B) \to B) \to (A \vee B) \tag{1}$$

which can be equivalently written as

$$((A - B) + B) \to (A \vee B)$$

and which is closely connected to the probability axiom

$$((P(A) - P(A \wedge B)) + P(B)) \to P(A \vee B).$$

Łukasiewicz logic is one of the most well-know and thoroughly studied mathematical fuzzy logics (see [14] for an overview of proof theoretic literature on mathematical fuzzy logics).

Nonetheless, the distinctive axiom of Łukasiewicz logic 1 is not **analytic-inductive** [11] (not even canonical) and it represents the main obstacle to a uniform and modular proof-theoretic treatment. Pivoting on an algebraic analysis of Łukasiewicz logic, we introduce a refinement of the general theory of display sequent calculi and algorithmic rule generation (as developed for instance in [8] and [11], respectively) aiming at overcoming this problem. In particular, we rely on the fact that Łukasiewicz operators are not only normal operators, but also regular operators in the following sense (in [10] and [9] such operators are called 'double quasioperators'):

normal binary diamond	normal binary box
$A \odot \mathbf{0} = \mathbf{0} = \mathbf{0} \odot A$ $(A \vee B) \odot C = (A \odot C) \vee (B \odot C)$ $C \odot (A \vee B) = (C \odot A) \vee (C \odot B)$	$A \oplus \mathbf{1} = \mathbf{1} = \mathbf{1} \oplus A$ $(A \wedge B) \oplus C = (A \oplus C) \wedge (B \oplus C)$ $C \oplus (A \wedge B) = (C \oplus A) \wedge (C \oplus B)$
$A \ominus \mathbf{1} = \mathbf{0} = \mathbf{0} \ominus A$ $(A \vee B) \ominus C = (A \ominus C) \vee (B \ominus C)$ $C \ominus (A \wedge B) = (C \ominus A) \vee (C \ominus B)$	$A \to \mathbf{1} = \mathbf{1} = \mathbf{0} \to A$ $(A \vee B) \to C = (A \to C) \wedge (B \to C)$ $C \to (A \wedge B) = (C \to A) \wedge (C \to B)$

regular binary diamond	regular binary box
$(A \vee B) \oplus C = (A \oplus C) \vee (B \oplus C)$ $C \oplus (A \vee B) = (C \oplus A) \vee (C \oplus B)$	$(A \wedge B) \odot C = (A \odot C) \wedge (B \odot C)$ $C \odot (A \wedge B) = (C \odot A) \wedge (C \odot B)$
$(A \wedge B) \to C = (A \to C) \vee (B \to C)$ $C \to (A \vee B) = (C \to A) \vee (C \to B)$	$(A \wedge B) \ominus C = (A \ominus C) \wedge (B \ominus C)$ $C \ominus (A \vee B) = (C \ominus A) \wedge (C \ominus B)$

Exploiting the previous observation, we introduce a language expansion where the different "personalities" (normal versus regular) of the operators are fully-fledged and, in turn, it becomes possible to introduce a sequent calculus with the so-called **relativized display property** (namely, every structure occurring in a derivable sequent is displayable). Moreover, all the logical introduction rules are standard and reflect the minimal order-theoretic properties of the operators, while the specific features of the logic are captured by so-called structural rules, so maintaining a neat division of labour that guarantees a modular treatment. At last, all the structural rules are automatically generated via (a specialisation of) the algorithm ALBA (to regular operators). Showing that the calculus enjoys (canonical) cut elimination is future work.

Below we expand on the treatment of the probability operator. The key idea is that the non-normal operators (like the conditional binary operator of conditional logics or the monotone unary modalities in non-normal modal logics) can be decomposed into the composition of normal modal operators [3]. In this work, we use a similar approach to deal with the probability operator P.

Let $\mathcal{B}$ be any set and $\mathcal{P}(\mathcal{B})$ be its power-set. Let $P : \mathcal{P}(\mathcal{B}) \to [0, 1]$ be a probability function on it. Let $R_\in, R_\notin \subseteq \mathcal{P}(\mathcal{B}) \times \mathcal{B}$ be defined as follows. For any $a \in \mathcal{B}$, $A \in \mathcal{P}(\mathcal{B})$,

$$AR_\in a \quad \text{iff} \quad a \in A \quad \text{and} \quad AR_\notin a \quad \text{iff} \quad a \notin A.$$

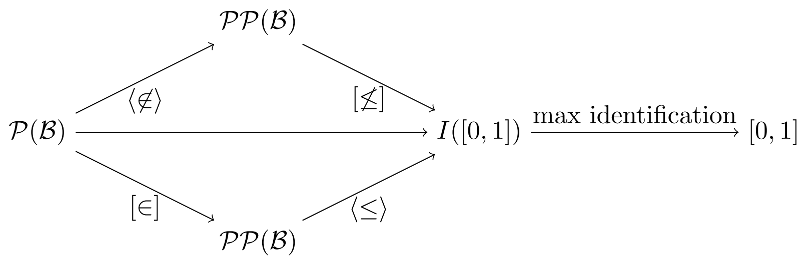

Figure 1: Decomposition of the probability operator P using normal operators

Let $R_{\leq}, R_{\nleq} \subseteq [0,1] \times \mathcal{P}(\mathcal{B})$ be defined as follows. For any $\alpha \in [0,1]$, $A \in \mathcal{P}(\mathcal{B})$,

$$\alpha R_{\leq} A \quad \text{iff} \quad \alpha \leq P(A) \quad \text{and} \quad \alpha R_{\nleq} A \quad \text{iff} \quad \alpha \nleq P(A).$$

Let $A \subseteq \mathcal{B}$, and $U \subseteq \mathcal{P}(\mathcal{B})$ be any subsets of $\mathcal{B}$ and $\mathcal{P}(\mathcal{B})$ respectively. Let $[\in](A) = [R_{\in}](A)$, $\langle\notin\rangle(A) = \langle R_{\notin}\rangle(A)$, $\langle\leq\rangle(U) = \langle R_{\leq}\rangle(U)$, and $[\nleq](U) = [R_{\nleq}](U)$. Then, we have

Lemma 1. *For any $A \subseteq \mathcal{B}$, and $U \subseteq \mathcal{P}(\mathcal{B})$,*

1. $[\in](A) = A^{\downarrow}$.
2. $\langle\notin\rangle(A) = (A^{\uparrow})^c$.
3. $\langle\leq\rangle(U) = [0, \max\{P(A) \mid A \in U\}]$.
4. $[\nleq](U) = [0, \min\{P(A) \mid A \in U^c\}]$.

The following corollary follows immediately from the Lemma.

Corollary 2. *For any $A \subseteq \mathcal{B}$, $P(A) = \max(\langle\leq\rangle[\in](A)) = \max([\nleq]\langle\notin\rangle(A))$.*

Thus, under the identification of an interval with its largest element above, the corollary shows that the probability operator P can be decomposed into the combination of normal operators $\langle\leq\rangle$, $[\in]$, $[\nleq]$, and $\langle\notin\rangle$ in two ways. This decomposition allows us to write the probability axioms in the language of Łukasewicz logic expanded with the above modal operators. Therefore, the axioms of probability logic can be expressed in the above multi-type normal modal logic.

Finally, the difficult (non-analytic) axiom in the probability theory is the inclusion-exclusion axiom. This axiom is very similar to the peculiar axiom of Łukasewicz logic discussed earlier (with the addition of the operator P). In this talk we will expand on the work in progress aiming at introducing a properly displayable multi-type calculus for probability logic. Showing that the calculus enjoys (canonical) cut elimination is future work.

We believe that these techniques would allow us to deal with other (non-classical) logics of uncertainty such as the logics for probabilities and belief functions over Belnap-Dunn logic introduced in [2].

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From contact relations to modal opetators, and back

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One of the standard axioms for Boolean contact algebras says that if a region x is in contact with the join of y and z, then x is in contact with at least one of the two regions. Our intention is to examine a stronger version of this axiom according to which if x is in contact with the supremum of some family S of regions, then there is y in S that is in contact with x.

Any Boolean algebra is turned into a *Boolean contact algebra* by expanding it to a structure $\mathfrak{B} = \langle B, \cdot, +, -, \mathbf{0}, \mathbf{1}, \mathsf{C}\rangle$ where $\mathsf{C} \subseteq B^2$ is a *contact* relation which satisfies the following five axioms:

$$\neg(\mathbf{0} \mathrel{\mathsf{C}} x), \tag{C0}$$

$$x \leq y \wedge x \neq \mathbf{0} \longrightarrow x \mathrel{\mathsf{C}} y, \tag{C1}$$

$$x \mathrel{\mathsf{C}} y \longrightarrow y \mathrel{\mathsf{C}} x, \tag{C2}$$

$$x \leq y \wedge z \mathrel{\mathsf{C}} x \longrightarrow z \mathrel{\mathsf{C}} y\,, \tag{C3}$$

$$x \mathrel{\mathsf{C}} y + z \longrightarrow x \mathrel{\mathsf{C}} y \vee x \mathrel{\mathsf{C}} z\,. \tag{C4}$$

In this study, we consider *complete* Boolean contact algebras in which the contact *completely* distributes over join, i.e., those that satisfy the following second-order constraint:

$$x \mathrel{\mathsf{C}} \bigvee_{i\in I} x_i \longrightarrow (\exists i \in I)\, x \mathrel{\mathsf{C}} x_i\,. \tag{C4$^{\mathrm{c}}$}$$

It is clear that (C4$^{\mathrm{c}}$) entails (C4), yet the converse implication is not true in general. The main objective of the talk is to present the consequences of adopting (C4$^{\mathrm{c}}$) as an axiom, provide several examples, and analyze a modal possibility operator that is definable in the class of contact algebras satisfying the aforementioned stronger version of (C4).

1 A modal operator

It is known that the relation of *subordination* on a Boolean algebra is a natural generalization of the notion of modal operator. For instance, modal operators give rise to special subordinations called by [1] *modally definable*. Moreover, the authors prove that modal operators on a Boolean algebra are in one-to-one correspondence with modally definable subordinations. Contact relations give rise to non-tangential inclusion that is a special case of the subordination relation:

$$x \ll y \;:\longleftrightarrow\; x \mathrel{\not\mathsf{C}} -y\,,^{1} \tag{$\mathtt{df}\ll$}$$

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and in the talk we show that there is a correspondence between contact and modal operators. The crucial observation is that if $\mathfrak{B}$ is a complete Boolean contact algebra, then $\mathfrak{B}$ satisfies (C4^{c}) iff for every region x there exists a unique region y such that $\mathsf{C}(x) = \mathsf{O}(y)$, where:

$$\mathsf{C}(x) \coloneqq \{y \in B \mid x \,\mathsf{C}\, y\}\,,$$
$$\mathsf{O}(x) \coloneqq \{y \in B \mid x \cdot y \neq \mathbf{0}\}\,.$$

The uniqueness property entails existence of an operation $m\colon B \longrightarrow B$ such that:

$$m(x) \coloneqq (\iota y)\ \mathsf{C}(x) = \mathsf{O}(y)\,.^{2} \tag{df m}$$

We prove that m is a modal possibility operator. Obviously, we have that:

$$x \,\mathsf{C}\, y \longleftrightarrow m(x) \cdot y \neq \mathbf{0}\,,$$

and so:

$$x \ll y \longleftrightarrow m(x) \leq y\,.$$

Recall that any modal algebra $\mathfrak{B} \coloneqq \langle B, \Diamond\rangle$ whose possibility operator satisfies the following two conditions:

$$x \leqslant \Diamond x\,, \tag{T$_\Diamond$}$$
$$\Diamond\Box x \leq x\,, \tag{B$_\Diamond$}$$

where $\Box \coloneqq -\Diamond-$, is a *KTB-algebra*.

If $\mathfrak{B}$ is a complete Boolean contact algebra that satisfies (C4^{c}), then $m\colon B \longrightarrow B$ is a completely additive modal possibility operator such that $\langle B, m\rangle$ is a KTB-algebra. So, under our assumptions the non-tangential inclusion is a modally definable subordination.

On the other hand, if $\mathfrak{B}$ is a complete KTB-algebra, then:

$$\mathsf{C}_\Diamond \coloneqq \{\langle x, y\rangle \mid x \cdot \Diamond y \neq \mathbf{0}\} \tag{df $\mathsf{C}_\Diamond$}$$

is a contact relation that satisfies (C4^{c}). Moreover, $\Diamond = m$, where m is the modal operator for $\mathsf{C}_\Diamond$ introduced by (df m).

2 The isomorphism of categories

Let $\mathbf{C4^c}$ be the class of complete Boolean contact algebras that satisfy (C4^{c}). We endow this class with certain morphisms in order to obtain a category. Given two algebras $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathbf{C4^c}$, a mapping $h\colon B_1 \to B_2$ is a *p-morphism*[3] iff it is a homomorphism such that:

$$h(x)\,\mathsf{C}_2\,h(y) \longrightarrow x\,\mathsf{C}_1\,y\,, \tag{P1}$$
$$h(z) \ll_2 y \longrightarrow (\exists x \in B_1)(z \ll_1 x \wedge h(x) \leq_2 y)\,. \tag{P2}$$

Of course, (P1) is equivalent to:

$$x \ll_1 y \longrightarrow h(x) \ll_2 h(y)\,.$$

[1] $-y$ is the Boolean complement of y.
[2] ι is the uniqueness operator, i.e., $(\iota x)\,\varphi(x)$ denotes the only object x that satisfies $\varphi(x)$.
[3] The idea of this comes from [2], where similar morphisms are called q-morphism.

The class $\mathsf{C4^c}$ together with p-morphisms form a category with the identity functions serving as the identity morphisms.

Also, let $\mathsf{KTB^c}$ be the class of all complete KTB algebras, that we turn into a category by taking as morphisms the standard Boolean homomorphisms preserving the possibility operator.

We show that there is a covariant functor $F\colon \mathsf{C4^c} \to \mathsf{KTB^c}$ which sends a complete BCA satisfying (C4^c) to a complete modal algebra, and such that for every $f \in \mathrm{Hom}_{\mathsf{C4^c}}(B_1, B_2)$, f is also an arrow in $\mathrm{Hom}_{\mathsf{KTB^c}}(B_1, B_2)$, i.e. $F(f) = f$. Analogously, there is a covariant functor $G\colon \mathsf{KTB^c} \to \mathsf{C4^c}$, which takes every $h \in \mathrm{Hom}_{\mathsf{KTB^c}}(B_1, B_2)$ to h itself.

Moreover, it is the case that:

$$G \circ F = 1_{\mathsf{C4^c}} \qquad \text{and} \qquad F \circ G = 1_{\mathsf{KTB^c}}\,,$$

where $1_{\mathsf{C4^c}}$ and $1_{\mathsf{KTB^c}}$ are the identity functors for the respective categories. In consequence we obtain that the categories $\mathsf{C4^c}$ and $\mathsf{KTB^c}$ are isomorphic.

3 Resolution contact algebras

Resolution contact algebras form a proper subclass of $\mathsf{C4^c}$ and serve as a spatial interpretation of both the contact relation that satisfies (C4^c) and the modal operator defined via the contact. The inspiration for this comes from [5], [6] and [3].

A *partition* of a Boolean algebra $\mathfrak{B}$ is any non-empty set P of non-zero and disjoint regions of B that add up to the unity: $\bigvee P = \mathbf{1}$. Let $\mathfrak{B}$ be a complete Boolean contact algebra, let $P \coloneqq \{p_i \mid i \in I\}$ be its partition. We define:

$$x \mathrel{\mathsf{C}_P} y \;:\longleftrightarrow\; (\exists i \in I)\,(x \mathrel{\mathsf{O}} p_i \wedge y \mathrel{\mathsf{O}} p_i)\,. \tag{df C_P}$$

C_P is a contact relation which satisfies (C4^c). For every element p_i of the partition, $\langle {\downarrow} p_i, \mathsf{C}_i\rangle$ where $\mathsf{C}_i \coloneqq \mathsf{C}_P \cap ({\downarrow} p_i \times {\downarrow} p_i)$ is a Boolean contact algebra with the full contact relation, so in particular, it satisfies (C4^c).

We adopt the following conventions: every partition of $\mathfrak{B}$ will be called its *resolution*[4], and the elements of the partition will be called *cells*. Any Boolean algebra expanded with C_P for a given partition P will be called *resolution contact algebra*. RCA is the class of such algebras, and $\mathsf{RCA^c}$ is its subclass composed of complete resolution algebras. In the case $x \mathrel{\mathsf{C}_P} y$ we will say that x is in *c-contact* with y.

The fineness of the partition is a counterpart of the precision with which we can discern regions and their mutual relations. For example, the regions x and y in Figure 1a are in c-contact, since they overlap a common cell from the sixteen element partition. From the perspective of the picture those regions may seem to be way apart, but we can think of the resolution as the frame of reference for comparison of regions with respect to C_P relation. The finer the resolution, the more precise approximation of contact between regions, as we can see in Figure 1b.

In every resolution algebra: $m(m(x)) = m(x)$, so m is a closure operator in every such algebra.

If $\mathfrak{B} \in \mathsf{RCA^c}$ has a finite resolution $P = \{p_i \mid i \leqslant n\}$ for some $n \in \mathbb{N}$, then the Kripke relation on the set $\mathrm{Ult}\, B$ is an equivalence relation and there is a one-to-one correspondence $f\colon P \to \mathrm{Ult}\, B/_R$ between cells and equivalence classes of ultrafilters.

Moreover, the contact determined by partition is related to S5 modal operators in the sense of the following:

[4]The name comes from [6], yet unlike there we do not limit it to finite partitions.

(a) (b)

Figure 1: On the left: regions x and y that are in contact with respect to a partition consisting of sixteen cells. On the right: regions x and y are no longer in contact if we take a finer partition as the frame of reference.

Theorem 3.1. *Given an S5 modal algebra $\mathfrak{B} = \langle B, \Diamond \rangle$, its expansion $\mathfrak{B}^\star = \langle B, \Diamond, \mathsf{C}_\Diamond \rangle$ can be embbeded into a modal expansion of a resolution algebra.*

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Weak Systems Have Intractable Theorems

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Given a proof system, how can we specify the "hardness" of its theorems? One way to tackle this problem is taking the lengths of proofs as the corresponding hardness measure. Following this route, we call a theorem hard when even its shortest proof in the system is "long" in a certain formal sense. Finding hard theorems in proof systems for classical logic has been an open problem for a long time. However, in recent years as significant progress, many super-intuitionistic and modal logics have been shown to have hard theorems. In this talk, we will extend the aforementioned result to also cover a variety of weaker logics in the substructural realm. We show that there are theorems in the usual calculi for substructural logics that are even hard for the intuitionistic systems.

In technical terms, for any proof system **P** at least as strong as Full Lambek calculus, **FL**, and polynomially simulated by the extended Frege system for some infinite branching super-intuitionistic logic, we present an exponential lower bound on the proof lengths. More precisely, we will provide a sequence of **P**-provable formulas $\{A_n\}_{n=1}^{\infty}$ such that the length of the shortest **P**-proof for A_n is exponential in the length of A_n. The lower bound also extends to the number of proof-lines (proof-lengths) in any Frege system (extended Frege system) for a logic between FL and any infinite branching super-intuitionistic logic. Finally, in the classical substructural setting, we will establish an exponential lower bound on the number of proof-lines in any proof system polynomially simulated by the cut-free version of $\mathbf{CFL_{ew}}$.

To be able to present the results formally, we need some ingredients. Let us start with defining substructural logics. For simplicity, we provide hard formulas for $\mathsf{FL_e}$. However, there are also hard theorem for the weaker logic FL [2]. The language we use is $\{0, 1, \wedge, \vee, *, \to\}$. Uppercase Greek letters denote multisets of formulas, and lower case Greek letters represent formulas. Consider the following sequent calculus:

$$\varphi \Rightarrow \varphi \qquad \Rightarrow 1 \qquad 0 \Rightarrow$$

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, 1 \Rightarrow \Delta}\,(1w) \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow 0, \Delta}\,(0w)$$

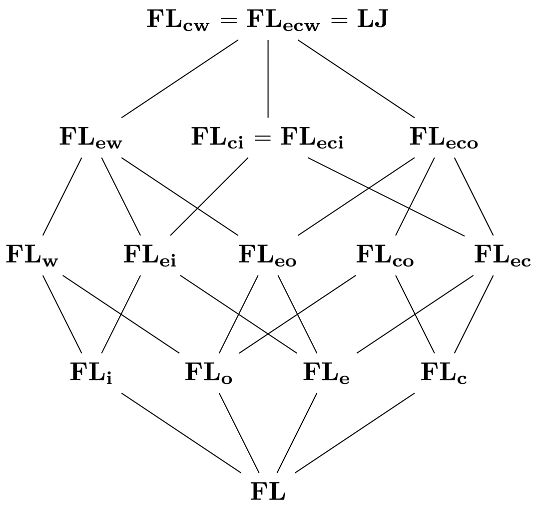
$$\frac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma, \varphi \wedge \psi \Rightarrow \Delta} \qquad \frac{\Gamma, \psi \Rightarrow \Delta}{\Gamma, \varphi \wedge \psi \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \varphi, \Delta \qquad \Gamma \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \varphi \wedge \psi, \Delta}$$

$$\frac{\Gamma, \varphi \Rightarrow \Delta \qquad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \varphi \vee \psi \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \varphi, \Delta}{\Gamma \Rightarrow \varphi \vee \psi, \Delta} \qquad \frac{\Gamma \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \varphi \vee \psi, \Delta}$$

$$\frac{\Gamma, \varphi, \psi \Rightarrow \Delta}{\Gamma, \varphi * \psi \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \varphi, \Delta \qquad \Sigma \Rightarrow \psi, \Lambda}{\Gamma, \Sigma \Rightarrow \varphi * \psi, \Delta, \Lambda}$$

$$\frac{\Gamma \Rightarrow \varphi, \Delta \qquad \Sigma, \psi \Rightarrow \Lambda}{\Gamma, \Sigma, \varphi \to \psi \Rightarrow \Delta, \Lambda} \qquad \frac{\Gamma, \varphi \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \varphi \to \psi, \Delta}$$

$$\frac{\Gamma \Rightarrow \varphi, \Delta \qquad \Sigma, \varphi \Rightarrow \Lambda}{\Gamma, \Sigma \Rightarrow \Delta, \Lambda}\,(cut)$$



The sequent calculus $\mathbf{FL_e}$ is the single-conclusion version of the sequent calculus presented above and $\mathbf{CFL_e}$ is the multi-conclusion version. The structural rules are as usual:

Weakening rules:

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, \varphi \Rightarrow \Delta}\,(i) \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \varphi, \Delta}\,(o)$$

Contraction rules:

$$\frac{\Gamma, \varphi, \varphi \Rightarrow \Delta}{\Gamma, \varphi \Rightarrow \Delta}\,(Lc) \qquad \frac{\Gamma \Rightarrow \varphi, \varphi, \Delta}{\Gamma \Rightarrow \varphi, \Delta}\,(Rc)$$

Adding these rules to the sequent calculi defined, result in various substructural calculi. It is worth mentioning that if we consider uppercase Greek letters to be *sequences* of formulas instead of multisets, i.e., the exchange rule is not present, then, we can introduce two implication-like connectives $\backslash$ and $/$, and include their respective rules. This system is called **FL**. The figure on top of this page shows the web of the sequent calculi between the full Lambek calculus **FL** and **LJ**, the usual sequent calculus for the intuitionistic logic IPC. Some other sequent calculi for which our result holds for are listed in Table 1.

Second, let us define Frege systems. They are the most natural calculi for propositional logic. A *(Frege) rule* is an expression of the form $\frac{\varphi_1, \ldots, \varphi_k}{\varphi}$ where $\varphi_1, \ldots, \varphi_k, \varphi$ are propositional formulas. Let $\mathbf{P}$ be a finite set of rules. A $\mathbf{P}$-proof of φ from a set of assumptions X, denoted by $X \vdash_{\mathbf{P}} \varphi$, is $\varphi_1, \ldots, \varphi_m = \varphi$ such that each $\varphi_i \in X$, or is inferred from some φ_j, $j < i$, by a substitution instance of rule in $\mathbf{P}$. The formulas φ_i are called *lines* of the proof.

A finite set of rules, $\mathbf{P}$, is called a *Frege system* for a logic L when

(1) $\mathbf{P}$ is strongly sound: if $\varphi_1, \ldots, \varphi_n \vdash_{\mathbf{P}} \varphi$, then $\varphi_1, \ldots, \varphi_n \vdash_{\mathsf{L}} \varphi$,

(2) $\mathbf{P}$ is strongly complete: if $\varphi_1, \ldots, \varphi_n \vdash_{\mathsf{L}} \varphi$, then $\varphi_1, \ldots, \varphi_n \vdash_{\mathbf{P}} \varphi$.

Third, and finally, we give a characterization of superintuitoinistic logics of infinite branching. Consider the following superintuitionistic (si) logics:

$$\mathsf{KC} = \mathsf{IPC} + \neg p \vee \neg\neg p \qquad , \qquad \mathsf{BD_n} = \mathsf{IPC}{+}BD_n$$

Table 1: Some sequent calculi with their definitions.

Sequent calculus	Definition
RL	$\mathbf{FL} + (0 \Leftrightarrow 1)$
CyFL	$\mathbf{FL} + (\varphi \backslash 0 \Leftrightarrow 0/\varphi)$
DFL	$\mathbf{FL} + (\varphi \wedge (\psi \vee \theta) \Leftrightarrow (\varphi \wedge \psi) \vee (\varphi \wedge \theta))$
$\mathbf{P_nFL}$	$\mathbf{FL} + (\varphi^n \Leftrightarrow \varphi^{n+1})$
psBL	$\mathbf{FL_w} + \{(\varphi \wedge \psi \Leftrightarrow \varphi * (\varphi \backslash \psi)), (\varphi \wedge \psi \Leftrightarrow (\psi/\varphi) * \varphi)\}$
HA	$\mathbf{FL_w} + (\varphi \Leftrightarrow \varphi^2)$
DRL	$\mathbf{RL} + (\varphi \wedge (\psi \vee \theta) \Leftrightarrow (\varphi \wedge \psi) \vee (\varphi \wedge \theta))$
IRL	$\mathbf{RL} + (\varphi \Rightarrow 1)$
CRL	$\mathbf{RL} + (\varphi * \psi \Leftrightarrow \psi * \varphi)$
GBH	$\mathbf{RL} + \{(\varphi \wedge \psi \Leftrightarrow \varphi * (\varphi \backslash \psi)), (\varphi \wedge \psi \Leftrightarrow (\psi/\varphi) * \varphi)\}$
Br	$\mathbf{RL} + (\varphi \wedge \psi \Leftrightarrow \varphi * \psi)$

where IPC is the intuitionistic logic and $BD_0 := \bot$ and $BD_{n+1} := p_n \vee (p_n \to BD_n)$. Jeřábek in [3] proved the following interesting theorem that a superintuitionistic logic L has infinite branching iff $\mathsf{L} \subseteq \mathsf{BD}_2$ or $\mathsf{L} \subseteq \mathsf{KC} + \mathsf{BD}_3$.

Now, let us give a sketch of how to prove the lower bound. In order to do so, we have to provide a sequence of formulas provable in $\mathbf{FL_e}$, such that every proof of them are long. This task requires two steps. The first step, which is the main task, is providing a sequence of $\mathbf{FL_e}$-tautologies. To achieve this goal we change the existing hard intuitionistic tautologies in a suitable way that they become provable in $\mathbf{FL_e}$, but remain hard. The next step, which is the easier part, is proving that these tautologies are hard. To do so, we use the canonical translation of the language of $\mathbf{FL_e}$ to the language of IPC, i.e., sending $\{0, 1, *\}$ to $\{\bot, \top, \wedge\}$, respectively and the other connectives to themselves. It is easy to see that this transformation takes polynomial time.

Let us mention the form of the hard intuitionistic tautologies. The following formulas, $\Theta_{n,k}$, are hard for IPC and they are negation-free and $\bot$-free. Small Roman letters denote atomic formulas and the formulas α_n^k and β_n^{k+1} are monotone, i.e., only consist of atoms, $\wedge$, $\vee$.

$$\Theta_{n,k} := \bigwedge_{i,j}(p_{i,j} \vee q_{i,j}) \to$$

$$[(\bigwedge_{i,l}(s_{i,l} \vee s'_{i,l}) \to \alpha_n^k(\bar{p}, \bar{s}, \bar{s'})) \vee (\bigwedge_{i,l}(r_{i,l} \vee r'_{i,l}) \to \beta_n^{k+1}(\bar{q}, \bar{r}, \bar{r'}))]$$

The result by Hrubeš [1] and Jeřábek [3] is the following theorem:

Theorem. The formulas $\Theta_{n,k}$ are IPC-tautologies and require IPC-Frege proofs with $2^{n^{\Omega(1)}}$ lines, for $k = \lfloor \sqrt{n} \rfloor$.

In the following we see the form of the hard $\mathsf{FL_e}$ tautologies:

$$\Theta^*_{n,k} := [\mathop{\circledast}_{i,j}((p_{i,j} \wedge 1) \vee (q_{i,j} \wedge 1))] \ \to$$

$$\big[(\mathop{\circledast}_{i,l}((s_{i,l} \wedge 1) \vee (s'_{i,l} \wedge 1)) \to \alpha_n^k) \ \vee (\mathop{\circledast}_{i,l}((r_{i,l} \wedge 1) \vee (r'_{i,l} \wedge 1)) \to \beta_n^{k+1})\big]$$

Now, we have all the ingredients to formally state our result:

Theorem. [2] The formulas $\Theta^*_{n,k}$ are $\mathsf{FL_e}$-tautologies. Moreover, for any substructural logic L and any superintuitionistic logic of infinite branching M such that $\mathsf{FL_e} \subseteq \mathsf{L} \subseteq \mathsf{M}$, the formulas $\Theta^*_{n,k}$ require L-Frege proofs with $2^{n^{\Omega(1)}}$ lines, for $k = \lfloor \sqrt{n} \rfloor$.

The concrete application of the theorem follows:

Corollary. Let $S \subseteq \{e, c, i, o\}$, and L be $\mathsf{FL_S}$, or any of the logics of the sequent calculi in Table 1. Then the length of every proof of Θ^*_n in any (extended) Frege system for L is exponential in n.

Let us end with the following question: what happens in the case of the classical versions of the above substructural logics? They are not included in IPC and hence our method does not work. However, for their cut-free versions we have the following theorem.

Theorem. The length of every proof of Θ^*_n in the sequent calculi $\mathbf{CFL}^-_{\mathbf{e}}, \mathbf{CFL}^-_{\mathbf{ei}}, \mathbf{CFL}^-_{\mathbf{eo}}$, and $\mathbf{CFL}^-_{\mathbf{ew}}$ is exponential in n, where the " $-$ " means without the cut rule.

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Abstract Model and Deduction System for Logic of Multiple Agent in Quantum Physics

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Quantum logic (QL) has been studied to handle the strange propositions of quantum physics. Moreover, numerous types of logic and structures have been proposed to represent and analyze these propositions [8] [9]. In particular, logic based on *orthomodular lattices*, namely, *orthomodular logic* (OML), has been studied since 1936, proposed by Birkhoff and Von Neumann [7] [11]. An orthomodular lattice is related to the closed subspaces of a Hilbert space, which is a state space of a particle in quantum physics. Instead of these lattices, the *Kripke model* (possible world model) of OML can be used, which is called the *orthomodular-model* (OM-model) [9] [12] [13] . Intuitively, each possible world of an OM-model expresses a one-dimensional subspace of a Hilbert space, corresponding to a *quantum state*.

To treat an agent's *knowledge* in quantum mechanics, some studies combine *epistemic logic* (EL) with QL. EL is a field of modal logic that treats the proposition of an agent's knowledge. In the Kripke model of EL, the *indistinguishability of states* is used to express knowledge. That is, if a formula ϕ is true at all states that are indistinguishable from the current state for agent i, then agent i knows that ϕ is true. Furthermore, *dynamic EL* (DEL) has been studied to handle the transitions of knowledge. In general, *public announcement logic* (PAL) is treated as the most basic and simple logic in DEL. Basic PAL includes only two types of modal symbols: the symbols for knowledge K_i of individual agents and the symbol $[\,]$ for public announcements. $[\phi]\psi$ can be read as "After a public announcement ϕ, ψ is true." For more details of DEL, see [10]. Ref [5] and [6] can be cited as one of the studies of logic that deal with the concept of knowledge with quantum physics. In these studies, the models which incorporate specific *quantum information* concepts were used. Ref [3] and [4] can be cited as studies of knowledge with more general concepts of quantum physics.

Although knowledge in quantum mechanics has been analyzed in some directions in logic, *abstract* model for this field wasn't much discussed, and *deduction systems* are not well constructed. That is, in general, QL has been developed using two primary methods. The first method is research using models that can express almost all properties of Hilbert spaces. In this context, the Hilbert space is often employed as a model. The second method is research using a simple model that uses only essential parts of a Hilbert space. Studies using orthomodular lattices formed by observational propositions of a Hilbert space belong to this category. The two methods have their advantages and disadvantages. The former method is suitable for detailed and diverse analysis of quantum mechanics because it can express almost all propositions for the states or values of physical quantities in quantum mechanics. However, it has the disadvantage that logical analysis is difficult because logical symbols and models become quite complex. In the latter method, although detailed analysis is impossible, essential properties can be abstractly treated. Further, because simple logical symbols and models are used, it is easy to perform logical analysis and comparison with other logic.

The former method is extremely common when considering propositions about complex notions in quantum mechanics such as agent's knowledge. Especially, to date, there are few logical analyses of knowledge of *multiple* agents (with multiple particles) using an abstract and simple model. One of the reasons is related to a problem using orthomodular lattices. An orthomodular lattice L can be developed by extracting the concept of the closed subspace of a Hilbert space H. If the state space of one particle is H, the state space of two particles is represented by $H \otimes H$, where $\otimes$ denotes the tensor products of spaces. However, intuitively, the tensor product of lattices $L \otimes L$ does not correspond to $H \otimes H$. The tensor products of orthomodular lattices cannot represent a linear combination in a vector space. For example, assume that H is a 2D Hilbert space. Then, $c(|0> \otimes |0> + |1> \otimes |1>)$ is a 1D closed subspace of H. However, an element corresponding to this space is not included in $L \otimes L$, i.e., $L \otimes L$ includes only elements corresponding to the states represented by the multiplication, such as $c|0> \otimes |0>$ or $c|1> \otimes |1>$). Therefore, when handling multiple particles, using the tensor product of orthomodular lattices does not include essential elements such as entanglement in the model [1]. This situation is the same even when using the OM-model.

The situation where multiple agents have their particles is common in quantum mechanics. Therefore, it is meaningful to develop a method that can abstractly discuss propositions in such situations. In this study, we propose some methods and models to overcome the above problem and construct and analyze new logic for knowledge of multiple agents or multiple particles in quantum mechanics.

As a new logic, MDEQL (Multi-particle dynamic epistemic quantum logic) is constructed and discussed. It is desirable to avoid the models that introduce the concept of certain notions of Hilbert space concretely. Therefore, for the basic model of MDEQL, OM-model is adopted the same as OML. By using the OML model and language almost as they are, it becomes easier to analyze and prove the theorem. Then, we limit models to those that satisfy important conditions of a tensor product Hilbert space, i.e., intuitively, it is assumed that a model already corresponds to the tensor product $H \otimes H$of Hilbert spaces, and several properties of individual Hilbert spaces H are represented by additional conditions. This method avoids the above-mentioned deficiencies in developing a tensor product model from models. Based on these models, the models of MDEQL are defined by adding modality relations of knowledge.

The language for MDEQL is defined as follows. We index propositional variables into multiple classes to indicate which particle's proposition each propositional variable represents. Such an expression method is often used [2]. Furthermore, for technical reasons, formulas are defined in two parts. One corresponds to the language of OML and the other corresponds to formulas for expressing knowledge.

q-formula $A ::= p_i \mid \ \sim A \mid A \wedge A$

g-formula $\phi ::= A \mid \neg\phi \mid \phi \wedge \phi \mid \mathsf{K}_i\phi \mid [A]\phi$

$\sim$ is quantum negation, whereas $\neg$ is classical negation. Only q-formulas can be placed in the modal symbol [ ] because only the situation of acquiring information on the quantum states is discussed.

We construct the deduction system which satisfies the soundness and completeness theorem with respect to the new models. *Sequent calculus* style of deduction system is constructed because it is compatible with OML [16] [17] [18].

As another approach to the problem, we consider another language for multiple particles.We want to treat the concepts as abstractly and simply as possible; therefore, we avoid symbols that primarily represent the concepts of quantum mechanics. However, there is a limit to what can be expressed with ordinary languages of OML itself. For instance, we can confirm that the following non implications of propositions of individual particles, which are important in multi particle systems, cannot be expressed. “Suppose that A is a proposition about the i-th space and B is a proposition about the j-th space. If $A \neq \top, A \neq \bot$, $B \neq \top$, and $B \neq \bot$, there exists states x, y such that $x \models A$, $x \not\models B$, $y \not\models A$, and $y \models B$”. That is, in tensor space $H_3 \otimes H_3$ of 3D Hilbert space H_3, consider the following propositions. “The value of a physical quantity (which is associated with $|0>, |1>, |2>$) of the first particle is 0.” “The value of a physical quantity (which is associated with $|0>, |1>, |2>$) of the second particle is 1.” At $|0> \otimes |0> \in H_3 \otimes H_3$, the first proposition is true but the second is false; moreover, at $|1> \otimes |1>$, the first proposition is false but the second proposition is true.

Because of these circumstances, we extend a language that is not as complicated as possible.

Importantly, we use the relationship between OML and modal logic **B**. **B** denotes the logic developed on the frame assuming symmetry and reflectivity in the binary relation of the Kripke frame. **B** and OML are associated by McKinsey-Tarski transfer [9]. This correspondence is the same as that of intuitionistic logic and modal logic **S4**. As in the case of **S4** and intuitionistic logic, the corresponding modal logic can express a finer concept via a formula. For example, in both OML and intuitionistic logic, negation can be decomposed into $\Box\neg$ in modal logic, and $\Box$ and $\neg$(classical negation) can be separately used. Because it is convenient to handle $\Box$ alone, we develop the language and models based on **B** rather than OML.

A quantification symbol $\forall p_i$ is used for propositional variables. That is, $\forall p_i(\phi)$ is added to the definition of formulas. This is necessary to express properties such as entanglement concisely. This conceptually belongs to the category of the *second-order* propositional logic; however, only the quantification of propositional variables is employed, and not the quantification of the entire formula. Therefore, intuitively, the complex problems in the second-order propositional logic do not occur and can be handled fairly simply.

Using this language has the disadvantage of being a bit more complicated than the previous language, but it has the advantage of increasing the expressiveness of the model's conditions. In this study, the correspondences between various formulas with the above new definition and model conditions for a Hilbert space are proven. Some examples are shown below.($A_i, B_i, \dots$ represent formulas which includes only $p_i, q_i, r_i, \dots$as propositional variables).

For all $x, y \in W$, there exists $z \in W$ such that xRz and zRy.

$$\Box\Box A \to \Box^n A \quad \text{(for each } n \in \mathbb{N}\text{)}$$

Each propositional variable represents a one-dimensional subspace of each Hilbert space.

$$(p_i \wedge A_i) \to \Box\Box(p_i \to A_i)$$

Non-implications of propositions of an individual particle.

$$(\neg\Box\Box A_i \wedge \neg\Box\Box\neg A_i \wedge \neg\Box\Box B_j \wedge \neg\Box\Box\neg B_j) \to \Diamond\Diamond(\neg A_i \wedge B_j) \wedge \Diamond\Diamond(A_i \wedge \neg B_j) \quad (i \neq j)$$

"Particle i and j are entangled"

$$\mathcal{E}_{i,j} = \forall p_i(\neg p_i) \wedge \forall q_j(\neg q_j) \wedge \forall p_i \exists q_j[p_i]q_j \wedge \forall q_j \exists p_i[q_l]p_i$$

The contributions of this study are the following.

1. New abstract logical frameworks and models for dealing with propositions about multiple agents and quantum particles are proposed.

2. A deduction system for new models that holds soundness and completeness is constructed.
3. We show that important conditions on models can be expressed with a little development of the language, and prove that these formulas are valid if a model satisfies specific conditions.

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Some properties of residuated lattices using two parameters derivations

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Residuation is a fundamental concept of ordered structures. Non classical logic is closely related to logical algebraic systems and it is well-known that the algebraic study of logical systems plays a significant role in artificial intelligence, formal concept analysis for example ([5],[14]). Such systems are usually modeled as partially ordered sets together with suitable operations reflecting their properties. Residuated lattices are obtained by adding a residuated monoid operation on lattices. Researches based on residuated lattices have shown them as valuable tools for solving both algebraic and logical problems ([7],[15]).

The notion of derivation, which comes from mathematical analysis, is useful for studying some structural properties of various kinds of algebra, in particular it has allowed to characterize distributive and modular lattices. Indeed it has been applied to theory of algebras with two operations + and . specially to the theory of commutative rings in 1957 [12] by Posner. For a ring $\mathcal{R} := (R; +, .)$, a map $d : R \to R$ is called a derivation if it satisfies the condition : For all $x, y \in R$,

$$\begin{aligned} d(x+y) &= d(x) + d(y) \\ d(x.y) &= d(x).y + x.d(y). \end{aligned}$$

It was applied to the theory of lattice $\mathcal{L} := (L; \vee, \wedge)$ by Szász in 1975 [13], where the operations + and . were interpreted as lattice operations $\vee$ and $\wedge$ respectively. Further, the concept of derivation is also applied to other algebra, such as BCI- algebra by Y. B. Jun and X. L. Xin in 2004 [8], later by Alsheri in MV-algebra in 2010 [2]. P. He proposed the notion of derivation in residuated lattices [6] in 2016. Based on [16], the concept of derivation was extended to f-derivation in lattices by Çeven and Özturk [4], in this work authors characterized distributive, modular lattices by using f-derivation. Maffeu et al introduced the concept of f-derivation in residuated multilattice in 2019 [11] . The concept was further explored in the form of (f, g)-derivations in lattices by Asci in 2008 [3], later by Alsatayhi on BL-algebras in 2017 [1]. In

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the same direction Keubeng et al have extended the notion of (f, g)-derivation in residuated multilattice [10].

The primary goal of this talk is to extend the notion of derivation by introducing two-parameter derivations in a bounded commutative residuated lattice. After defining this notion, we illustrate it with some examples and study the properties of some related notions. We give the condition for a (f, g)-multiplicative derivation to be monotone. Moreover, the set of fixed points is defined using the notion of (f, g)- multiplicative derivation of bounded commutative residuated lattices and the conditions for this set to be a down closed set and an ideal are given. We conclude with the characterization of set of complemented elements in terms of its derivations.

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Modal Information Logic: Decidability and Completeness

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The present abstract studies formal properties of Modal Information Logic (MIL), a modal logic proposed in [1] as a way of using possible-worlds semantics to model a theory of information. It does so by extending the language of propositional logic with a single binary modality defined in terms of being the supremum of two states.

First proposed in 1996, MIL has been around for some time, yet not much is known: [2, 3] pose two central open problems, namely (1) axiomatizing the logic and (2) proving (un)decidability.

While the majority of this abstract is spent on motivations and definitions, the first novel part of this abstract is concerned with these two problems. We solve both, (1) by providing an axiomatization and completeness proof and (2) by proving decidability. In proving the latter, we emphasize our method as a general heuristic on proving decidability 'via completeness' for semantically introduced logics.

If time allows, we will also be presenting the second novel part of this abstract. It is concerned with axiomatizing a kindred logic, where the supremum-modality is interpreted on join-semilattices. Besides the result being of interest per se, we believe the ideas involved in the axiomatization can be used when trying to axiomatize other logics. By higlighting these ideas, a general theme of this abstract will be a study in (Kripke) completeness.

Defining the logic

We continue by formally defining Modal Information Logic.

Definition 1 (Language)**.** The language $\mathcal{L}_M$ of Modal Information Logic is defined using a countable set of proposition letters $\mathbf{P}$ and a binary modality $\langle \sup \rangle$. The formulas $\varphi \in \mathcal{L}_M$ are then given by the following BNF-grammar

$$\varphi ::= \bot \mid p \mid \neg\varphi \mid \varphi \vee \psi \mid \langle \sup \rangle \varphi\psi,$$

where $p \in \mathbf{P}$ and $\bot$ is the falsum constant. ⊣

Modal Information Logic is defined by semantical means. That is, as the set of $\mathcal{L}_M$-validities of a class of structures, namely preorders. Formally we define as follows.

Definition 2 (Frames and models)**.** A (Kripke) *preorder-frame* for $\mathcal{L}_M$ is a pair $\mathbb{F} = (W, \leq)$ where

- W is a set; and
- $\leq$ is a preorder on W, that is: reflexive and transitive.

*The following abstract is based on some preliminary results from my MSc thesis at the *ILLC* at *University of Amsterdam*, supervised by Johan van Benthem and Nick Bezhanishvili. (At the time of writing, the thesis has neither been defended nor submitted.) I am indebted to both. I would also like to thank two anonymous referees for helpful comments.

A (Kripke) *preorder-model* for $\mathcal{L}_M$ is a triple $\mathbb{M} = (W, \leq, V)$ where

- $(W, \leq)$ is a preorder-frame; and
- V is a valuation on W, that is: a function $V : \mathbf{P} \to \mathcal{P}(W)$. ⊣

Having defined the structures in which to interpret $\mathcal{L}_M$-formulas, we are about to define the actual semantics. In order to do so, we provide the following definition generalizing the notion of supremum from partial orders to preorders.

Definition 3 (Supremum)**.** Given a preorder-frame $(W, \leq)$ and worlds $u, v, w \in W$, we say that w is *a supremum* of u, v and write $w \in \sup(u, v)$ iff

- w is an upper bound of u, v, i.e. $u \leq w$ and $v \leq w$; and
- $w \leq x$ for all upper bounds x.

In general, $\sup(u, v)$ denotes the set of suprema of $\{u, v\}$, and if this happens to be a singleton $\{w\}$, we may write $w = \sup(u, v)$. ⊣

Definition 4 (Semantics)**.** Given a preorder-model $\mathbb{M} = (W, \leq, V)$ and a world $w \in W$, *satisfaction* of a formula $\varphi \in \mathcal{L}_M$ at w in $\mathbb{M}$ (written $\mathbb{M}, w \Vdash \varphi$) is defined using the following recursive clauses on φ:

$\mathbb{M}, w \nVdash \bot$,		
$\mathbb{M}, w \Vdash p$	**iff**	$w \in V(p)$,
$\mathbb{M}, w \Vdash \neg\varphi$	**iff**	$\mathbb{M}, w \nVdash \varphi$,
$\mathbb{M}, w \Vdash \varphi \vee \psi$	**iff**	$\mathbb{M}, w \Vdash \varphi$ *or* $\mathbb{M}, w \Vdash \psi$,
$\mathbb{M}, w \Vdash \langle\sup\rangle\varphi\psi$	**iff**	there exist $u, v \in W$ such that $\mathbb{M}, u \Vdash \varphi$, $\mathbb{M}, v \Vdash \psi$, and $w \in \sup(u, v)$.

Notions like *global truth*, *validity*, etc. are defined as usual in possible-worlds semantics. ⊣

With these notions laid out, Modal Information Logic is defined as follows:

Definition 5. Modal Information Logic is denoted by MIL_{Pre}, and defined as

$$MIL_{Pre} := \{\varphi \in \mathcal{L}_M : (W, \leq) \Vdash \varphi \ \textit{ for all preorder-frames } (W, \leq)\}.$$

That is, MIL_{Pre} is the set of $\mathcal{L}_M$-validities on the class of all preorder-frames. ⊣

Having formally defined our logic, we end this section defining natural variations of Modal Information Logic obtained by considering kindred structures, e.g.:

MIL_{Pos}, which is the logic of poset-frames, i.e. frames $(W, \leq)$ where '$\leq$' is a partial order; and

MIL_{Sem}, which is the logic of frames $(W, \leq)$ where '$\leq$' is a join-semilattice.

Results

Having formally set out the logic and semantics, we present the results obtained. Firstly, we have shown that

Proposition 6. *MIL_{Pre} does not have the FMP w.r.t. preorder-frames.*

Proof (sketch). This is witnessed by the formula

$$\psi_N := HP\langle\text{sup}\rangle pp \wedge HP\neg\langle\text{sup}\rangle pp,$$

where

$$P\varphi := \langle\text{sup}\rangle\varphi\top$$

and $H := \neg P \neg$ is the dual of P. □

At first glance, this might make decidability appear unlikely. However, we circumvent this problem as follows. We (1) axiomatize the logic, (2) use this to show the logic to be complete with respect to another class of structures (where the ternary relation of $\langle\text{sup}\rangle$ won't necessarily be the supremum-relation of a preorder, but something more general), and then (3) prove that the logic enjoys the FMP on this other class of structures, from which we can deduce decidability.

That is, first, we provide an axiomatization:

Definition 7 (Axiomatization)**.** Let $\mathbf{MIL_{Pre}}$ be the least normal modal logic in the language of $\mathcal{L}_M$ containing the following axioms:

(Re.) $p \wedge q \to \langle\text{sup}\rangle pq$

(4) $PPp \to Pp$

(Co.) $\langle\text{sup}\rangle pq \to \langle\text{sup}\rangle qp$

(Dk.) $(p \wedge \langle\text{sup}\rangle qr) \to \langle\text{sup}\rangle pq$ ⊣

Theorem 8 (Completeness)**.** $\mathbf{MIL_{Pre}}$ *is sound and strongly complete w.r.t.* MIL_{Pre}*. So, in particular,* $\mathbf{MIL_{Pre}} = MIL_{Pre}$*.*

Further, as a corollary, we get that

Corollary 9. $MIL_{Pre} = MIL_{Pos}$*.*

Second, we define a class of structures $\mathcal{C}$, which is seen to be complete w.r.t. $\mathbf{MIL_{Pre}}$:

Definition 10. Let $\mathcal{C}$ be the class of pairs (W, C), where W is a set and C is a ternary relation on W satisfying the following four conditions:

(Re.f) $\forall w\,(Cwww)$

(4.f) $\forall w, v, u\,(Cwwv \wedge Cvvu \to Cwwu)$

(Co.f) $\forall w, v, u\,(Cwvu \to Cwuv)$

(Dk.f) $\forall w, v, u\,(Cwvu \to Cwwv)$ ⊣

Proposition 11. $\mathbf{MIL_{Pre}}$ *is sound and (strongly) complete w.r.t.* $\mathcal{C}$*.*

And, third, we show the following:

Theorem 12. $\mathbf{MIL_{Pre}}$ *admits filtration w.r.t. the class* $\mathcal{C}$*. Thus,*

$$\mathbf{MIL_{Pre}} = \text{Log}(\mathcal{C}_F),$$

where $\text{Log}(\mathcal{C}_F)$ *denotes the NML of the class of finite* $\mathcal{C}$*-frames.*

Using this, we deduce that

Corollary 13. *Modal Information Logic is decidable.*

Afterwards, if time allows, we turn our attention to axiomatizing MIL_{Sem}. We do so by syntactically defining a logic $\mathbf{MIL_{Sem}}$ extending $\mathbf{MIL_{Pre}}$ via an infinite axiom-scheme. We then show

Theorem 14 (Completeness). $\mathbf{MIL_{Sem}} = MIL_{Sem}$.

When presenting this last result, we highlight some of the techniques and ideas going into it, especially (a) how the infinite extension-scheme can be intuited as capturing ever-more of the algebraic structure of a given join-semilattice, and (b) how we apply König's Lemma in the completeness proof.

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Finite Characterisations of Modal Formulas

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Abstract

We initiate the study of finite characterizations and exact learnability of modal languages. A finite characterization of a modal formula w.r.t. a set of formulas is a finite set of finite models (labelled either positive or negative) which distinguishes this formula from every other formula from that set. A modal language $\mathcal{L}$ is finitely characterizable if every $\mathcal{L}$-formula has a finite characterization w.r.t. $\mathcal{L}$. This definition can be applied not only to the basic modal logic **K**, but to arbitrary normal modal logics. We show that a normal modal logic is finitely characterizable (for the full modal language) iff it is locally tabular. This shows that finite characterizations with respect to the full modal language are rare, and hence motivates the study of finite characterizations for fragments of the full modal language. Our main result is that the positive modal language without the truth-constants $\top$ and $\bot$ is finitely characterizable. Moreover, we show that this result is essentially optimal: finite characterizations no longer exist when the language is extended with the truth constant $\bot$ or with all but very limited forms of negation.

1 Introduction

We study the existence of finite characterizations of modal formulas. A finite characterization of a formula φ w.r.t. a set of formulas $\mathcal{L}$ is a finite set of finite models that distinguishes φ from every other formula in $\mathcal{L}$. Such finite characterizations are a precondition for the existence of *exact learning algorithms* for 'reverse-engineering' a hidden goal formula from examples in Angluin's model of exact learning with membership queries [1]. Our interest in exact learnability is motivated by applications in description logic. But besides learnability, the generation of examples consistent with a given formula can be used for e.g. query visualization and debugging (see e.g. [7] for a more detailed discussion of such applications). The exhaustive nature of the examples is of additional value, as they essentially display all 'ways' in which the query can be satisfied or falsified.

In this extended abstract, we only provide a high level description of our results and proof techniques. Detailed proofs can be found here: https://bit.ly/3LCtmQt.

2 Preliminaries

Given a set of propositional variables Prop and a set of connectives $C \subseteq \{\wedge, \vee, \Diamond, \Box, \top, \bot\}$, let $\mathcal{L}_C$[Prop] (or simply $\mathcal{L}_C$ when Prop is clear from context) denote the collection of all modal formulas generated from literals (i.e. positive or negated propositional variables) from Prop, using the connectives in C. Note that all such formulas are in negation normal form, i.e. negations may only occur in front of propositional variables. Thus, $\mathcal{L}_{\Box,\Diamond,\wedge,\vee,\top,\bot}$[Prop] is the set of all modal formulas with variables in Prop in negation normal form. Further, for any modal

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fragment $\mathcal{L}$ as defined above, $\mathcal{L}^+$ and $\mathcal{L}^-$ denote the set of positive, respectively negative $\mathcal{L}$ formulas, where a formula φ is *positive* if no $p \in var(\varphi)$ occurs negated, and *negative* if all $p \in var(\varphi)$ occur only negated. We will use *modal language* to refer to any such fragment. By the *full modal language* we will mean $\mathcal{L}_{\Box,\Diamond,\wedge,\vee,\top,\bot}[\text{Prop}]$.

For a modal formula φ, let $var(\varphi)$ denote the set of variables occurring in φ and $d(\varphi)$ its *modal depth*, i.e. the nesting depth of $\Diamond$'s and $\Box$'s in φ.

A *normal modal logic* is a collection of modal formulas containing all instances of the K-axiom $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$ and closed under uniform substitution, modus ponens and generalisation.

A (Kripke) *model* is a triple $M = (dom(M), R, v)$ where $dom(M)$ is the a set of 'possible worlds', $R \subseteq dom(M) \times dom(M)$ a binary 'accessibility' relation and a valuation $V : \text{Prop} \to \mathcal{P}(W)$. A *pointed model* is a pair M, s of a Kripke model M together with a state $s \in dom(M)$. A (Kripke) *frame* is a model without its valuation.

3 Finite Characterizations

First, we define what a finite characterization means in the context of modal logic.

Definition 1. (Finite characterizations) A *finite characterization* of a formula $\varphi \in \mathcal{L}[\text{Prop}]$ w.r.t. $\mathcal{L}[\text{Prop}]$ is a pair of finite sets of finite pointed models $\mathbb{E} = (E^+, E^-)$ such that (i) φ *fits* (E^+, E^-), i.e. $E, e \models \varphi$ for all $(E, e) \in E^+$ and $E, e \not\models \varphi$ for all $(E, e) \in E^-$ and (ii) φ is the only formula in $\mathcal{L}[\text{Prop}]$ which fits (E^+, E^-), i.e. if $\psi \in \mathcal{L}[\text{Prop}]$ satisfies condition (i) then $\varphi \equiv \psi$. A modal language $\mathcal{L}$ is *finitely characterizable* if for every finite set of propositional variables Prop, every $\varphi \in \mathcal{L}[\text{Prop}]$ has a finite characterization w.r.t. $\mathcal{L}[\text{Prop}]$.

Thus if (E^+, E^-) is a finite characterization of a formula $\varphi \in \mathcal{L}[\text{Prop}]$ w.r.t. $\mathcal{L}[\text{Prop}]$, then for every $\psi \in \mathcal{L}[\text{Prop}]$ with $\varphi \not\equiv \psi$, E^+ contains a finite model of $\varphi \wedge \neg\psi$ or E^- contains a finite model of $\neg\varphi \wedge \psi$.

For example, the formula $p \wedge q$ has a finite characterization w.r.t. $\mathcal{L}^+_\wedge[\text{Prop}]$ with Prop $= \{p, q, r\}$, namely $(\{\cdot_{p,q}\}, \{\cdot_p, \cdot_q\})$, where "$\cdot_P$" is the single point model where all $p \in P$ are true.

Our motivation for studying finite characterizations, comes from *computational learning theory*. Specifically, finite characterizability is a necessary precondition for *exact learnability with membership queries* in Dana Angluin's interactive model of exact learning [1]. In our context, exact learnability with membership corresponds to a setting in which the learner has to identify a formula by asking question to an oracle, where each question is of the form "is the formula true or false in pointed model (M, w)?" This can also be viewed as a 'reverse engineering' task, where a formula has to be identified based on its behaviour on only a finite set of models. Exact learnability has recently gained a renewed interest in the description logic literature. We comment more on the connection with description logic in Section 4.

Our starting observation is:

Theorem 1. *The full modal language is not finitely characterizable.*

Proof. It suffices to give one counterexample, so suppose that e.g. $\varphi = \Box\bot$ had a finite characterization (E^+, E^-) w.r.t. the full modal language. Observe that for each n, $M, s \models \Box^{n+1}\bot \wedge \Diamond^n\top$ iff $height(M, s) = n$, where the *height* of a pointed model M, s is the length of the longest path in M starting at s, or ∞ if there is no finite upper bound. Every finite characterization can only contain pointed models up to some bounded height $< n$ (by choice of n) or must contain some model of infinite (∞) height. In either case, it follows that no negative example $(E, e) \in E^-$ satisfies $\Box^{n+1}\bot \wedge \Diamond^n\top$. Hence for large enough n, $\varphi \vee (\Box^{n+1}\bot \wedge \Diamond^n\top)$ also fits (E^+, E^-), yet is clearly not equivalent to φ. □

In fact, by a variation of the same argument, we can show that *no* modal formula has a finite characterization w.r.t. the full modal language. Theorem 1 raises two questions, namely: *do finite characterizations exist in other modal logics than* **K**, and *which fragments of modal logic admit finite characterizations*. We address each of these two questions next.

We first generalize Definition 1 as follows (whereby Theorem 1 becomes a result about the special case of the basic normal modal logic **K**): a finite characterization of a modal formula φ with $var(\varphi) \subseteq \text{Prop}$ w.r.t. a normal modal logic L is a finite set (E^+, E^-) of finite pointed models based on L frames such that (i) φ fits (E^+, E^-) and (ii) if ψ with $var(\psi) \subseteq \text{Prop}$ fits (E^+, E^-) then $\varphi \equiv_L \psi$, where $\varphi \equiv_L \psi$ iff $\varphi \leftrightarrow \psi \in L$. We say that a normal modal logic L is finitely characterizable if for every finite set Prop, every modal φ with $var(\varphi) \subseteq \text{Prop}$ has a finite characterization w.r.t. L. We can give a complete characterization over which modally definable frame classes the full modal language is finitely characterizable.

It turns out that only very few normal modal logics are uniquely characterizable. A normal modal logic L is *locally tabular* if for every finite set Prop of propositional variables, there are only finitely many formulas φ with $var(\varphi) \subseteq \text{Prop}$ up to L-equivalence.

Theorem 2. *A normal modal logic L is finitely characterizable iff it is locally tabular.*

In other words the full modal language is only finitely characterizable in the degenerate case where there are only finitely many formulas to distinguish from (up to equivalence). This result motivates the investigation of finite characterizability for modal fragments. Specifically, inspired by previous work on finite characterizability of the positive existential fragment of first order logic [2], we consider positive fragments of the full modal language.

Note that, in the remainder of this section, we only consider again the modal logic **K**.

The proof of Theorem 1 can easily be modified to show the following:[1]

Theorem 3. *$\mathcal{L}^+_{\Box,\Diamond,\wedge,\vee,\bot}$ is not finitely characterizable.*

On the other hand, based on results in [2], we can show that:

Theorem 4 (From [2])**.** *$\mathcal{L}^+_{\Diamond,\wedge}$ is finitely characterizable. Indeed, given a formula in $\mathcal{L}^+_{\Diamond,\wedge}$, we can construct a finite characterization in polynomial time.*

More precisely, it was shown in [2] that finite characterizations can be constructed in polynomial time for "c-acyclic conjunctive queries", a fragment of first-order logic that includes the standard translations of $\mathcal{L}^+_{\Diamond,\wedge}$-formulas.

Our main result here extends Theorem 4 by showing that $\mathcal{L}^+_{\Box,\Diamond,\wedge,\vee}$ is finitely characterizable.

Theorem 5. *$\mathcal{L}^+_{\Box,\Diamond,\wedge,\vee}$ is finitely characterizable.*

Theorem 3 above shows that this is essentially optimal; we leave open the question whether the fragment without $\bot$ but with $\top$ is finitely characterizable.

In the rest of this section, we outline the ideas behind the proof of Theorem 5. A key ingredient is the novel notion of *weak simulation*, which we obtain by weakening the back and forth clauses of the *simulations* studied in [3]. Simulations are themselves a weakening of bisimulations. It was shown in [3] that $\mathcal{L}^+_{\Box,\Diamond,\wedge,\vee,\top,\bot}$ is characterized by preservation under simulations.

[1] It suffices to replace $\top$ by a fresh propositional variable q in the proof of Theorem 1.

A *weak simulation* between two pointed models $(M,s),(M',s')$ is a relation $Z \subseteq M \times M'$ such that for all $(t,t') \in Z$:

(atom)	$M,s \models p$ implies $M',s' \models p$
(forth$'$)	If $R^M tu$, either $M,u \underline{\leftrightarrow} \circlearrowright_\emptyset$ or there is a u' with $R^{M'}t'u'$ and $(u,u') \in Z$
(back$'$)	If $R^{M'}t'u'$, either $M',u' \underline{\leftrightarrow} \circlearrowright_{\text{Prop}}$ or there is a u with $R^M tu$ and $(u,u') \in Z$

where $\circlearrowright_\emptyset$ denotes the single reflexive point with empty valuation, $\circlearrowright_{\text{Prop}}$ denotes the single reflexive point with full valuation and $\underline{\leftrightarrow}$ denotes bisimulation. If such Z exists, we say that M',s' *weakly simulates* M,s. The crucial weakening is witnessed by the fact that the deadlock model, i.e. the single point with no successors, weakly simulates $\circlearrowright_\emptyset$, but does not simulate it.

Because weak simulations are closed under relational composition, which is associative, the collection of pointed models and weak simulations forms a category with $\circlearrowright_\emptyset$ and $\circlearrowright_{\text{Prop}}$ as weak initial and final objects, respectively.

Theorem 6. *$\mathcal{L}^+_{\Box,\Diamond,\wedge,\vee}$ is preserved under weak simulations.*

In high level terms, the proof of Theorem 5 proceeds as follows: given a formula $\varphi \in \mathcal{L}^+_{\Box,\Diamond,\wedge,\vee}$, we show how to construct positive and negative examples $(E^+_\varphi, E^-_\varphi)$ that φ fits and which form a *duality* (a generalisation of the notion of *splittings* in lattice theory [6]) in the category of pointed models and weak simulations. By this, we mean that every pointed model either weakly simulates some positive example in E^+ or is weakly simulated by some negative example in E^-. More specifically, we show that every model of φ weakly simulates some positive example in E^+ and that every non-model of φ is weakly simulated by some negative example in E^-. It follows by Theorem 6 that any $\mathcal{L}^+_{\Box,\Diamond,\wedge,\vee}$-formula that fits E^+ is implied by φ, while every formula that fits E^- implies φ. Combined, this shows that $(E^+_\varphi, E^-_\varphi)$ is a finite characterization of φ w.r.t. $\mathcal{L}^+_{\Box,\Diamond,\wedge,\vee}$.

This proof technique was inspired by results in [7], which established a similar connection between finite characterizations for GAV schema mappings (or, equivalently, unions of conjunctive queries) and dualities in the category of finite structures and homomorphisms.

See `https://bit.ly/3LCtmQt` for more details and further results.

4 Discussion

Our construction, although effective, is non-elementary. For this reason, we cannot obtain from it an efficient exact learning algorithm. On the other hand, it follows from the results in [2] that $\mathcal{L}^+_{\Diamond,\wedge}$-formulas are polynomial-time exactly learnable with membership queries. We leave it as future work to prove matching lower bounds for our construction, and to understand more precisely which modal fragments admit polynomial-sized finite characterizations and/or are polynomial-time exactly learnable with membership queries.

Variants of Theorem 5 can be obtained for $\mathcal{L}^-_{\Box,\Diamond,\wedge,\vee}$ and, more generally, for *uniform* modal formulas, where certain propositional variables only occur positive and others only negatively.

As we mentioned, our immediate motivation for this work came from a renewed interest in exact learnability in description logic. In particular, in [2], exact learnability with membership queries is studied for the description logic $\mathcal{ELI}$. These results are extended to results on learning $\mathcal{ELI}$ concepts under DL-Lite ontologies (i.e. background theory) [4] and temporal instance queries formulated in fragments of linear time logic LTL [5]. We expect that our proof of Theorem 5 can be lifted to the poly-modal case without major changes, with implications for some description logics under the closed-world assumption.

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Many-valued coalgebraic modal logic with a semi-primal algebra of truth-degrees

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Abstract

In this talk we report on our work in progress on a general coalgebraic approach to many-valued modal logic with a semi-primal bounded lattice expansion of truth-values. In particular, we illustrate how it relates to classical modal logic and discuss how our approach generalizes various many-valued modal logics from the literature.

1 Setting

In his generalized 'Boolean' theory of universal algebras [4] Foster introduced primal algebras. Generalizing the functional completeness of the two-element Boolean algebra $\mathbf{2}$, an algebra $\mathbf{L}$ is *primal* if every operation on its carrier set L is term-definable. During the second half of the 20th century, various weakenings of this property have been studied [9]. Since the algebras thus arising are still 'close to $\mathbf{2}$', it is reasonable to consider them as algebras of truth-values for many-valued logic. In the talk we focus on *semi-primality* [5].

Definition 1. A finite algebra $\mathbf{L}$ is *semi-primal* if every operation $f\colon L^n \to L$ which preserves subalgebras[1] is term-definable in $\mathbf{L}$.

In a slogan, semi-primal algebras are like primal algebras which allow proper subalgebras. Prominent examples from logic are finite Łukasiewicz chains or finite Łukasiewicz-Moisil chains. Further examples of semi-primal (or, more generally, quasi-primal) algebras which are not based on chains can be found among the FL_{ew}-algebras or among the *pseudo-logics*, that is, bounded lattices with an additional unary operator swapping 0 and 1. The framework of our talk is the following.

Assumption 2. Let $\mathbf{L}$ be a semi-primal algebra with underlying bounded lattice $\mathbf{L}^\flat = (L, \wedge, \vee, 0, 1)$ where $0 \neq 1$. Let $\mathcal{A} = \mathbb{HSP}(\mathbf{L})$ be the variety generated by $\mathbf{L}$.

Abstractly, $\mathbf{2}$-valued coalgebraic modal logic for an endofunctor $\mathsf{T}\colon \mathsf{Set} \to \mathsf{Set}$ is summarized in the following picture based on Stone duality after 'forgetting topology':

$$\mathsf{T} \circlearrowright \mathsf{Set} \rightleftarrows \mathsf{BA} \circlearrowleft \mathsf{A} \qquad (1)$$

For example, if $\mathsf{T} = \mathcal{P}$ is the covariant powerset functor, then the category of $\mathcal{P}$-coalgebras $\mathsf{Coalg}(\mathcal{P})$ corresponds to the category Kripke frames with bounded morphisms. Similarly the category of A-algebras $\mathsf{Alg}(\mathsf{A})$ corresponds to the variety of modal algebras.

*Speaker
[1] If $\mathbf{S}$ is a subalgebra of $\mathbf{L}$ then $a_1 \dots a_n \in S \Rightarrow f(a_1, \cdots a_n) \in S$.

To get a similar picture for our variety $\mathcal{A}$, we apply the duality for semi-primal varieties due to Keimel and Werner [7] (also see [3]) which asserts that $\mathcal{A}$ is dually equivalent to the category $\mathsf{Stone}_{\mathbf{L}}$ defined as follows

Definition 3. Objects of $\mathsf{Stone}_{\mathbf{L}}$ are of the form $(X, \mathbf{v})$ where $X \in \mathsf{Stone}$ and $\mathbf{v} \colon X \to \mathbb{S}(\mathbf{L})$ is continuous. Morphisms $f \colon (X, \mathbf{v}) \to (Y, \mathbf{w})$ in $\mathsf{Stone}_{\mathbf{L}}$ are continuous maps satisfying $\mathbf{w}(f(x)) \leq \mathbf{v}(x)$.

Let $\mathsf{Set}_{\mathbf{L}}$ be the category obtained from $\mathsf{Stone}_{\mathbf{L}}$ after 'forgetting topology'. There is a canonical way to lift T from diagram (1) to an endofunctor $\mathsf{T}' \colon \mathsf{Set}_{\mathbf{L}} \to \mathsf{Set}_{\mathbf{L}}$. We ultimately aim to describe the modal logic abstractly characterized by

$$\mathsf{T}' \circlearrowright \mathsf{Set}_{\mathbf{L}} \rightleftarrows \mathcal{A} \circlearrowleft \mathsf{A}' \tag{2}$$

This also yields the more commonly investigated case

$$\mathsf{T} \circlearrowright \mathsf{Set} \rightleftarrows \mathcal{A} \circlearrowleft \mathsf{A}' \tag{3}$$

obtained after composing by the forgetful functor $\mathsf{U} \colon \mathsf{Set}_{\mathbf{L}} \to \mathsf{Set}$ and its left adjoint.

2 Examples

Example 4. In our first example, let $\mathsf{T} = \mathcal{P}$. The coalgebras for the lifted functor $\mathsf{Coalg}(\mathcal{P}')$ correspond to *crisp* $\mathbf{L}$*-frames*. That is, to triples $\mathfrak{F} = (W, R, \mathbf{v})$ where (W, R) is a Kripke frame and $\mathbf{v} \colon W \to \mathbb{S}(\mathbf{L})$ satisfies the compatibility condition

$$wRw' \Rightarrow \mathbf{v}(w') \subseteq \mathbf{v}(w)$$

For the $\mathbf{L}$-*models* over $\mathfrak{F}$ we only allow valuations $Val : W \times \mathsf{Prop} \to L$ which always satisfy

$$Val(w, p) \in \mathbf{v}(w).$$

In this case, diagram (2) is closely related to work by Maruyama [8]: the algebras $\mathsf{Alg}(\mathsf{A}')$ correspond to what is therein called $\mathbb{ISP}_{\mathbb{M}}(\mathbf{L})$. The non-restricted case where all valuations are allowed corresponds to diagram (3) and arises if $\mathbf{v}(w) = \mathbf{L}$ everywhere. Here, in the special case $\mathbf{L} = \mathbf{Ł}_n$ it corresponds to modal extensions of Łukasiewicz many-valued logic as described in [6].

Example 5. For another example, we hint at the case where $\mathsf{T} = \mathcal{L}$ is the covariant functor which generalizes $\mathcal{P}$, that is, it is defined on objects by $\mathcal{L}(X) = L^X$ and on morphisms $f : X \to Y$ by

$$\begin{aligned} \mathcal{L}f : L^X &\to L^Y \\ h &\mapsto (y \mapsto \bigvee \{h(x) \mid f(x) = y\}). \end{aligned}$$

Now in (2) the coalgebras for the lifted endofunctor $\mathsf{Coalg}(\mathcal{L}')$ correspond to the $\mathbf{L}$-*labeled* $\mathbf{L}$-*frames*, that is, $(W, R, \mathbf{v})$ similar to the crisp $\mathbf{L}$-frames except that now the accessibility relation

$$R \colon W \to L^W$$

is many-valued as well. Diagram (3) corresponds again to $\mathbf{L}$-labeled frames without further restrictions. This, in the case $\mathbf{L} = \mathbf{Ł}_n$ corresponds to the frames that have been recently investigated by algebraic means in [2] (see also [1]).

3 Content of the talk

In the talk, we will report about our work in progress on the investigation of the modal logics arising from diagrams (2) and (3) in the general case, and discuss some examples which arise by specifying to some particular functors T.

We explain how our logics relate to the classical case given by diagram (1). Afterwards we may discuss our results regarding completeness, definability and expressivity of these logics.

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What is the cost of cut?

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Abstract

In [8, 7] we looked at substructural calculi from a game semantic point of view, guided by certain intuitions about resource conscious and, more specifically, cost conscious reasoning. This culminated in labelled extensions of (intuitionistic, affine) linear logic with multimodalities (subexponentials), which allowed for an elegant interpretation of the *dereliction* rule. In this work, we investigate the proof theoretical effect of costs in the cut-elimination process.

Introduction. Various kinds of game semantics have been introduced to characterize computational features of substructural logics, in particular fragments and variants of linear logic (LL) [6]. This line of research can be traced back to the works of Blass [3], Abramsky and Jagadeesan [1] among several others.

Our particular view of game semantics is that it is not just a technical tool for characterizing provability in certain calculi, but rather a playground for illuminating specific semantic intuitions underlying certain proof systems. Specially, we aim at a better understanding of *resource conscious* reasoning, which is often cited as a motivation for substructural logics.

As presented in [8], in a first step, we characterize a version of linear logic (exponential-free affine inuitionistic linear logic **aIMALL**, or, equivalently, Full Lambek Calculus with exchange and weakening FLew) by a game, where the difference between additive and multiplicative connectives is modeled as sequential versus parallel continuation in game states that directly correspond to sequents. More precisely, every branching rule for a multiplicative connective corresponds to a game rule that splits the current run of the game into two independent subgames. Player **P**, who seeks to establish the validity of a given sequent, has to win all the resulting subgames. In contrast, a branching rule for an additive connective is modeled by a choice of player **O** between two possible succeeding game states, corresponding to the premises of the sequent rule in question. Note that this amounts to a deviation from the paradigm "formulas as games", underlying the game semantic tradition initiated by Blass [3]. Our games are, at least structurally, closer to Lorenzen's game for intuitionistic logic [9], where a state roughly corresponds to a situation in which a proponent seeks to defend a particular statement against attacks from an opponent, who, in general, has already granted a bunch of other statements. This kind of semantics for linear logic (but without the sequential/parallel distinction) was first explored in [5].

As long as we only care about the existence of winning strategies, the distinction between sequential and parallel subgames is redundant. However, our model not only highlights the intended semantics, but it also has concrete effects once we introduce *costs* for resources (represented by formulas) into the game. This is done via the unary operator $!^a$, $a \in \mathbb{R}^+$, called *subexponential* in LL (SELL [4, 10]). The intuition is that, from $!^a A$ we can obtain A as often as we want, each time paying the price a. We lift our game to the extended language by enriching game states with a *budget* that is decreased whenever a price is paid. Different strategies for proving the same endsequent can then be compared by the budget

*Speaker.

which they require to be run safely, i.e. without getting into debts. This form of resource consciousness not only enhances the game, but it also translates into a novel sequent system, where cost bounds for proofs are attached as labels to sequents. In [8], we only considered resources in *assumptions*. This is translated to sequents by restricting *negatively* the occurrences of the modalities $!^a$. Thus a promotion rule was not present and the proof-theoretic properties of the proposed systems, such as cut-elimination, can be mimicked by the ones of **aIMALL**. Here we move towards two possible generalizations, allowing modalities also in positive contexts: (i) we propose an admissible cut rule for a restricted form of cut formulas; (ii) we propose an alternative notion of the cut rule itself, with a cost-continuation kind of style.

A game model of branching with costs We will denote by $C^\ell(\mathbb{R}^+)$ the SELL system with labelled sequents of the form $b : \Gamma \longrightarrow \Delta$ where Γ, Δ are multisets of formulas and $b \in \mathbb{R}^+$. Formulas are built from the grammar $A ::= p \mid \mathbf{0} \mid \mathbf{1} \mid A_1 \multimap A_2 \mid A_1 \otimes A_2 \mid A_1 \,\&\, A_2 \mid A_1 \oplus A_2 \mid !^a A$, where p stands for atomic propositions (variables); $\mathbf{0}/\mathbf{1}$ are the false/true units; $\multimap$ denotes linear implication; $\otimes/\&$ are the multiplicative/additive conjunctions; $\oplus$ is the additive disjunction; and $!^a A$ is a subexponential with $a \in \mathbb{R}^+$. The rules for $C^\ell(\mathbb{R}^+)$ are depicted in Fig 1.

The game described by $C^\ell(\mathbb{R}^+)$ is formally defined as follows.

Definition 1 (The game $\mathcal{G}_C(\mathbb{R}^+)$)**.** *$\mathcal{G}_C(\mathbb{R}^+)$ is a game of two players,* **P** *and* **O**. *Game states are tuples (H, b), where H is a finite multiset of extended sequents and $b \in \mathbb{R}$ is a "budget". $\mathcal{G}_C$ proceeds in rounds, initiated by* **P***'s selection of an extended sequent S from the current game state. The successor state is determined according to rules that fit one of the two following schemes:*

$$\begin{array}{lll} (1) & (G \cup \{S\}, b) & \rightsquigarrow & (G \cup \{S'\}, b') \\ (2) & (G \cup \{S\}, b) & \rightsquigarrow & (G \cup \{S^1\} \cup \{S^2\}, b) \end{array}$$

A round proceeds as follows: After **P** *has chosen an extended sequent $S \in H$ among the current game state, she chooses a rule instance r of $C(\mathbb{R}^+)$ such that S is the conclusion of that rule. Depending on r, the round proceeds as follows:*

1. *If r is a unary rule different from $!_L$ with premise S', then the game proceeds in the game state $(G \cup \{S'\}, b)$.*
2. **Budget decrease:** *If $r = !_L$ with premise S' and principal formula $!^a A$, then the game proceeds in the game state $(G \cup \{S'\}, b - a)$.*
3. **Parallelism:** *If r is a binary rule with premises S_1, S_2 pertaining to a* multiplicative *connective, then the game proceeds as $(G \cup \{S_1\} \cup \{S_2\}, b)$.*
4. **O-choice:** *If r is a binary rule with premises S_1, S_2 pertaining to an* additive *connective, then* **O** *chooses $S' \in \{S_1, S_2\}$ and the game proceeds in the game state $(G \cup \{S'\}, b)$.*

A **winning state** *(for* **P***) is a game state (H, b) such that all $S \in H$ are initial sequents of $C(\mathbb{R}^+)$ and $b \geq 0$.*

We write $\models_{\mathcal{G}_C(\mathbb{R}^+)} (H, b)$ if **P** has a w.s. in the $\mathcal{G}_C(\mathbb{R}^+)$-game starting on (H, b). The intuitive reading of $\models_{\mathcal{G}_C(\mathbb{R}^+)} (H, b)$ is: *The budget b suffices to win the game H.*

The following result states the strong adequacy for $\mathcal{G}_C(\mathbb{R}^+)$ w.r.t $C^\ell(\mathbb{R}^+)$.

Theorem 1. $\models_{\mathcal{G}_C(\mathbb{R}^+)} (\{\Gamma \longrightarrow A\}, b)$ *iff* $\vdash_{C^\ell(\mathbb{R}^+)} b : \Gamma \longrightarrow A$.

The problem with cut-admissibility. Due to the tight relationship between $C^\ell(\mathbb{R}^+)$ and SELL, it is clear that $C(\mathbb{R}^+)$ inherits the admissibility of the following cut rule, for some $c \in \mathbb{R}$.

$$\frac{a : !\Gamma, \Delta_1 \longrightarrow A \quad b : !\Gamma, \Delta_2, A \longrightarrow C}{c : !\Gamma, \Delta_1, \Delta_2 \longrightarrow C} \; cut$$

labelled sequent system for $C^\ell(\mathbb{R}^+)$

$$\frac{b : \Gamma, A, B \longrightarrow C}{b : \Gamma, A \otimes B \longrightarrow C}\ \otimes_L \qquad \frac{a : !\Gamma, \Delta_1 \longrightarrow A \quad b : !\Gamma, \Delta_2 \longrightarrow B}{a + b : !\Gamma, \Delta_1, \Delta_2 \longrightarrow A \otimes B}\ \otimes_R$$

$$\frac{a : !\Gamma, \Delta_1 \longrightarrow A \quad b : !\Gamma, \Delta_2, B \longrightarrow C}{a + b : !\Gamma, \Delta_1, \Delta_2, A \multimap B \longrightarrow C}\ \multimap_L \qquad \frac{b : \Gamma, A \longrightarrow B}{b : \Gamma \longrightarrow A \multimap B}\ \multimap_R$$

$$\frac{b : \Gamma, A_i \longrightarrow B}{b : \Gamma, A_1 \,\&\, A_2 \longrightarrow B}\ \&_{L_i} \qquad \frac{a : \Gamma \longrightarrow A \quad b : \Gamma \longrightarrow B}{\max\{a, b\} : \Gamma \longrightarrow A \,\&\, B}\ \&_R$$

$$\frac{a : \Gamma, A \longrightarrow C \quad b : \Gamma, B \longrightarrow C}{\max\{a, b\} : \Gamma, A \oplus B \longrightarrow C}\ \oplus_L \qquad \frac{b : \Gamma \longrightarrow A_i}{b : \Gamma \longrightarrow A_1 \oplus A_2}\ \oplus_{R_i}$$

$$\frac{b : \Gamma, !^a A, A \longrightarrow C}{b + a : \Gamma, !^a A \longrightarrow C}\ !_L \qquad \frac{b : \Gamma^{a\leq}, A \longrightarrow A}{b : \Gamma \longrightarrow !^a A}\ !_R$$

$$\frac{}{b : \Gamma, p \longrightarrow p}\ I \qquad \frac{}{b : \Gamma \longrightarrow \mathbf{1}}\ \mathbf{1}_R \qquad \frac{}{b : \Gamma, \mathbf{0} \longrightarrow A}\ \mathbf{0}_L \qquad \frac{a : \Gamma \longrightarrow A}{b : \Gamma \longrightarrow A}\ w_\ell (b \geq a)$$

Figure 1: The labelled sequent system $C^\ell(\mathbb{R}^+)$

The question then is: how to calculate c? The following result shows that it is not possible to define a function for determining the label of the conclusion depending exclusively on the labels of the premises.

Theorem 2. *There is no function $f : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ such that the following rule is admissible in $C^\ell(\mathbb{R}^+)$.*

$$\frac{!\Gamma, \Delta_1 \longrightarrow_a A \quad !\Gamma, \Delta_2, A \longrightarrow_b C}{!\Gamma, \Delta_1, \Delta_2 \longrightarrow_{f(a,b)} C}\ cut$$

Proof: Let p, q be different propositional variables, and let $A^{\otimes n}$ denote the n-fold multiplicative conjunction of a formula A. The sequents $!^{1/k} p \longrightarrow_a !^{1/k} p^{\otimes(k\cdot a)}$ and $!^{1/k} p^{\otimes(k\cdot a)} \longrightarrow_b p^{\otimes(k\cdot k\cdot a\cdot b)}$ are provable in $C^\ell(\mathbb{R}^+)$ for all natural numbers a, b, k. The smallest label f which makes their cut conclusion $!^{1/k} p \longrightarrow_f p^{\otimes(k\cdot k\cdot a\cdot b)}$ provable in $C^\ell(\mathbb{R}^+)$ is $k \cdot a \cdot b$, which is not a function on the premise labels a, b. □

Some alternative paths. We finish this text by discussing two alternatives for defining a notion of cuts with costs: first by restricting the cut-formulas; second by enhancing the notion of cut rule.

The following cut rule is admissible for a restricted form of the cut formula [7].

Theorem 3. *If A is bang-free and $c \neq 0$, then the following cut rule is admissible in $C^\ell(\mathbb{R}^+)$:*

$$\frac{a : !\Gamma, \Delta_1 \longrightarrow !^c A \quad b : !\Gamma, \Delta_2, !^c A \longrightarrow C}{f(a, b, c) : !\Gamma, \Delta_1, \Delta_2 \longrightarrow C}\ cut_\ell \quad \text{where } f(a, b, c) = b + \lfloor b/c \rfloor .a$$

Note that, in the particular case where the cut formula itself has no bangs from the beginning, then $f(a, b) = a + b$. On the other hand, the general case where A is not bang-free is an open problem.

Finally, Thm. 2 leaves open the possibility that cut is admissible w.r.t. a function f which takes more information of the premises into account than just their labels. The next definition formalizes this process.

Definition 2. *Let $\mathcal{E} = \{a_b \mid a, b \in \mathbb{R}^+\}$ be such that*

1. $a_b +_{\mathcal{E}} c_d = a + b + c + d$.
2. $a_b \geq_{\mathcal{E}} a_c$ *(i.e., the ordering* $\geq_{\mathcal{E}}$ *ignores the subindices).*
3. $a_b >_{\mathcal{E}} c_d$ *iff* $a > c$.

For any formula $F \in C^\ell(\mathbb{R}^+)$*, we define* $[F]_c$ *as the formula that substitutes any modality* $!^{a_b}$ *with* $!^{a_{b+c}}$.

Hence $C^\ell(\mathbb{R}^+)$ can be slightly modified so that sequent labels belong to $\mathbb{R}^+$, while modal labels belong to $\mathcal{E}$. Due to the ordering above, the promotion of $!^{a_0}$ has the same effect/constraints that the promotion of $!^{a_b}$. However, the dereliction of the latter requires a greater budget ($a + b$ instead of a). Moreover, the equivalence $!^{a_b}F \equiv !^{a_c}F$ can be proven, each direction requiring a different budget. Finally, note that $\mathcal{E}_0 = \{a_0 \mid a \in \mathbb{R}^+\} \simeq \mathbb{R}^+$, that is, each element $a \in \mathbb{R}^+$ can be seen as the equivalence class of a_0 in $\mathbb{R}^+ \times \mathbb{R}^+$ modulo $\mathbb{R}^+$. We will abuse the notation and continue representing the resulting system by $C^\ell(\mathbb{R}^+)$, also unchanging the representation of sequents. The following has a straightforward proof.

Lemma 1. *If* $b : \Gamma, [F]_c \longrightarrow G$ *then* $b : \Gamma, F \longrightarrow G$ *with* $b \geq b'$*. More generally, if* $b : \Gamma, [F]_c \longrightarrow C$ *and* $c \geq c'$ *then* $b : \Gamma, [F]_{c'} \longrightarrow C$ *with* $b \geq b'$.

The next definition restricts the appearance of unbounded modalities only under linear implication.

Definition 3. *F is* $\multimap$*-linear if for all subformulas of the form* $A \multimap B$*, A doesn't have occurrences of* $!^a$.

The following result presents the admissibility of an extended form of the cut rule, where the budget information from the left premise is passed to the cut-formula in the right premise. Observe that the label of the conclusion is now a function of the labels of the premises.

Theorem 4 ($\multimap$-linear cut)**.** *The following rule is admissible*

$$\frac{a : !\Gamma, \Delta_1 \longrightarrow F \quad b : !\Gamma, \Delta_2, [F]_a \longrightarrow C}{a + b : !\Gamma, \Delta_1, \Delta_2 \longrightarrow C} \; cut_{LL} \quad F \text{ is a } \multimap\text{-linear formula}$$

Moreover, if $a : \Gamma \longrightarrow C$ *is provable using* cut_{LL}*, then there is a cut-free proof of* $a' : \Gamma \longrightarrow C$ *with* $a \geq a'$.

For future work, we expect that the study of costs of proofs and cut-elimination in labeled calculi may indicate a relationship between labels and bounds of computation as in [2].

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Free algebras in all subvarieties of the variety generated by the MG t-norm

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Basic Fuzzy Logic (BL for short) was introduced by Hájek in [7] to formalize fuzzy logics in which the conjunction is interpreted by a continuous t-norm on the real segment $[0,1]$ and the implication by its corresponding adjoint. The equivalent algebraic semantics for BL, in the sense of Blok and Pigozzi, is the variety of BL-algebras $\mathcal{BL}$. Many algebraic properties of BL-algebras correspond to logical properties of BL. For example, and what is our concern, the elements of free algebras in $\mathcal{BL}$ are in correspondence with equivalence classes of formulas in the logic. This is why many attempts to study free BL-algebras have been accomplished in the last decades. Some of these studies, like [6] and [4], describe free algebras in subvarieties of BL-algebras from an structural point of view, considering the representation of the algebra as weak boolean product of directly indecomposable BL-algebras over the Stone Space corresponding to a free Boolean algebra. Some others, as [9] and [2] present a functional description of the elements of the free algebra.

In [5] there is a functional representation of the free algebra in the subvariety of BL-algebras generated by a chain which is the ordinal sum of the standard MV-algebra $[0,1]_{\mathbf{MV}}$ and a basic hoop $\mathbf{H}$, that is, generated by $[0,1]_{\mathbf{MV}} \oplus \mathbf{H}$. The main advantage of this approach, is that unlike the work done in [3] and [2], when the number n of generators of the free algebra increase the generating chain remains fixed. This provides a clear insight of the role of the two main blocks of the generating chain in the description of the functions in the free algebra: the role of the regular elements and the role of the dense elements.

We will focus on the particular case of the variety $\mathcal{MG}$, the variety generated by the ordinal sum of the standard algebra $[0,1]_{\mathbf{MV}}$ and the Gödel hoop $[0,1]_{\mathbf{G}}$. As a logical counterpart, this variety $\mathcal{MG}$ has also an equational characterization as a subvariety of $\mathcal{BL}$, given by adding the equation

$$(\neg\neg x \to x)^2 = \neg\neg x \to x$$

to the axioms of Basic Logic. This equation show that in our variety, the dense elements are idempotent. The t-norm that generates this variety (which we call MG t-norm) is the function $t : [0,1]^2 \to [0,1]$ defined by

$$t(x,y) = \begin{cases} max\{0, x+y-\frac{1}{2}\} & \text{if } x,y \in [0,\frac{1}{2}); \\ min\{x,y\} & \text{otherwise.} \end{cases}$$

The functions in the free algebra $Free_{\mathcal{MG}}(n)$ can be described by decomposing the domain of the functions $([0,1]_{\mathbf{MV}} \oplus [0,1]_{\mathbf{G}})^n$ in a finite number of pieces. On each piece a function $F \in Free_{\mathcal{MG}}(n)$ coincides either with a function in $Free_{\mathcal{MV}}(n)$ or a function in $Free_{\mathcal{G}}(m)$, for some $m \leq n$.

*Speaker.

It is well known that there exists a one-to-one correspondence between subvarieties of BL-algebras and axiomatic extensions of the BL-logic, through a natural translation between algebraic equations and logical axioms. Therefore, when we study a family of subvarieties of $\mathcal{BL}$ and their equational bases, we also obtain the corresponding axiomatic extensions of BL. In [1] there is a description of the lattice of subvarieties of $\mathcal{BL}$, but in our case we will completely describe the lattice of subvarieties of a particular subvariety: $\mathcal{MG}$. For this purpose, we will first characterize the join-irreducible subvarieties in the lattice and then show that every subvariety is a join of finite join-irreducible varieties in the lattice. With that results, we will then give an equational characterization for every subvariety in the lattice.

Finally, we give a characterization for every subvariety of $\mathcal{MG}$ as a finite product of algebras given by the restriction of the functions in $Free_{\mathcal{MG}}(n)$ over some rational points and their neighbourhoods. This result, extends the characterization given by G. Panti in [10] for all subvarieties of $\mathcal{MV}$ and the description of the algebra $Free_{\mathcal{MG}}(n)$ given in [8], since we give in this case a description of the free algebra for every subvariety of $\mathcal{MG}$.

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The quasivariety given by the class of possibilistic Ł$_n$-valued Kripke frames

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Possibility theory is an uncertainty theory devoted to the handling of incomplete information, and was originally introduced by Zadeh in [7]. However, only later on *possibilistic logic* emerges as a logic utilizing classical formulas associated with degrees of certainty [3, 4].

Denote by Ł$_n$ the n-element MV-chain as a bounded residuated lattice, that is, the set

$$\text{Ł}_n = \left\{0, \frac{1}{n-1}, \ldots, \frac{n}{n-1}, 1\right\}$$

is equipped with the constants $0, 1$ and the operations $\wedge$, $\vee$, $*$ and $\rightarrow$, where the last two are given by $x * y = \max\{0, x + y - 1\}$ and $x \rightarrow y = \min\{1, 1 - x + y\}$.

An Ł$_n$-valued possibilistic Kripke frame is a structure $\langle W, \pi \rangle$ where W is a non-empty set whose elements are called *possible worlds* and $\pi : W \rightarrow \text{Ł}_n$ is a function on W, known as a *possibility distribution*. A possibility distribution π is normalized when $\sup_{w \in W} \pi(w) = 1$, or equivalently for Ł$_n$, when there is a $w \in W$ such that $\pi(w) = 1$.

For each Ł$_n$-valued Kripke frame $\langle W, \pi \rangle$, the *possibilistic complex algebra* $\langle \text{Ł}_n^W, \forall^\pi, \exists^\pi \rangle$ associated with $\langle W, \pi \rangle$ is the MV-algebra Ł_n^W of functions from W to Ł$_n$ and MV-operations defined pointwise, endowed with two unary operators $\forall^\pi$ and $\exists^\pi$ defined for each $x \in \text{Ł}_n^W$ by:

$$\forall^\pi(x)(i) = \bigwedge_{j \in W} \{\pi(j) \rightarrow x(j)\},$$
$$\exists^\pi(x)(i) = \bigvee_{j \in W} \{\pi(j) * x(j)\}.$$

This is the way the operators are defined in a more general setting by Hájek as the fuzzification of the logic of belief KD45 [5]. In our case, since MV-algebras are involutive, actually only one operator is needed, as $\forall^\pi(x) = \neg\exists^\pi(\neg x)$.

This possibilistic framework is a particular case of Ł$_n$-valued Kripke frames, that consists of structures $\langle W, R \rangle$ where $R : W \times W \rightarrow \text{Ł}_n$ is a binary non-classical relation on W, known as an *accessibility relation*. Possibility distributions can be seen as accesibility relations R satisfying that for each $j \in W$, $R(i, j) = R(k, j)$ for each $i, k \in W$. In the more general setting, for each Ł$_n$-valued Kripke frame $\langle W, R \rangle$, the *complex algebra* $\langle \text{Ł}_n^W, \forall^R, \exists^R \rangle$ associated with $\langle W, R \rangle$ is the

MV-algebra $Ł_n^W$ endowed with two unary operators $\forall^R$ and $\exists^R$ defined for each $x \in Ł_n^W$ by:

$$\forall^R(x)(i) = \bigwedge_{j \in W} \{R(i,j) \to x(j)\},$$
$$\exists^R(x)(i) = \bigvee_{j \in W} \{R(i,j) * x(j)\}.$$

As before, in the case of MV-algebras, only one operator is needed.

In [2], the authors give an algebraic semantics for the quasivariety generated by the complex algebras, ML_n-algebras, based on results for the minimum modal logic over finite residuated lattices presented in [1] .

For each fixed $n \in \mathbb{N}$, $n \geq 2$, and each rational number $0 \leq \alpha \leq 1$, McNaughton's Theorem guarantees the existence of a unary term $\eta_\alpha : Ł_n \to \{0,1\}$ that satisfies

$$\eta_\alpha^{Ł_n}(a) = \begin{cases} 1 & \text{if } \alpha \leq a, \\ 0 & \text{if } a < \alpha. \end{cases}$$

These unary operators where introduced by Moisil in [6] to define some algebraic structures related to MV$_n$-algebras. We will make use of these terms for the case $\alpha \in Ł_n$.

An algebra $(\mathbf{A}, \forall)$ is an ML_n-algebra if $\mathbf{A}$ is an MV$_n$-algebra and $\forall$ is a unary operator satisfying:

(PMV$\forall$) $\forall 1 = 1$,
(PMV2) $\forall(x \wedge y) = \forall x \wedge \forall y$,
($\mathbf{R}_a^*$) the quasiequations ($\mathbf{R}_a^*$) for each $a \in Ł_n \setminus \{0\}$.

These quasiequations are defined as follows: for each $n \in \mathbb{N}$, $n \geq 2$, let $a_2 = \frac{1}{n-1}$, $a_3 = \frac{2}{n-1}$, ..., $a_{n-1} = \frac{n-2}{n-1}$ and $a_n = 1$, which are all the non-zero elements of the finite chain $Ł_n$. Given a and b in the chain $Ł_n$, we settle

$$\epsilon_{a,b}(x_2, \ldots, x_n, y) = (\eta_{a_2 * b}(x_2) \wedge \eta_{a_3 * b}(x_3) \wedge ... \wedge \eta_{a_n * b}(x_n)) \to \eta_{a * b}(y)$$

$$\delta_a(x_2, \ldots, x_n, y) = (\eta_{a_2}(\forall x_2) \wedge \eta_{a_3}(\forall x_3) \wedge ... \wedge \eta_{a_n}(\forall x_n)) \to \eta_a(\forall y)$$

Then for each $a \in Ł_n \setminus \{0\}$,

$$(\boldsymbol{R_a^*}) \qquad \bigwedge_{b > \neg a} (\epsilon_{a,b} \approx 1) \to (\delta_a \approx 1).$$

We define $\mathbb{ML}_n$ as the quasivariety of ML_n-algebras.

The intuition behind these quasiequations (see [1, 2]) is that the normality axiom K

$$\forall(x \to y) \to (\forall x \to \forall y) \approx 1$$

does not hold for complex algebras associated with $Ł_n$-valued Kripke frames, but the weaker version given by the rules ($\mathbf{R}_a^*$) does hold, and thus every complex algebra $\langle Ł_n^W, \forall^R \rangle$ associated with an $Ł_n$-valued Kripke frame $\langle W, R \rangle$ is in $\mathbb{ML}_n$. Moreover,

Theorem 1. *[2, Theorem 3.10] Every ML_n-algebra belongs to the quasivariety generated by the complex algebras. Therefore the quasivariety $\mathbb{ML}_n$ is generated by the complex algebras.*

In this work, we will consider the class $\mathcal{P}_\pi$ of complex algebras associated with normalized possibilistic Ł$_n$-frames.

It is not difficult to see that each complex algebra in $\mathcal{P}_\pi$ is an algebra in $\mathbb{ML}_n$ that additionally satisfies:

(PMV1) $\forall x \to \exists x = 1$,

(PMV3) $\exists(x * \exists y) = \exists x * \exists y$.

We call $\mathbb{P}_n$ the subquasivariety of $\mathbb{ML}_n$ satisfying these additional equations.

Our main goal is to prove the following theorem:

Theorem 2. *The quasivariety $\mathbb{P}_n$ is generated by possibilistic complex algebras, i.e., the algebras associated to possibilistic Ł$_n$-frames.*

To do this, we define the following properties for Kripke frames $\langle W, R\rangle$:

(J1) For each $i \in W$ there exists $j \in W$ with $R(i,j) = R(j,j) = 1$.

(J2) For each $i, j, k \in W$, $R(i,j) * R(j,k) = R(i,j) * R(i,k)$.

And we will prove the following lemmas:

Lemma 1. *Let $\langle W, R\rangle$ be an Ł$_n$-valued Kripke frame satisfying **J1** and **J2**. Then there exists $\{W_s\}_{s\in S}$ a partition of W and functions $\pi_s : W_s \to$ Ł$_n$ such that*

i. For each $s \in S$, there exists $j \in W_s$ such that $\pi_s(j) = 1$.

ii. For each $s \in S$ and $i, j \in W_s$, we have that $R(i,j) = \pi_s(j)$.

Moreover,

$$\langle L_n^W, \forall^R\rangle \cong \prod_{s\in S} \langle L_n^{W_s}, \forall^{\pi_s}\rangle.$$

Lemma 2. *Every complex algebra in $\mathbb{P}_n$ satisfies **J1** and **J2**.*

From these results, every complex algebra in $\mathbb{P}_n$ is a product of complex algebras in $\mathcal{P}_\pi$. This together with the fact that complex algebras generate $\mathbb{ML}_n$, imply that $\mathbb{P}_n$ is the quasivariety generated by complex algebras in $\mathcal{P}_\pi$.

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Modal Nelson lattices and their associated twist structures

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This work is about one of the most challenging trends of research in non-classical logic which is the attempt to combine different non-classical approaches together, in our case many-valued and modal logic. This kind of combination offers the skill of dealing with modal notions like belief, knowledge, and obligations, in interaction with other aspects of reasoning that can be best handled using many-valued logics, for instance, vagueness, incompleteness, and uncertainty. In fact, the study that we are going to introduce could be especially interesting from the point of view of Theoretical Computer Science and Artificial Intelligence.
One of the best-known logical systems proposed for handling uncertainty is perhaps Possibilistic logic [3, 4] which is able to reason with graded (epistemic) beliefs on classical propositions by means of necessity and possibility measures. Many authors have proposed generalizations for many-valued propositions [7, 5] but most of these settings have a limited scope since either only apply over finite truth-values (some times expanded with truth-constants and the Monteiro-Baaz's Δ operator) or only consider a language with finitely many variables or where the logic is defined over a two-tiered language, i.e. a flat modal language. Here, we are going to consider full modal logics defined over a *Nilpotent Minimum algebra* which allows interpreting conjunction in terms of min and negation in an involutive way.
In fact, by attempting to be as broad as possible, we introduce a more general approach based on modal Nelson lattices. Later, we show that modal Nilpotent algebras are a subvariety of them.

In order to reach our goal, we will first introduce an extension for modal setting of the one well-known construction of Nelson lattices called twist structures, whose importance has been growing in recent years within the study of algebras related to non-classical logics (see [1, 6, 9]). Our proposed extension is more general than others considered in the literature because it is not required to be monotone with respect to modal operators (see [8]).

We assume the reader know the main properties and definitions about residuated lattices and Heyting algebras. In addition, a residuated lattice is called involutive if it is bounded and it satisfies the double negation equation:

$$a = \neg\neg a.$$

A Nelson residuated lattice or simply Nelson lattice (N3) is an involutive residuated lattice satisfying:

$$((a^2 \to b) \wedge ((\neg b)^2 \to \neg a)) \to (a \to b) = \top.$$

Definition 1. Given a Heyting algebra $\mathbf{A}$, we shall denote by $D(\mathbf{A})$ the filter of dense elements of $\mathbf{A}$, i.e. $D(\mathbf{A}) = \{a \in A : \neg a = \bot\}$.

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A filter F of $\mathbf{A}$ is said to be Boolean provided the quotient $\mathbf{A}/F$ is a Boolean algebra. It is well known and easy to check that a filter F of the Heyting algebra $\mathbf{A}$ is Boolean if and only if $D(\mathbf{A}) \subseteq F$. The Boolean filters of $\mathbf{A}$, ordered by inclusion, form a lattice, having the improper filter A as the greatest element and $D(\mathbf{A})$ as the smallest element.
With all these elements, we can reproduce the twist-structures corresponding to N3-lattices.

Theorem 2. *(Sendlewski + Theorem 3.1 in [1].) Given a Heyting algebra*

$$\mathbf{H} = \langle H, \wedge, \vee, \to, \top, \bot \rangle$$

and a Boolean filter F of $\mathbf{H}$ let

$$R(\mathbf{H}, F) := \{(x, y) \in H \times H : x \wedge y = \bot \text{ and } x \vee y \in F\}.$$

Then we have:

1. $\mathbf{R}(\mathbf{H}, F) = \langle R(\mathbf{H}, F), \wedge, \vee, *, \Rightarrow, \bot, \top \rangle$ *is a Nelson lattice, when the operations are defined as follows:*
   - $(x, y) \vee (s, t) = (x \vee s, y \wedge t)$,
   - $(x, y) \wedge (s, t) = (x \wedge s, y \vee t)$,
   - $(x, y) * (s, t) = (x \wedge s, (x \to t) \wedge (s \to y))$,
   - $(x, y) \Rightarrow (s, t) = ((x \to s) \wedge (t \to y), x \wedge t)$,
   - $\top = (\top, \bot)$, $\bot = (\bot, \top)$.
2. $\neg(x, y) = (y, x)$,
3. *Given a Nelson lattice $\mathbf{A}$, there is a Heyting algebra $\mathbf{H}_{\mathbf{A}}$, unique up to isomorphisms, and a unique Boolean filter $F_{\mathbf{A}}$ of $\mathbf{H}_A$ such that $\mathbf{A}$ is isomorphic to $\mathbf{R}(\mathbf{H}_{\mathbf{A}}, F_{\mathbf{A}})$.*

Remark 3. Let $\mathbf{A}$ be a Nelson lattice. Let us consider $H = \{a^2 : a \in \mathbf{A}\}$ with the operations $a \star^* b = (a \star b)^2$ for every binary operation $\star \in \mathbf{A}$. Then,

$$\mathbf{H}^* = \langle H, \vee^*, \wedge^*, \to^*, 0, 1 \rangle$$

is a Heyting algebra ([10]).

Now, for our aim, we need to introduce some definitions of modal algebras.

Definition 4. A modal Heyting algebra $M\mathbf{A}$ is an algebra $\langle \mathbf{A}, \Box, \Diamond \rangle$ such that the reduct $\mathbf{A}$ is an Heyting algebra, $\Box$ and $\Diamond$ are two binary operators and, for all $a, b \in A$,

$$\text{if } a \wedge b = \bot \text{ then } \Box a \wedge \Diamond b = \bot. \tag{1}$$

Modal Heyting algebras obviously form a quasivariety and, at the present, we do not know whether this class is in fact a variety or not. However, there is well known extension of this quasi-variety that is a variety called *normal* modal Heyting algebra. It is obtaned by including the following equations:

3. $\neg \Diamond a = \Box \neg a$,
4. $\Box(a \to b) \to (\Box a \to \Box b) = \top$,

5. $\Box\top = \top$.

Note that (1) implies that $\Box a \wedge \Diamond\neg a = \bot$ and $\Box\neg a \wedge \Diamond a = \bot$, therefore, we can conclude $\Diamond\neg a \leq \neg\Box a$ and $\Box\neg a \leq \neg\Diamond a$. In addition, if (5) is assumed, we have $\Diamond\bot = \bot$.

Definition 5. A modal N3-lattice (for short MN3-lattice) is an algebra $\langle \mathbf{A}, \blacksquare, \blacklozenge \rangle$ such that the reduct $\mathbf{A}$ is an N3-lattice and, for all $a, b \in A$,

1. $\blacklozenge a = \neg\blacksquare\neg a$,
2. if $a^2 = b^2$ then $(\blacksquare a)^2 = (\blacksquare b)^2$ and $(\blacklozenge a)^2 = (\blacklozenge b)^2$,
3. if $(a \wedge b)^2 = \bot$ then $(\blacksquare a \wedge \blacklozenge b)^2 = \bot$.

In addition, $\mathbf{A}$ is said to be regular if it satifies the following:

4. $\blacksquare(a \wedge b) = \blacksquare a \wedge \blacksquare b$.

Moreover, if $\mathbf{A}$ is a regular modal N3-lattice (for short RMN3-lattice) by using (1) and (4), we can conclude:

4′. $\blacklozenge(a \vee b) = \blacklozenge a \vee \blacklozenge b$.

Finally, we say that a modal Nelson lattice is normal if it is regular and, in addition, satisfies:

5. $\blacksquare\top = \top$.

In this case, we can reproduce the following classical result on RMN3-lattices:

Lemma 6. *If* $\mathbf{N}$ *is a regular modal N3-lattice then it satisfies the next monotony properties:*

$$\text{if } a^2 \leq b \text{ then } (\blacksquare a)^2 \leq \blacksquare b, \quad \text{and} \quad \text{if } (\neg a)^2 \leq \neg b \text{ then } (\neg\blacksquare a)^2 \leq \neg\blacksquare b.$$

Now we are ready to formulate the first result of this work.

Theorem 7. *Let* $\mathbf{H}$ *and* F *be a modal Heyting algebra as defined in 4 and a Boolean filter satisfying:*

$$\text{if } a \wedge b = \bot \text{ and } a \vee b \in F \text{ then } \Box a \vee \Diamond b \in F.$$

Then, $\mathbf{R}(\mathbf{H}, F) = \langle R(\mathbf{H}, F), \wedge, \vee, *, \Rightarrow, \bot, \top, \blacksquare, \blacklozenge \rangle$ *is a Modal Nelson lattice, where the operators* $\blacksquare, \blacklozenge$ *are defined as follows:*

$$\blacksquare(x, y) = (\Box x, \Diamond y), \quad \text{and} \quad \blacklozenge(x, y) = (\Diamond x, \Box y).$$

Now, we are going to extend the representation of Nelson lattice in terms of Heyting algebras from Theorem 2 to the modal context. First we need introduce the next result.

Lemma 8. *Let* $\mathbf{N}$ *be a MN3-lattice. Consider* $\mathbf{H}^* = \langle H, \vee^*, \wedge^*, \rightarrow^*, \bot, \top, \Box^*, \Diamond^* \rangle$ *with* $H = \{a^2 : a \in N\}$ *and operators* $\vee^*, \wedge^*, \rightarrow^*$ *as in Remark 3 and modal operators as follows*

$$\Box^* a = (\blacksquare a)^2, \quad \text{and} \quad \Diamond^* a = (\blacklozenge a)^2$$

for every $a \in H$*. Then* $\mathbf{H}^*$ *is a modal Heyting algebra. In addition, if we take* $F = \{(a \vee \neg a)^2 : a \in N\}$*, then* F *is a Boolean filter satisfying*

$$\text{if } a \vee^* b \in F \text{ and } a \wedge^* b = \bot \text{ then } \Box^* a \vee^* \Diamond^* b \in F$$

for every $a, b \in H$*.*

A direct consequence of previous Lemma is our main result:

Theorem 9. *Let* $\mathbf{N}$ *be a modal N3-lattice. Then* $\mathbf{N}$ *is isomorphic to* $\mathbf{R}(\mathbf{H}^*, F)$ *as defined in Theorem 2 by taking* F *as in the previous lemma.*

Now, we would like to consider an important class of bounded residuated lattices which is the variety $\mathcal{MTL}$ determined by the prelinearity equation:

$$(a \to b) \vee (b \to a) = \top.$$

The involutive members of $\mathcal{MTL}$ satisfying the following equation are called nilpotent minimum algebras:

$$(a * b \to \bot) \vee (a \wedge b \to a * b) = \top.$$

In addition, it is well-known that every Nelson lattice satisfying prelinearity is a nilpotent minimum algebra (see [1, Theorem 6.16]). As usual, a Gödel algebra is a Heyting algebra that satisfies the prelinearity equation. Obviously, we can adapt Definition 4 for modal Gödel algebras and Definition 5 for modal nilpotent minimum algebra (MNM-algebras for short). Furthermore, it is easy to reproduce Theorem 7 giving a twist-construction of modal nilpotent minimum algebras in terms of modal Gödel algebras.
Obviously, MNM-algebras form a quasivariety. However, we are interesting in considering one of their subvarieties that we call pseudo-monadic nilpotent minimum algebras (see [2]). These algebras give us one of the possible algebraic semantics of Possibilistic logic.

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On equational completeness theorems

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Abstract

A logic is said to admit an equational completeness theorem when it can be interpreted into the equational consequence relative to a class of algebras. We characterize logics admitting an equational completeness theorem that have at least one tautology. As a consequence, a protoalgebraic logic admits an equational completeness theorem precisely when it has a matrix semantics validating a nontrivial equation. While the problem of determining whether a logic admits an equational completeness theorem is shown to be decidable both for logics presented by a finite set of finite matrices and for locally tabular logics presented by a finite Hilbert calculus, it becomes undecidable for arbitrary logics presented by finite Hilbert calculi.

1 Introduction

By a *logic* [11] we understand a consequence relation $\vdash$ on the set of formulas Fm (built up with a denumerable set of variables) of some algebraic language that, moreover, is *substitution invariant* in the sense that for every $\Gamma \cup \{\varphi\} \subseteq Fm$ and every substitution σ,

$$\text{if } \Gamma \vdash \varphi, \text{ then } \sigma[\Gamma] \vdash \sigma(\varphi).$$

A logic $\vdash$ admits an *equational (soundness and) completeness theorem* if there are a set of equations $\boldsymbol{\tau}(x)$ and a class of similar algebras K such that for every set of formulas $\Gamma \cup \{\varphi\}$,

$$\begin{aligned}\Gamma \vdash \varphi \iff &\text{for every } \boldsymbol{A} \in \mathsf{K} \text{ and } \vec{a} \in A,\\ &\text{if } \boldsymbol{A} \vDash \boldsymbol{\tau}(\gamma^{\boldsymbol{A}}(\vec{a})) \text{ for every } \gamma \in \Gamma, \text{ then } \boldsymbol{A} \vDash \boldsymbol{\tau}(\varphi^{\boldsymbol{A}}(\vec{a})).\end{aligned}$$

In this case, K is said to be an *algebraic semantics* for $\vdash$ (or a $\boldsymbol{\tau}$*-algebraic semantics* for $\vdash$). Accordingly, a logic admits an equational completeness theorem precisely when it has an algebras semantics.

For instance, the well-known equational completeness theorem for the classical propositional calculus **CPC** states that for every set of formulas $\Gamma \cup \{\varphi\}$,

$$\begin{aligned}\Gamma \vdash_{\mathbf{CPC}} \varphi \iff &\text{for every Boolean algebra } \boldsymbol{A} \text{ and } \vec{a} \in A,\\ &\text{if } \boldsymbol{A} \vDash \gamma^{\boldsymbol{A}}(\vec{a}) \approx 1 \text{ for every } \gamma \in \Gamma, \text{ then } \boldsymbol{A} \vDash \varphi^{\boldsymbol{A}}(\vec{a}) \approx 1.\end{aligned}$$

Thus, the variety of Boolean algebras is an algebraic semantic for **CPC**.

The notion of an algebraic semantics was introduced by Blok and Pigozzi in the study of *algebraizable logics* [5], i.e., logics that are *equivalent* to equational consequences in the sense of [1, 2]. From this point of view, a logic has a $\boldsymbol{\tau}$-algebraic semantics K when it satisfies one half of this equivalence, namely it can be interpreted into the equational consequence relative to K by translating formulas into equations by means of the set of equations $\boldsymbol{\tau}(x)$.

Intrinsic characterizations of logics with an algebraic semantics have proved elusive, partly because equational completeness theorems can take unexpected forms. For instance, in view of Glivenko's theorem [12], for every set of formulas $\Gamma \cup \{\varphi\}$ of **CPC**,

$$\Gamma \vdash_{\mathbf{CPC}} \varphi \Longleftrightarrow \{\neg\neg\gamma \colon \gamma \in \Gamma\} \vdash_{\mathbf{IPC}} \neg\neg\varphi,$$

where **IPC** stands for the intuitionistic propositional calculus. Since Heyting algebras form an $\{x \approx 1\}$-algebraic semantics for **IPC**, one obtains

$$\begin{aligned}\Gamma \vdash_{\mathbf{CPC}} \varphi \Longleftrightarrow &\text{ for every Heyting algebra } \boldsymbol{A} \text{ and } \vec{a} \in A,\\ &\text{if } \boldsymbol{A} \vDash \neg\neg\gamma^{\boldsymbol{A}}(\vec{a}) \approx 1 \text{ for every } \gamma \in \Gamma, \text{ then } \boldsymbol{A} \vDash \neg\neg\varphi^{\boldsymbol{A}}(\vec{a}) \approx 1.\end{aligned}$$

Consequently, the variety of Heyting algebras is also an algebraic semantics for **CPC**, although certainly not the intended one [7, Prop. 2.6].

The fragility of the property of having an algebraic semantics was confirmed by Blok and Rebagliato, who showed that every logic possessing an idempotent connective admits an algebraic semantics [7, Thms. 3.1]. On the other hand, the existence of logics that do not possess any algebraic semantics is known since [3]. It is therefore sensible to wonder whether an intelligible characterization of logics with an algebraic semantics could possibly be obtained [16]. In this talk we provide a positive answer to this question for a wide family of logics.

2 Main results

We shall describe large families of logics with an algebraic semantics. To this end, it is convenient to isolate some limits cases: a logic is said to be *graph-based* when its language comprises only constant symbols and, possibly, a single unary connective. Needless to say, most interesting logics in the literature are *not* graph-based.

To tackle the case of logics that are not graph-based, we first introduce a general method for constructing algebraic semantics based on a universal algebraic trick known as *Maltsev's Lemma*, which provides a description of congruence generation in arbitrary algebras. More precisely, we establish the following, where $Var(\varphi)$ denotes the set of variables occurring in the formula φ.

Theorem 1. *Let $\vdash$ be a logic that is not graph-based. If $\vdash$ has a matrix semantics validating a nontrivial equation $\varphi \approx \psi$ such that $Var(\varphi) \cup Var(\psi) = \{x\}$, then $\vdash$ has an algebraic semantics.*

A logic $\vdash$ is said to be *locally tabular* if it has a matrix semantics whose algebraic reducts generate a locally finite variety.

Corollary 2. *If a logic is locally tabular and not graph-based, then it has an algebraic semantics.*

Another application of Theorem 1 consists in a description of logics with theorems possessing an algebraic semantics. Recall that a formula φ is said to be a *theorem* of a logic $\vdash$ when $\emptyset \vdash \varphi$. Furthermore, a logic $\vdash$ is called *assertional* [15] when it has a matrix semantics M for which there is a unary formula $\gamma(x)$ such that for every $\langle \boldsymbol{A}, F \rangle \in \mathsf{M}$, the term-function $\gamma^{\boldsymbol{A}} \colon A \to A$ is a constant function and its unique value a is such that $F = \{a\}$. Intermediate logics, as well as global consequences [13] of normal modal logics, are known to be assertional.

Theorem 3. *Let $\vdash$ be a nontrivial logic with a theorem φ such that $Var(\varphi) \neq \emptyset$. Then $\vdash$ has an algebraic semantics if and only if either $\vdash$ is assertional and graph-based or it is not graph-based and has a matrix semantics validating a nontrivial equation $\epsilon \approx \delta$ such that $Var(\epsilon) \cup Var(\delta) = \{x\}$.*

A logic $\vdash$ is said to be *protoalgebraic* [4, 8, 9, 10] if there exists a set of formulas $\Delta(x, y)$ such that $\emptyset \vdash \Delta(x, x)$ and $x, \Delta(x, y) \vdash y$. Nontrivial protoalgebraic logics are not graph-based and possess at least a theorem φ such that $Var(\varphi) \neq \emptyset$. This makes them amenable to the above theorem which, moreover, can be improved as follows:

Corollary 4. *A nontrivial protoalgebraic logic has an algebraic semantics if and only if it has a matrix semantics validating a nontrivial equation.*

In view of the above result, almost all reasonable protoalgebraic logics have an algebraic semantics. It is therefore natural to wonder whether they have also a *natural* algebraic semantics. There is, however, evidence against this conjecture, since, while the local consequence [13] of the normal modal logic **K** (resp. **K4** and **S4**) has an ad hoc algebraic semantics in view of the above corollary, it does not possess one based on the variety of modal algebras (resp. K4-algebras and interior algebras).

We conclude our journey among equational completeness theorems with some computational observations:

Theorem 5. *The following holds:*

(i) *The problem of determining whether logics presented by a finite set of finite matrices in a finite language have an algebraic semantics is decidable;*

(ii) *The problem of determining whether locally tabular logics presented by a finite set of finite rules in a finite language have an algebraic semantics is decidable;*

(iii) *The problem of determining whether logics presented by a finite set of finite rules in a finite language have an algebraic semantics is undecidable.*

The last item of the above result is established by means of a reduction to the classical halting problem for Turing machines [17]. This talk is based on the paper [14].

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Quantificational issues in Prawitzian validity

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While standard model-theoretic semantics explains meaning and validity in terms of truth-preservation under interpretations over - mainly set-theoretic - structures, constructive semantics focuses on provability-preservation over relevant deductive - usually atomic - systems.

Prawitz's semantics is a constructive setup where proofs are understood as valid arguments - see mainly [6] - or epistemic grounds - see mainly [8]. It stems from Prawitz's normalisation theorems in Gentzen's natural deduction [7], and in particular from what Schroeder-Heister [11] called the "fundamental corollary" of Prawitz's results: closed derivations in suitable systems - e.g. intuitionistic logic - reduce to *canonical* form, that is, to closed derivations ending by an introduction. This may confirm Gentzen's [3] claim that introductions define the meaning of the logical constants, whereas eliminations can be shown to be unique functions of the introductions via reductions such as those used by Prawitz for proving normalisation, e.g.

$$\dfrac{\dfrac{\begin{matrix}[\alpha]\\ \mathscr{D}_1\\ \beta\end{matrix}}{\alpha \to \beta}\,(\to_I) \qquad \begin{matrix}\mathscr{D}_2\\ \alpha\end{matrix}}{\beta}\,(\to_E) \qquad \overset{\phi^{\to}}{\Longrightarrow} \qquad \begin{matrix}\mathscr{D}_2\\ [\alpha]\\ \mathscr{D}_1\\ \beta\end{matrix}$$

or, in Curry-Howard equivalent λ-form,

$$\mathtt{App}(\lambda x.T(x), U) = T(U) \tag{1}$$

for $x, U : \alpha$ and $T : \beta$. In Prawitz's semantics, the "fundamental corollary" becomes Dummett's [2] *fundamental assumption*: if α is provable, then α is canonically provable. This implies moving from derivations to proofs, i.e. abstracting from specific systems, and allowing argument structures or linear forms to involve (types for) arbitrary inferences. So, reductions/equations act upon maximal formulas or redexes *latu sensu*, e.g. for disjunctive syllogism

$$\dfrac{\begin{matrix}\mathscr{D}_1\\ \alpha \vee \beta\end{matrix} \qquad \begin{matrix}\mathscr{D}_2\\ \neg\alpha\end{matrix}}{\beta}\,(DS) \qquad \overset{\phi^{DS}}{\Longrightarrow} \qquad \dfrac{\begin{matrix}\mathscr{D}_1\\ \alpha \vee \beta\end{matrix} \qquad \dfrac{\dfrac{[\alpha] \qquad \begin{matrix}\mathscr{D}_2\\ \neg\alpha\end{matrix}}{\bot}\,(\to_E)}{\beta}\,(\bot) \qquad [\beta]}{\beta}\,(\vee_E)$$

or, in a Curry-Howard alternative linear form,

$$DS(\mathtt{inj}_2^{\alpha}(T), U) = T \tag{2}$$

for $T : \beta$ and $U : \neg\alpha$. The interpretation of the non-logical signs is attained by introducing atomic systems Σ, i.e. sets of (production) rules

$$\frac{\alpha_1, ..., \alpha_n}{\beta}$$

where $\alpha_i \neq \bot$ and β are atomic ($i \leq n$) - plus other restrictions. Given an argument structure or linear form τ with (types for) arbitrary inferences, and a set $\mathscr{J}$ of reductions/equations for maximal formulas/redexes *latu sensu*, one then says that $\langle \tau, \mathscr{J} \rangle$ is, respectively, a valid argument or ground over Σ iff τ reduces through $\mathscr{J}$ to a canonical form whose immediate sub-structures or sub-terms are, respectively, valid arguments or grounds over Σ - possibly under closure of unbound variables and assumptions. An inference (rule) R can be said to be valid on Σ iff there is (a reduction/equation defining) a function ϕ such that, for every suitable valid argument or ground $\langle \tau, \mathscr{J} \rangle$ over Σ, $\langle \tau^*, \mathscr{J} \cup \{\phi\} \rangle$ is valid over Σ, where τ^* is the result of appending R to τ.

Prawitz's semantics has been often understood as a natural formal framework for Dummett's verificationist theory of meaning [2]. Dummett's anti-realist arguments seem to imply that intuitionistic logic - in short, IL - is the "correct" logic. Therefore, intuitionistic logic is expected to be complete with respect to Prawitz's semantics. This is what *Prawitz's conjecture* claims. But logical validity in Prawitz's framework can be defined in two ways, depending on the order in which we quantify over reductions/equations and atomic systems.

The first possibility is *system-rooted* validity - in short, PS-validity. R is PS-valid iff, for every atomic system Σ, there is (a reduction/equation defining) a function ϕ such that R is valid over Σ. The second possibility is *schematic* validity - in short, P-validity. R is P-valid iff there is (a reduction/equation defining) a function ϕ such that, for every atomic system Σ, R is valid over Σ. This can be extended to a more traditional relation between (sets of) formulas, i.e. $\Gamma \models_{\mathtt{PS/P}} \alpha$ iff there is a PS/P-valid inference from Γ to α. Given the easily provable correcteness result with respect to both notions, i.e. $\Gamma \vdash_{\mathtt{IL}} \alpha \Rightarrow \Gamma \models_{\mathtt{PS/P}} \alpha$, Prawitz's conjecture then claims that the inverse also holds, i.e. $\Gamma \models_{\mathtt{PS/P}} \alpha \Rightarrow \Gamma \vdash_{\mathtt{IL}} \alpha$. Of course, whether the conjecture is true or not may vary depending on whether we choose PS- or P-validity.

In fact, Piecha and Schroeder-Heister [5] have proved that, if by logical validity we mean PS, Prawitz's conjecture fails - Harrop's rule being a counterexample. The importance of Piecha and Schroeder-Heister's proof does not only rely upon *what* it shows, but also on the fact that - contrarily to previous approaches, see e.g. [1] - it abstracts from specific restrictions on atomic systems - e.g. from whether we allow bindings at the atomic level, or higher-level atomic rules, see e.g. [10, 9]. Piecha and Schroeder-Heister introduce a number of principles which are shown to be sufficent for framing various notions of consequence, and for classifying various necessary or sufficient conditions of constructive (in)completeness.

However, some of these principles seem to crucially fail over P-validity, e.g. a sort of semantic *admissibility principle* $\Gamma \models_{\mathtt{P}} \alpha \Leftrightarrow (\models_{\mathtt{P}} \Gamma \Rightarrow \models_{\mathtt{P}} \alpha)$, and a sort of semantic *disjunction property* $\Gamma \models_{\mathtt{P}} \alpha \vee \beta \Leftrightarrow (\Gamma \models_{\mathtt{P}} \alpha$ or $\Gamma \models_{\mathtt{P}} \beta)$ for $\vee, \exists$ not occurring in Γ. So, Piecha and Schroeder-Heister's proof may not apply to P-validity, and Prawitz's conjecture would remain open.

In my talk, I first of all aim at highlighting the (often overlooked) distinction between PS- and P-validity relative to Piecha and Schroeder-Heister's proof. Based on this primary goal, I then address two further issues, leading to two derived notions of prawitzian validity.

First, what I call *choice validity* - in short, C-validity. This simply amounts to the idea that we can "extract" P-validity from PS-validity by allowing a choice-function $\mathscr{F}$ in the class of our reductions/equations. Suppose $\Gamma \models_{\mathtt{PS}} \alpha$, then there is a PS-valid inference R from Γ to α, then for every atomic system Σ we can find (a reduction/equation defining) a function ϕ_Σ such that R is valid over Σ. Thus, given any atomic system Σ, and any valid arguments or grounds $\mathscr{D}_\Sigma$ over Σ for the elements of Γ, we may state

$$\begin{array}{c} \mathscr{D}_\Sigma \\ \dfrac{\Gamma}{\alpha}\, R \end{array} \quad \overset{\mathscr{F}}{\Longrightarrow} \quad \phi_\Sigma(\mathscr{D}^*_\Sigma)$$

where $\mathscr{D}^*_\Sigma$ is the result of appending R to $\mathscr{D}_\Sigma$. In other words, $\mathscr{F}$ "picks" the right function for R relative to the system where the input lives. Thus $\Gamma \models_{\texttt{PS}} \alpha \Rightarrow \Gamma \models_{\texttt{C}} \alpha$, and the claim would now be that, since we always use "one and the same" reduction/equation defining $\mathscr{F}$ to validate R over all atomic systems, this is "sufficiently schematic" for having $\Gamma \models_{\texttt{C}} \alpha \Rightarrow \Gamma \models_{\texttt{P}} \alpha$. This would mean that Prawitz's conjecture is refuted also over P-validity. But a crucial difference occurs between $\mathscr{F}$ and $\phi^{\rightarrow}$ or ϕ^{DS}. The latter are specified without referring to atomic systems and functions over these systems, whilst in the former these parameters must explicitly occur. So $\mathscr{F}$ is a function, not only of valid arguments or grounds, *but also of atomic systems.* This is much more evident if we move to a Curry-Howard linear form, say

$$\mathscr{F}(T, \Sigma) = \texttt{h}(\Sigma)(T) \tag{3}$$

for $T : \Gamma$ and $\texttt{h} : \texttt{S} \hookrightarrow \texttt{E}$, where S is the class of atomic systems and E is the class of functions for R over elements of S - so h is our choice function. Contrarily to equations (1) and (2), equation (3) checks where its proof-input comes from, thus involving an additional parameter Σ and a kind of "meta-function" from systems to functions for R. Its proof-output depends not only on the proof-input, but also on the domain of this proof-input.

Choice may be *in principle* acceptable, but two problems seem to affect our specific case here. First, the overall class of atomic systems may not be "sufficiently constructive" to allow for an acceptable usage of Choice. Secondly, when one thinks of schematic reductions/equations, one seemingly thinks of functions which only operate on proof-inputs, and which only generate proof-outputs, without additionally operating on the whole systems which these inputs and outputs belong to. If one accepts these objections, one should also reject the implication from C-validity to P-validity: additional restrictions should be put on schematic reductions/equations for them to respect what we mean by P-validity.

A truly fine-grained notion of P-validity requires specifying in greater detail what "schematic" means, i.e. defining more precisely the class of schematically acceptable reductions/equations. We know that a reduction/equation for R must be such that the function ϕ associated to R is linear over substitutions - i.e. $\phi(x[\star/\bullet]) = (\phi(x))[\star/\bullet]$ - and yields an output with the same type as, and no more variables and assumptions than the input. Given our previous discussion about C-validity, we know we must also have some limitations on the kind of inputs, whose range should be somehow bound to proof-objects, and exclude (functions on) structures where proof-objects live. But that said, it is anything but clear whether and when our restrictions-list can be said to be complete, nor is it clear how to formulate the restrictions in a rigorous way.

My proposal is that "schematic" is understood as "provably valid through logic and general proof-principles only", i.e.: that the reduction/equation for R is schematic means that we can *prove*, with *no other means* than logic and general proof-principles, that such reduction/equation defines a function ϕ which validates R. No linguistic component referring to atomic systems can occur in such a proof, and in particular no such linguistic component can occur in the reduction/equation defining ϕ. The proof goes through for every atomic system, without speaking of any system; hence, the same holds for the reduction/equation at issue.

For this proposal to make sense, we need systems where facts about inferences and reductions/equations can be proved. To this end, I build upon a class of systems for epistemic grounding that I introduced elsewhere [4], and define what I call *minimal grounding systems* - in short, MGS. An MGS relies upon a multi-sorted language where one can quantify over proofs, and contains:

- Gentzen's introductions and eliminations;
- generalised eliminations that fix an equational constraint for proofs of α to reduce to canonical proofs of α - namely, Dummett's fundamental assumption;

- syntactically schematic equations for eliminating redexes *latu sensu*, like (1) and (2) above.

This leads to the following definition of provable validity - in short, PR-validity. Let R be an inference (rule) from assumptions $\alpha_1, ..., \alpha_n$ to β. Then, R is PR-valid iff there is an equation ε defining a function ϕ for R such that, given an MGS containing ε we have

$$\vdash_{\texttt{MGS}} \forall x_1...x_n(x_1 : \alpha_1 \wedge ... \wedge x_n : \alpha_n \rightarrow \phi(x_1, ..., x_n) : \beta) \quad (4)$$

- where some apparent variables x_i may need to be replaced by functional variables $\texttt{h}_i$ taking as arguments variables y (for assumptions γ) bound by R on index i, in which case $x_i : \alpha_i$ in the antecedent is replaced by $\forall y_1 y_2(y_2 : \gamma \rightarrow \texttt{h}_i(y_1, y_2) : \alpha_i)$ $(i \leq n)$. This definition raises some issues, with which I conclude my talk:

- PR-validity requires limiting to linear forms, which may be unproblematic insofar as valid arguments and reductions can be Curry-Howard translated to grounds and equations. But, given we are reasoning in extended frameworks, how can we grant that we have a full equivalence between the two approaches?
- Which logic should we choose for MGS? Does this influence PR-completeness?
- If we look at the MGS-s as a class, we can seemingly order rules through an order-relation $\Re$, and set a rank for sequences of rules in $\Re$, say R is of degree 0 iff (4) is obtained in an MGS with no other equations than that for R, and it is of degree $i+1$ iff the MGS where (4) is proved is one containing equations for rules whose highest degree is i. This may in turn permit to reformulate Prawitz's conjecture, say IL is PR-complete iff, for each PR-valid R, there is an $\Re$-path ending with R and whose mininimal elements are rules of IL.

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Some Proof-theoretical aspects of non-associative, non-commutative multi-modal linear logic

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Abstract

Adding multi-modalities (called *subexponentials*) to linear logic enhances its power as a logical framework, which has been extensively used in the specification of *e.g.* proof systems and programming languages. Initially, subexponentials allowed for classical, linear, affine or relevant behaviors. Recently, this framework was enhanced so to allow for commutativity as well. In a work just accepted to IJCAR 2022, we have closed the cycle by considering associativity. In this proposal, we will show two undecidability results for fragments/variations of acLL_Σ in [5], and present a preliminary focused version for that system.

Introduction. Resource aware logics have been object of passionate study for quite some time now. The motivations for this passion vary: resource consciousness are adequate for modeling steps of computation; logics have interesting algebraic semantics; calculi have nice proof theoretic properties; multi-modalities allow for the specification of several behaviors; there are many interesting applications in linguistics, etc.

With this variety of subjects, applications and views, it is not surprising that different groups developed different systems based on different principles. For example, the Lambek calculus (L) [10] was introduced for mathematical modeling of natural language syntax, and it extends a basic categorial grammar [2, 4] by a concatenation operator. Linear logic (LL) [9], originally discovered by Girard from a semantical analysis of the models of polymorphic λ-calculus, turned out to be a refinement of classical and intuitionistic logic, having the dualities of the former and constructive properties of the latter. The key point is the presence of the *modalities* !, ?, called *exponentials* in LL. In the intuitionistic version of LL, denoted by ILL, only the ! exponential is present.

L and LL were compared in [1], when Abrusci showed that Lambek calculus coincides with a variant of the non-commutative, multiplicative version of ILL [11]. This correspondence can be lifted for considering also the additive connectives: Full (multiplicative-additive) Lambek calculus FL relates to non-commutative multiplicative-additive version of ILL, here denoted by cLL.

In the paper just accepted to IJCAR [5], we have proposed the sequent based system acLL_Σ, a conservative extension of cLL, where associativity is allowed only for formulas marked with a special kind of modality, determined by a *subexponential signature* Σ. The core fragment of acLL_Σ (*i.e.*, without the subexponentials) corresponds to the non-associative version of full Lambek calculus, FNL [6]. This extended abstract presents the two undecidability results of [5] and proposes a focused version for acLL_Σ.

*Speaker.

Non-associative, non-commutative multi-modal linear logic. The language of acLL_Σ consists of a denumerable infinite set of propositional variables $\{p, q, r, \ldots\}$, the unities $\{1, \top\}$, the binary connectives for additive conjunction and disjunction $\{\&, \oplus\}$, the non-commutative multiplicative conjunction $\otimes$, the non-commutative linear implications $\{\to, \leftarrow\}$, and the unary subexponentials $!^i$, with i belonging to a pre-ordered set of labels $(I, \preceq)$.

Roughly speaking, subexponentials [8] are substructural multi-modalities. In LL, $!A$ indicates that the linear formula A behaves *classically*, that is, it can be contracted *and* weakened. Labeling ! with indices allows moving one step further: The set I can be partitioned so that, in $!^i A$, A can be contracted *and/or* weakened. In this work, we consider not only weakening and contraction, but also commutativity and associativity, all such substructural properties determined by the axioms:

$$\mathsf{C}: \ !^iF \to !^iF \otimes !^iF \quad \mathsf{W}: \ !^iF \to 1 \quad \mathsf{E}: \ (!^iF) \otimes G \equiv G \otimes (!^iF)$$
$$\mathsf{A1}: \ !^iF \otimes (G \otimes H) \to (!^iF \otimes G) \otimes H \quad \mathsf{A2}: \ (G \otimes H) \otimes !^iF \to G \otimes (H \otimes !^iF)$$

The signature Σ of acLL_Σ contains $(I, \preceq)$ together with a function stating which of those axioms are assumed for each label. Pre-ordering the labels (together with an upward closeness requirement) guarantees cut-elimination [5]. Sequents have a *nested structure*, corresponding to trees of formulas, here called *structures*. And rules are applied deeply in such structures. Formally:

Definition 1 (Structured sequents)**.** Structures *are formulas or pairs containing structures:* $\Gamma, \Delta := F \mid (\Gamma, \Gamma)$*, where the constructors may be empty but never a singleton. The notation* $!^j\Gamma$ *will represent a structure where every formula* $F \in \Gamma$ *is such that* $F = !^j F'$.

An n-ary context $\Gamma\{^1\}\cdots\{^n\}$ *is a context that contains n pairwise distinct numbered* holes $\{\,\}$ *wherever a formula may otherwise occur. Given n contexts* $\Gamma_1, \ldots, \Gamma_n$*, we write* $\Gamma\{\Gamma_1\}\cdots\{\Gamma_n\}$ *for the context where the k-th hole in* $\Gamma\{^1\}\cdots\{^n\}$ *has been replaced by* Γ_k *(for* $1 \leq k \leq n$*). If* $\Gamma_k = \emptyset$ *the hole is removed. A* structured sequent *(or simply* sequent*) has the form* $\Gamma \Rightarrow F$ *where* Γ *is a structure and* F *is a formula.*

Definition 2 (SDML)**.** *Let* $\mathcal{A}$ *be a set of a xioms. A (non-associative/commutative)* simply dependent multimodal logical system *(*SDML*) is given by a triple* $\Sigma = (I, \preccurlyeq, f)$*, where I is a set of indices,* $(I, \preccurlyeq)$ *is a pre-order, and f is a mapping from I to* $2^{\mathcal{A}}$.
If Σ *is a* SDML*, then the* logic described by Σ *has the modality* $!^i$ *for every* $i \in I$*, with the rules of* FNL *depicted in Fig. 1, together with rules for the axioms* $f(i)$ *and the interaction axioms* $!^jA \to !^iA$ *for every* $i, j \in I$ *with* $i \preccurlyeq j$*. Finally, every* SDML *is assumed to be upwardly closed w.r.t.* $\preceq$*, that is, if* $i \preceq j$ *then* $f(i) \subseteq f(j)$ *for all* $i, j \in I$.

Fig. 2 presents the structured system acLL_Σ, for the logic described by the SDML determined by Σ, with $\mathcal{A} = \{\mathsf{C}, \mathsf{W}, \mathsf{A1}, \mathsf{A2}, \mathsf{E}\}$ where, in the subexponential rule for $\mathsf{S} \in \mathcal{A}$, the respective $s \in I$ is such that $\mathsf{S} \in f(s)$ (*e.g.* the subexponential symbol e indicates that $\mathsf{E} \in f(e)$). As usual, $\Gamma^{\preceq i}$ represents the context with the tree structure inherited by Γ, with all the subexponentials greater or equal to i.

(Un)decidability results. Non-associativity makes a significant difference in decidability and complexity matters. For our system acLL_Σ, its decidability or undecidability depends on its signature Σ. If for every $i \in I$, $\mathsf{C} \notin f(s)$, then acLL_Σ is clearly decidable, since the cut-free proof search space is finite. Therefore, for undecidability it is necessary to have at least one subexponential which allows contraction.

For FNL with only one fully-powered exponential modality s, undecidability was proven in a preprint by Tanaka [12]. In [5], we have refined Tanaka's result by showing that acLL_Σ containing the multiplicatives $\otimes, \to$, the additive $\oplus$ and one *classical* subexponential is undecidable.

Theorem 1. *If there exists such* $s \in I$ *that* $f(s) \supseteq \{\mathsf{C}, \mathsf{W}\}$*, then the derivability problem in* acLL_Σ *is undecidable. Moreover, this holds for the fragment with only* $\otimes$, $\to$, $\oplus$, $!^s$.

In the second undecidability result, we keep two subexponentials, but with a minimalist configuration: the implicational fragment of the logic plus two subexponentials: the "main" one allowing for contraction,

$$\frac{\Gamma\{(F,G)\} \Rightarrow H}{\Gamma\{F \otimes G\} \Rightarrow H}\ \otimes L \qquad \frac{\Gamma_1 \Rightarrow F \quad \Gamma_2 \Rightarrow G}{(\Gamma_1,\Gamma_2) \Rightarrow F \otimes G}\ \otimes R \qquad \frac{\Gamma\{F\} \Rightarrow H \quad \Gamma\{G\} \Rightarrow H}{\Gamma\{F \oplus G\} \Rightarrow H}\ \oplus L$$

$$\frac{\Gamma \Rightarrow F_i}{\Gamma \Rightarrow F_1 \oplus F_2}\ \oplus R_i \qquad \frac{\Gamma\{F_i\} \Rightarrow G}{\Gamma\{F_1 \,\&\, F_2\} \Rightarrow G}\ \& L_i \qquad \frac{\Gamma \Rightarrow F \quad \Gamma \Rightarrow G}{\Gamma \Rightarrow F \,\&\, G}\ \& R$$

$$\frac{\Delta \Rightarrow F \quad \Gamma\{G\} \Rightarrow H}{\Gamma\{(\Delta, F \to G)\} \Rightarrow H} \to L \qquad \frac{(F,\Gamma) \Rightarrow G}{\Gamma \Rightarrow F \to G} \to R \qquad \frac{\Delta \Rightarrow F \quad \Gamma\{G\} \Rightarrow H}{\Gamma\{(G \leftarrow F, \Delta)\} \Rightarrow H}\quad L$$

$$\frac{(\Gamma, F) \Rightarrow G}{\Gamma \Rightarrow G \leftarrow F}\quad R \qquad \frac{\Gamma\{\,\} \Rightarrow F}{\Gamma\{1\} \Rightarrow F}\ 1L \qquad \frac{}{\Rightarrow 1}\ 1R \qquad \frac{}{\Gamma \Rightarrow \top}\ \top R$$

$$\frac{}{F \Rightarrow F}\ \mathsf{init} \qquad \frac{\Delta \Rightarrow F \quad \Gamma\left\{{}^{1}F\right\}\ldots\left\{{}^{n}F\right\} \Rightarrow G}{\Gamma\left\{{}^{1}\Delta\right\}\ldots\left\{{}^{n}\Delta\right\} \Rightarrow G}\ \mathsf{mcut}$$

Figure 1: Structured system FNL for non-associative, full Lambek calculus.

$$\frac{\Gamma^{\preceq i} \Rightarrow F}{\Gamma \Rightarrow\, !^i F}\ !^i R \qquad \frac{\Gamma\{F\} \Rightarrow G}{\Gamma\left\{!^i F\right\} \Rightarrow G}\ \mathsf{der}$$

$$\frac{\Gamma\{((!^a\Delta_1, \Delta_2), \Delta_3)\} \Rightarrow G}{\Gamma\{(!^a\Delta_1, (\Delta_2, \Delta_3))\} \Rightarrow G}\ \mathsf{A1} \qquad \frac{\Gamma\{(\Delta_1, (\Delta_2, !^a\Delta_3))\} \Rightarrow G}{\Gamma\{((\Delta_1, \Delta_2), !^a\Delta_3)\} \Rightarrow G}\ \mathsf{A2} \qquad \frac{\Gamma\{(\Delta_2, !^e\Delta_1)\} \Rightarrow G}{\Gamma\{(!^e\Delta_1, \Delta_2)\} \Rightarrow G}\ \mathsf{E1}$$

$$\frac{\Gamma\{(!^e\Delta_2, \Delta_1)\} \Rightarrow G}{\Gamma\{(\Delta_1, !^e\Delta_2)\} \Rightarrow G}\ \mathsf{E2} \qquad \frac{\Gamma\{\,\} \Rightarrow G}{\Gamma\{!^w\Delta\} \Rightarrow G}\ \mathsf{W} \qquad \frac{\Gamma\left\{{}^{1}!^c\Delta\right\}\ldots\left\{{}^{n}!^c\Delta\right\} \Rightarrow G}{\Gamma\left\{{}^{1}\right\}\ldots\left\{{}^{k}!^c\Delta\right\}\ldots\left\{{}^{n}\right\} \Rightarrow G}\ \mathsf{C}$$

Figure 2: Structured system acLL_Σ for the logic described by Σ.

exchange, and associativity (weakening is optional), and an "auxiliary" one allowing only associativity. This is a variation of Chaudhuri's result [7] (in the non-associative, non-commutative case), making use of fewer connectives (tensor is not needed) and less powerful subexponentials.

Theorem 2. *If there are $a, c \in I$ such that $f(a) = \{\mathsf{A1}, \mathsf{A2}\}$ and $f(c) \supseteq \{\mathsf{C}, \mathsf{E}, \mathsf{A1}, \mathsf{A2}\}$, then the derivability problem in acLL_Σ is undecidable. Moreover, this holds for the fragment with only $\to$, $!^a$, $!^c$.*

Focusing. The focusing discipline [3] is determined by the alternation of *focused* and *unfocused* phases in the proof construction. In the unfocused phase, inference rules can be applied eagerly and no backtracking is necessary; in the focused phase, on the other hand, either context restrictions apply, or choices within inference rules can lead to failures for which one may need to backtrack. These phases are totally determined by the polarities of formulas: provability is preserved when applying right/left rules for negative/positive formulas respectively, but not necessarily in other cases.

The importance of focusing is due to the fact that it gives a notion of *normal forms* for proofs. In the case of acLL_Σ, the following polarization is proposed.

Definition 3 (Polarized Syntax)**.** *Let $\mathcal{P}$ be the set propositional variables and $\mathcal{P}^+ \cap \mathcal{P}^-$ a partition of $\mathcal{P}$, with $A^+ \in \mathcal{P}^+$ and $A^- \in \mathcal{P}^-$. The* polarized formulas *are given by the following grammar*

$$\begin{array}{lcl@{\qquad}lcl} P, Q & := & A^+ \mid 1 \mid F \otimes F \mid F \oplus F \mid F \to F \mid F \leftarrow F \mid !^i F & N, M & := & A^- \mid \top \mid F \,\&\, F \\ L & := & A^+ \mid N & R & := & A^- \mid P \end{array}$$

A negative structure*, denoted by* Λ*, is given by* $\Lambda := L \mid (\Lambda, \Lambda)$*. A* polarized structured sequent *has one of the forms:* $\Gamma \Rightarrow F \quad \Lambda\{\langle F \rangle\} \Rightarrow R \quad \Lambda \Rightarrow \langle F \rangle$ *where the first is an* unfocused sequent *and the last two are* focused*, with* $\langle F \rangle$ *indicating that the formula F is under focus.*

The proposed focused system facLL_Σ is depicted in Figure 3, where the structural rules are restricted to neutral formulas only. Our ongoing work is to show that facLL_Σ is sound and complete w.r.t. acLL_Σ. We plan to apply the result in the analysis of natural language syntax.

$$\frac{\Lambda_1 \Rightarrow \langle F \rangle \quad \Lambda_2 \Rightarrow \langle G \rangle}{(\Lambda_1, \Lambda_2) \Rightarrow \langle F \otimes G \rangle} \otimes R \qquad \frac{\Lambda \Rightarrow \langle F_i \rangle}{\Lambda \Rightarrow \langle F_1 \oplus F_2 \rangle} \oplus R_i \qquad \frac{\Lambda\{\langle F_i \rangle\} \Rightarrow R}{\Lambda\{\langle F_1 \,\&\, F_2 \rangle\} \Rightarrow R} \,\&L_i$$

$$\frac{\Lambda' \Rightarrow \langle F \rangle \quad \Lambda\{\langle G \rangle\} \Rightarrow R}{\Lambda\{\langle (\Lambda', F \rightarrow G) \rangle\} \Rightarrow R} \rightarrow L \qquad \frac{\Lambda' \Rightarrow \langle F \rangle \quad \Lambda\{\langle G \rangle\} \Rightarrow R}{\Lambda\{\langle (G \quad F, \Lambda') \rangle\} \Rightarrow R} \quad L$$

$$\frac{\Lambda^{\leq i} \Rightarrow F}{\Lambda \Rightarrow \left\langle !^i F \right\rangle} \, !^i R \qquad \frac{}{\Rightarrow \langle 1 \rangle} \, 1R \qquad \frac{}{P \Rightarrow \langle P \rangle} \, \mathsf{init}+ \qquad \frac{}{\langle N \rangle \Rightarrow N} \, \mathsf{init}-$$

$$\frac{\Gamma\{(F, G)\} \Rightarrow H}{\Gamma\{F \otimes G\} \Rightarrow H} \otimes L \qquad \frac{\Gamma\{F\} \Rightarrow H \quad \Gamma\{G\} \Rightarrow H}{\Gamma\{F \oplus G\} \Rightarrow H} \oplus L \qquad \frac{\Gamma \Rightarrow F \quad \Gamma \Rightarrow G}{\Gamma \Rightarrow F \,\&\, G} \,\&R$$

$$\frac{(F, \Gamma) \Rightarrow G}{\Gamma \Rightarrow F \rightarrow G} \rightarrow R \qquad \frac{(\Gamma, F) \Rightarrow G}{\Gamma \Rightarrow G \leftarrow F} \quad R \qquad \frac{\Gamma\{\,\} \Rightarrow F}{\Gamma\{1\} \Rightarrow F} \, 1L \qquad \frac{}{\Gamma \Rightarrow \top} \, \top R$$

$$\frac{\Lambda\{\langle N \rangle\} \Rightarrow R}{\Lambda\{N\} \Rightarrow R} \, \mathsf{D}L \qquad \frac{\Lambda \Rightarrow \langle P \rangle}{\Lambda \Rightarrow P} \, \mathsf{D}R \qquad \frac{\Lambda\{\langle F \rangle\} \Rightarrow R}{\Lambda\left\{!^i F\right\} \Rightarrow R} \, \mathsf{der} \qquad \frac{\Lambda\{P\} \Rightarrow R}{\Lambda\{\langle P \rangle\} \Rightarrow R} \, \mathsf{R}L \qquad \frac{\Lambda \Rightarrow N}{\Lambda \Rightarrow \langle N \rangle} \, \mathsf{R}R$$

Figure 3: Structured system facLL_Σ for focused acLL_Σ.

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Logics of upsets of De Morgan lattices

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Even a very cursory review of the existing literature on non-classical logics will quickly reveal two facts. Firstly, many of the non-classical logics which have attracted the most attention among the community of algebraic logicians have a conjunction which is interpreted by a binary meet operation in some algebra with a distributive lattice reduct. Secondly, logics with such a lattice conjunction are almost inevitably assumed to satisfy the rule of *adjunction*:

$$x, y \vdash x \wedge y.$$

This rule, together with the rules $x \wedge y \vdash x$ and $x \wedge y \vdash y$, ensures that the designated sets of these logics form lattice filters in some appropriate class of distributive lattice-ordered algebras.

In this contribution, we develop tools which will enable us to study logics with a distributive lattice conjunction where the rule of adjunction fails. In other words, we will be concerned with *logics of upsets*, rather than logics of lattice filters.

As a case study, we shall consider logics determined by a class of matrices of the form $\langle \mathbf{A}, F \rangle$ where $\mathbf{A}$ is a De Morgan lattice and F is an upset of $\mathbf{A}$. However, the results stated below are much more general. The only feature of De Morgan lattices which we use is that they are generated as a quasivariety by a finite algebra, namely the four-element subdirectly-irreducible De Morgan lattice $\mathbf{DM}_1$, and that each prime filter on De Morgan lattice is a homomorphic preimage of a certain prime filter Q_1 on $\mathbf{DM}_1$, namely the filter $\{\mathsf{t}, \mathsf{b}\}$.

Logics of filters of De Morgan lattices have in fact recently been studied under the name *super-Belnap logics* [4, 1, 3]. The results presented below can be interpreted as extending the super-Belnap universe to cover natural logics such as Shramko's logic of "anything but falsehood" [5] which do not validate the rule of adjunction but which fit in well with the rest of the super-Belnap family in terms of their motivation. Indeed, extending the notion of a super-Belnap logic to cover such logics was first proposed by Shramko [6].

Our main results are the following two finite basis theorems. Their proofs are constructive: we provide an algorithm which finds the required axiomatizations. The second theorem yields finite Gentzen-style calculi even for logics which have no finite Hilbert-style calculus, such as the extension of Belnap–Dunn logic by the infinite set of rules $(x_1 \wedge \neg x_1) \vee \dots \vee (x_n \wedge \neg x_n) \vdash y$, which is complete with respect to an eight-element matrix.

Theorem 1. *Each logic determined by a finite set of finite matrices of the form $\langle \mathbf{A}, F \rangle$, where $\mathbf{A}$ is a De Morgan lattice and F is a prime upset of $\mathbf{A}$, has a finite Hilbert-style axiomatization.*

Theorem 2. *Each logic determined by a finite set of finite matrices of the form $\langle \mathbf{A}, F \rangle$, where $\mathbf{A}$ is a De Morgan lattice and F is a lattice filter of $\mathbf{A}$, has a finite Gentzen-style axiomatization.*

The key tool in proving these theorems will be the notion of an n-filter. The theorems will follow easily once we extend basic facts about filters on distributive lattices to n-filters.

An upset F of a distributive lattice $\mathbf{A}$ will be called an *n-filter*, for $n \geq 1$, if for each non-empty finite $X \subseteq \mathbf{A}$

$$\bigwedge Y \in F \text{ for each } Y \subseteq_n X \implies \bigwedge X \in F,$$

where we use the notation

$$X \subseteq_n Y \iff X \subseteq Y \text{ and } 1 \leq |X| \leq n.$$

We may restrict without loss of generality to $|X| = n + 1$ and $|Y| = n$ in this definition. Equivalently, F is an n-filter if the matrix $\langle \mathbf{A}, F \rangle$ validates the rule of *n-adjunction*:

$$\{\bigwedge_{j \neq i} x_j \mid 1 \leq i \leq n + 1\} \vdash x_1 \wedge \cdots \wedge x_{n+1},$$

where $\bigwedge_{j \neq i} x_j$ denotes the submeet of $x_1 \wedge \cdots \wedge x_{n+1}$ obtained by omitting x_i. For example, 1-adjunction is the ordinary rule of adjunction, while 2-adjunction is the rule

$$x \wedge y, y \wedge z, z \wedge x \vdash x \wedge y \wedge z.$$

Of course, each m-filter is an n-filter for $m \leq n$.

Because n-filters are closed under arbitrary intersection, we may talk about the n-filter $[U]_n$ *generated* by a subset U of $\mathbf{A}$. While understanding filter generation in arbitrary lattices is easy, we only have a good description of n-filter generation for $n > 1$ in distributive lattices.

Lemma 3. *Let U be an upset of a distributive lattice $\mathbf{A}$. Then $a \in [U]_n$ if and only if there is a non-empty finite set $X \subseteq \mathbf{A}$ such that $\bigwedge Y \in U$ for each $Y \subseteq_n X$ and $\bigwedge X \leq a$.*

Understanding how n-filters are generated allows us to prove the following theorem.

Theorem 4. *Each n-filter on a distributive lattice is an intersection of prime n-filters.*

An easy way of constructing n-filters is to take the union of a family of at most n filters. This does not suffice to construct all n-filters, but it does suffice to construct all prime n-filters. Here an upset U is called *prime* if $a \vee b \in U$ implies that either $a \in U$ or $b \in U$.

Theorem 5. *Each prime n-filter on a distributive lattice is a union of at most n prime filters.*

It remains to describe unions of at most n prime filters as the homomorphic preimages of a certain fixed upset. To this end, the *dual product* construction is useful. Given a family of matrices $\langle \mathbf{A}_i, F_i \rangle$ for $i \in I$, its dual product $\bigotimes_{i \in I} \langle \mathbf{A}_i, F_i \rangle$ is the matrix $\langle \mathbf{A}, F \rangle$ with $\mathbf{A} := \prod_{i \in I} \mathbf{A}_i$ and $F := \bigcup_{i \in I} \pi_i^{-1}[F_i]$, where $\pi \colon \mathbf{A} \to \mathbf{A}_i$ are the projection maps. In other words, a tuple $a \in \mathbf{A}$ is designated in the dual product if and only if some component $a_i \in \mathbf{A}_i$ of this tuple is designated in $\langle \mathbf{A}_i, F_i \rangle$. Let $\langle \mathbf{B}_n, P_n \rangle$ be the n-th dual power of the matrix $\langle \mathbf{B}_1, P_1 \rangle$. That is, $a \in P_n$ if and only if $a > \mathsf{f}$ in $\mathbf{B}_n$, where f denotes the bottom element of $\mathbf{B}_n$.

Lemma 6. *An upset U of a distributive lattice is a union of at most n prime filters if and only if it is a homomorphic preimage of the upset P_n of $\mathbf{B}_n$.*

Summing up: n-filters on distributive lattices are defined syntactically as upsets which satisfy the rule of n-adjunction, but they an also be characterized semantically as the intersections of homomorphic preimages of the prime n-filter $P_n \subseteq \mathbf{B}_n$.

This allows us to describe all logics of upsets of distributive lattices, i.e. logics determined by some class of matrices of the form $\langle \mathbf{A}, F \rangle$ where $\mathbf{A}$ is a distributive lattice and F is an upset of $\mathbf{A}$. These are precisely the extensions of the logic $\mathcal{DL}_\infty$ of all upsets of distributive lattices. Let $\mathcal{DL}_n$ be the extension of $\mathcal{DL}_\infty$ by the rule of n-adjunction, or equivalently let $\mathcal{DL}_n$ be the logic of all n-filters of distributive lattices. It will be convenient to take $\mathbf{B}_0$ to be the trivial lattice, 0-adjunction to be the rule $x \vdash y$, and P_0 to be the empty set.

Theorem 7. *The logic $\mathcal{DL}_n$ is complete with respect to the matrix $\langle \mathbf{B}_n, P_n \rangle$. Moreover, the logics $\mathcal{DL}_n$ for $n \in \omega$ are the only non-trivial proper extensions of $\mathcal{DL}_\infty$.*

Moving to the setting of De Morgan lattices, much of the above argument remains valid if we replace the prime filter P_1 on $\mathbf{B}_1$ by a prime filter Q_1 on $\mathbf{DM}_1$. (This filter consists of the top element and one of the fixpoints of negation.) We again define the matrix $\langle \mathbf{DM}_n, Q_n \rangle$ to be the n-th dual power of the matrix $\langle \mathbf{DM}_1, Q_1 \rangle$ and obtain the following completeness theorems for the logics $\mathcal{BD}_n$ of n-filters of De Morgan lattices, which extend the logic $\mathcal{BD}_\infty$ of all upsets of De Morgan lattices by the rule of n-adjunction.

Theorem 8. *The logic $\mathbf{DM}_n$ is complete with respect to the matrix $\langle \mathbf{DM}_n, Q_n \rangle$.*

The problem of axiomatizing the logic given by a finite set of prime upsets of De Morgan lattices reduces to the problem of axiomatizing the logic $\mathcal{L}$ given by a set S of submatrices of the finite matrix $\langle \mathbf{DM}_n, Q_n \rangle$ for some n: each upset of a finite De Morgan lattice is in fact an n-filter for some n, and if it is moreover prime, then it is a homomorphic image of Q_n. Furthermore, for each submatrix $\langle \mathbf{A}, F \rangle$ of $\langle \mathbf{DM}_n, Q_n \rangle$ there is either a finitary semantic construction of $\langle \mathbf{A}, F \rangle$ in terms of matrices from S witnessing that it is a model of $\mathcal{L}$ or a finitary rule which fails in $\langle \mathbf{A}, F \rangle$ but holds in $\mathcal{L}$. This yields a finite set of finitary rules R such that $\mathcal{L}$ is the smallest extension of $\mathcal{BD}_n$ which validates each rule in R and which is complete with respect to a class of prime upsets. This is equivalent to the claim that $\mathcal{L}$ is axiomatized relative to $\mathcal{BD}_n$ by what we call the *disjunctive variants* of the rules in R. This yields a finite Hilbert-style axiomatization for each logic determined by a finite set of prime upsets of De Morgan lattices.

As a concrete application of the algorithm sketched above, we obtain an axiomatization of the logic "anything but falsehood" introduced recently by Shramko [5] as the semantic dual to the logic of "nothing but the truth" introduced by Pietz and Rivieccio [2]. This is the logic determined by the matrix $\langle \mathbf{DM}_1, \{\mathsf{t}, \mathsf{n}, \mathsf{b}\} \rangle$, where n and b are the two fixpoints of negation in $\mathbf{DM}_1$ and t is the top element. The last rule in the axiomatization below is what we call the disjunctive variant of the rule $x, \neg x \vdash x \wedge \neg x$.

Theorem 9. *The logic of the structure $\langle \mathbf{DM}_1, \{\mathsf{t}, \mathsf{n}, \mathsf{b}\} \rangle$ is the extension of $\mathcal{BD}_\infty$ by the 2-adjunction rule, the law of the excluded middle $\emptyset \vdash x \vee \neg x$, and the rule $x \vee y, \neg x \vee y \vdash (x \wedge \neg x) \vee y$.*

To obtain the following theorem, it now suffices to observe that a finitary extension $\mathcal{L}$ of $\mathcal{BD}_\infty$ is complete with respect to some class of matrices of the form $\langle \mathbf{A}, F \rangle$ where F is a prime upset if and only if it satisfies the *proof by cases property (PCP)*:

$$\Gamma, \varphi_1 \vee \varphi_2 \vdash_{\mathcal{L}} \psi \iff \Gamma, \varphi_1 \vdash_{\mathcal{L}} \psi \text{ and } \Gamma, \varphi_2 \vdash_{\mathcal{L}} \psi.$$

Theorem 10. *The following are equivalent for each extension $\mathcal{L}$ of $\mathcal{BD}_\infty$:*

(i) $\mathcal{L}$ is a finitary extension of $\mathcal{BD}_n$ with the PCP,

(ii) $\mathcal{L}$ is complete with respect to some set of substructures of $\langle \mathbf{DM}_n, Q_n \rangle$,

(iii) $\mathcal{L}$ is complete with respect to some finite set of finite structures of the form $\langle \mathbf{L}, F \rangle$ where $\mathbf{L}$ is a De Morgan lattice and F is a prime n-filter of $\mathbf{L}$.

Some such n exists whenever $\mathcal{L}$ has the PCP and is complete w.r.t. a finite set of finite matrices.

The case of logics determined by a finite set of filters (rather than prime upsets) of De Morgan lattices admits an analogous analysis, but we need to consider n-prime filters (rather than prime

n-filters). A filter F on a distributive lattice $\mathbf{A}$ wil be called *n-prime* if it is a meet n-prime element of the lattice of all filters on $\mathbf{A}$, i.e. if for each non-empty finite family of filters $\mathcal{F}$ on $\mathbf{A}$

$$\bigcap \mathcal{F} \subseteq F \implies \bigcap \mathcal{G} \subseteq F \text{ for some } \mathcal{G} \subseteq_n \mathcal{F}.$$

Equivalently, n-prime filters are precisely the complements of prime n-ideals.

A finitary extension $\mathcal{L}$ of $\mathcal{BD}_1$ is complete with respect to a class of n-prime filters if and only if it satisfies what we call the *n-proof by cases property (n-PCP)*:

$$\Gamma, \bigvee_{j \neq 1} \varphi_j \vdash_{\mathcal{L}} \psi \text{ and } \dots \text{ and } \Gamma, \bigvee_{j \neq n+1} \varphi_j \vdash_{\mathcal{L}} \psi \implies \Gamma, \varphi_1 \vee \dots \vee \varphi_{n+1} \vdash_{\mathcal{L}} \psi.$$

In particular, the 2-PCP states the following:

$$\Gamma, \varphi_1 \vee \varphi_2 \vdash_{\mathcal{L}} \psi \text{ and } \Gamma, \varphi_2 \vee \varphi_3 \vdash_{\mathcal{L}} \psi \text{ and } \Gamma, \varphi_3 \vee \varphi_1 \vdash_{\mathcal{L}} \psi \implies \Gamma, \varphi_1 \vee \varphi_2 \vee \varphi_3 \vdash_{\mathcal{L}} \psi.$$

We now obtain the following theorem in a manner entirely analogous to the previous one.

Theorem 11. *The following are equivalent for each extension $\mathcal{L}$ of $\mathcal{BD}_1$:*

(i) $\mathcal{L}$ is a finitary and enjoys the n-PCP,

(ii) $\mathcal{L}$ is complete with respect to some set of substructures of $(\mathbb{DM}_1)^n$,

(iii) $\mathcal{L}$ is complete with respect to some finite set of finite structures of the form $\langle \mathbf{L}, F \rangle$ where $\mathbf{L}$ is a De Morgan lattice and F is an n-prime upset of $\mathbf{L}$.

Some such n exists whenever $\mathcal{L}$ is complete w.r.t. a finite set of finite matrices.

In this case, $\mathcal{L}$ is the smallest logic satisfying the n-PCP and a certain finite set of finitary rules R. This description of $\mathcal{L}$ cannot, in general, be transformed into a finite Hilbert-style axiomatization of $\mathcal{L}$: some logics determined by a filter on a finite De Morgan lattice do not admit any finite Hilbert-style axiomatization. We do, however, obtain a finite Gentzen-style axiomatization of $\mathcal{L}$, the key Gentzen-style rule being the n-PCP.

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Some Completeness Results in Derivational Modal Logic

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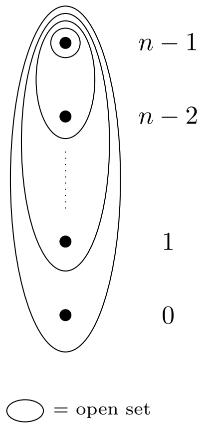
While modal logic is often associated with Saul Kripke's relational semantics, it also enjoys a topological interpretation which is becoming increasingly influential. This semantics can be traced back to the work of McKinsey and Tarski [MT44], who proposed to interpret $\Diamond$ as the topological closure operator, hence introducing the *closure semantics*, or *c-semantics* – recall that the closure $\mathsf{Cl}(A)$ of A is the set of all points x such that every open neighbourhood of x intersects A. They then proved the celebrated result that **S4** is the logic of any separable metric dense-in-itself space. This was subsequently strengthened by Rasiowa and Sikorski [RS63] who eliminated the separability condition – for a good survey of these results see [vBB07]. Since open sets can naturally be interpreted as *pieces of observation* [Vic96], this approach has recently gained momentum in fields such as formal epistemology [BBÖS19] [Özg17] and learning theory [dBY10]. A less known close kin of the c-semantics is the *derivational semantics*, or *d-semantics*. It is obtained by interpreting $\Diamond$ not as the closure, but as the *derivative* operator attributed to Georg Cantor and defined by $\mathsf{d}(A) := \{x \mid x \in \mathsf{Cl}(A \setminus \{x\})\}$. This variant was also introduced by McKinsey and Tarski, and further investigated by Esakia and others – see e.g., [Esa81][Esa01]. It is more expressive than the c-semantics, in the sense that any modally expressible property with respect to the c-semantics, is also modally expressible with respect to the d-semantics. The d-semantics semantics thus enables a more refined classification of spaces. Further, while the logic of the c-semantics is **S4**, the logic of the d-semantics is $\mathbf{wK4} := \mathbf{K} + \Diamond\Diamond p \to p \vee \Diamond p$, as proved by Esakia [Esa01]. Since **wK4** is weaker than **S4**, it has more extensions, and thus more logics that can be studied with regard to the d-semantics.

In spite of these compelling features, the d-semantics has received much less attention than the c-semantics, and our knowledge of it is largely incomplete: the interpretation of many standard logics is missing, and so are proofs of their completeness. One example is the axiom bd_n (for any natural integer n) which characterizes the Kripke frames that contain no path of length greater than n, that is, those with *depth bounded by n*. The topological interpretation of the concept of depth, however, is not obvious. Bezhanishvili et al. [BBLBvM17] resolved this question in the c-semantics, by introducing for any space X a number called the *modal Krull dimension* $\mathsf{mdim}(X)$ of X. The modal Krull dimension of X is initially defined as the size of a maximal stack of nested non-empty nowhere dense subspaces of X, but a more intuitive definition can be obtained with interior maps. An *interior map* from a space X to a space Y is a function $f : X \to Y$ satisfying $f[\mathsf{Cl}(A)] = \mathsf{Cl}(f[A])$ for all $A \subseteq X$. Then, a path of reflexive points of length n – called the *reflexive n-chain* – can be seen as the topological space C_n depicted below:



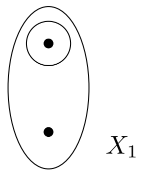
The modal Krull dimension of X is then the greatest n such that there is no surjective interior map from X to C_n. In words, $\mathsf{mdim}(X)$ is the greatest n such that X does not "contain" the reflexive $n+2$-chain. Such a formulation is already much closer to the initial graph-theoretic notion of depth. In [BBLBvM17] it is proven that $X \vDash \mathsf{bd}_n$ if and only if $\mathsf{mdim}(X) \leq n-1$. This yields a notion of depth for spaces, and this kind of parameter is of great interest when it comes to classification: spaces with finite depth are generally easier to deal with, and their logics tend to have good properties – like the finite model property. In the d-semantics, the interpretation of bd_n used to be unknown but presumably corresponds to some "derivative modal dimension" yet to be defined. We show that the appropriate measure for this purpose is, in fact, the modal Krull dimension itself: the equivalence $X \vDash \mathsf{bd}_n \iff \mathsf{mdim}(X) \leq n-1$ holds in the d-semantics too. This means that the c-semantics and the d-semantics of bd_n coincide. We also provide an alternative definition of mdim in terms of *d-morphisms*, which constitute the relevant notion of morphisms for the d-semantics, just as interior maps are morphisms for the c-semantics. We show that $\mathsf{mdim}(X) \leq n$ if and only if there is no surjective d-morphism from some subspace of X to *some* $n+2$-chain – not only the reflexive one. Arguably, this characterization is better suited to the framework of the d-semantics. These results show the relevance of topological depth for the derivational framework, and will hopefully lead to the apparition of this parameter in future classifications.

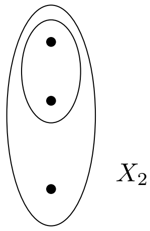
We then turn our attention to the well known axioms

$$\begin{array}{rrcl} & .2 & := & \Diamond\Box p \to \Box\Diamond p \\ \text{and} & .3 & := & \Box(\Box p \to q) \vee \Box(\Box q \to p) \end{array}$$

which have already been studied in the c-semantics [vBB07], but whose interpretation in the d-semantics had remained unexplored so far[1]. We recall that in the c-semantics, the axiom .2 defines the class of *extremally disconnected spaces*, i.e., those wherein the closure of any open set is also open. In the d-semantics, this is more complicated since the following two spaces are extremally disconnected but falsify .2:

[1]Note that .2 and .3 are usually defined as respectively $\Diamond(p \wedge \Box q) \to \Box(p \vee \Diamond q)$ and $\Box((p \wedge \Box p) \to q) \vee \Box((q \wedge \Box q) \to p)$ [BRV01, CZ97], so our definitions are somewhat unorthodox. Obviously in each case the two variants coincide over **S4**, but not in general. For the name of $\Box(\Box p \to q) \vee \Box(\Box q \to p)$ we occasionally find sc [CZ97] and D1 [CH12]. For some reason the "standard" definitions of .2 and .3 present little interest in the d-semantics, so this motivates the choice of our alternative formulations.





In fact, .2 turns out to define the class of extremally disconnected spaces that do not contain any pattern of the form X_1 or X_2. To be precise: $X \vDash .2$ if and only if X is a topological sum of the form

$$X = Y \cup \bigcup_{i \in I} \mathbf{1}_i \cup \bigcup_{i \in J} \mathbf{2}_i$$

where each $\mathbf{1}_i$ is a one-element space, each $\mathbf{2}_i$ is a two-element space with the coarsest topology, and Y is extremally disconnected and *strongly dense-in-itself*, i.e., every non-empty open set of Y contains at least three elements. The case of .3 is also very interesting. In the c-semantics, we know that $X \vDash .3$ if and only X is *hereditarily extremally disconnected*, i.e., every subspace of X is extremally disconnected. We also observe that from .3 we can derive the simpler axiom $\mathsf{aT} := \Box(p \to \Diamond p)$, which is a tautology in **S4**, but not in **wK4**. While the Kripke semantics of aT is rather unimpressive, its topological semantics is quite intriguing. We show that aT defines the class of what we call *accumulative* spaces. A space X is accumulative if for all $A \subseteq X$ such that $\mathsf{d}(A) \neq \emptyset$, there exists an open set U such that $\emptyset \neq A \cap U \subseteq \mathsf{d}(A)$. Examples of accumulative spaces include the set of natural numbers with the cofinal topology, as well as other pre-ordered sets with a similar topology. We then prove that .3 defines the class of hereditarily extremally disconnected accumulative spaces. Note that the axioms .2 and .3 are known to be related to the axioms of *bounded width* bw_n (with $n \in \mathbb{N}$) [CZ97, sec. 3.5], so in some way they talk about the width of spaces, and thus accompany very well our work on bounded depth. More precisely, .3 is merely equivalent to bw_1, so a natural line of research would be to generalize our results to bw_n.

Finally, we address the completeness of all of these logical systems. This question raises the particular challenge of turning Kripke frames into appropriate topological spaces, that is, in a way that preserves the truth of formulas with respect to the d-semantics. With this operation, one can transfer results of completeness from the Kripke semantics to the topological semantics. While this is straightforward in the c-semantics, the case of the d-semantics presents many difficulties related to reflexive points. A solution used notably in [BBFD21] is that of *unfolding*, but applying it correctly requires precision. This technique consists in replacing every reflexive point w of a frame by countably many copies of w, and to arrange them all into a dense-it-itself subspace, so that to mimic the reflexivity of w in the d-semantics. However the standard construction is not suitable for a number of logics (typically those containing aT), as it yields spaces that do not satisfy the T_1 separation condition. To bypass this problem we introduce a variant of unfolding with more open sets, so that to guarantee sufficient separation. We then successfully prove that $\mathbf{wK4} + \mathsf{bd}_n$, $\mathbf{wK4} + .2$, $\mathbf{wK4} + .3$, and various extensions of these logics are topologically complete. This demonstrates the richness of the method of unfolding, and may be a source of inspiration to future work.

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Nelson conuclei and nuclei: the twist construction beyond involutivity

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Until recently, twist-products (also known, in the literature, as *twist-structures* or *twist-algebras*) have been almost exclusively employed to construct and represent algebras (of non-classical logics) that carried an *involutive* negation, i.e. one satisfying the double negation identity ($\sim\sim x = x$). Prominent examples include various classes of bilattices and residuated structures, such as *Nelson algebras* (models of Nelson's constructive logic with strong negation: see e.g. [19]) and *N4-lattices* (models of the paraconsistent version of Nelson's logic: see [6, 7]). While the twist-product indeed provides an easy way to introduce an involutive negation, this feature is not essential to the construction, either from a technical or a conceptual point of view (concerning this latter aspect, see in particular [4, 5]). This observation is developed in a series of recent papers which explore the applicability of various non-involutive twist-product constructions: see [9, 8, 10, 11, 12, 13, 14, 15, 16, 17, 18]. In particular, the papers [18, 11] show how to generalize the definitions and twist representations of Nelson algebras and N4-lattices to a non-involutive setting; the resulting classes of algebras have been dubbed *quasi-Nelson algebras* and *quasi-N4-lattices*, respectively.

In the present contribution we extend the non-involutive twist construction so as to encompass yet more general classes of algebras. Observing that both Nelson and quasi-Nelson algebras are (special subclasses of) bounded commutative integral distributive residuated lattices, it seems natural to look at which among these properties (e.g. commutativity, integrality, distributivity) may be relaxed without compromising the twist representation. We accomplish our task considering two different approaches to twist-products: the original approach due to Rasiowa and the more recent one based on residuated lattices.

The algebraic models of Nelson's logic were first introduced (by H. Rasiowa, after D. Nelson's original presentation of the logic) in a language which featured a (non-residuated) intuitionistic-like implication (known in the literature as *weak implication*, and usually denoted by $\to$), and were later shown to be term equivalent to a class of integral residuated lattices. In the setting of Nelson algebras no problem arises, for the neutral element of the semigroup operation – which, by integrality, coincides with the top element of the associated lattice order – is a term definable algebraic constant; in fact, Nelson algebras can be characterized as precisely those N4-lattices on which the defining term ($x \to x$) is constant. In the non-integral setting of N4-lattices, however, the neutral element is no longer definable (such an element may not exist at all), and must therefore be introduced through a primitive nullary operation – if one wishes, that is, to study the models of paraconsistent Nelson's logic within the theory of residuated lattices. The class of N4-lattices enriched with such an extra constant is investigated in [1] under the

*Speaker.

name of *eN4-lattices* (another paper [3] uses the more suggestive name of *Nelson paraconsistent lattices*)[1].

This problem necessarily carries over to more general algebras. The paper [2] introduced a twist construction which determines a class of residuated lattices (dubbed *Kalman lattices*) that are commutative and involutive, but not necessarily integral nor distributive. The Kalman lattices of [2] include as subvarieties eN4-lattices but not N4-lattices[2].

The preceding considerations entail that the two approaches to twist-products – the one based on the strong implication (by which one obtains residuated structures) and the one based on the weak implication (which generalizes directly Rasiowa's construction of Nelson algebras) – grow further apart as we consider more general structures. This, in turn, suggests that it may be appropriate to pursue both approaches separately. In the present contribution we shall do so, drawing inspiration directly from the twist constructions presented in [2, 3] and extending them to a non-involutive setting. In particular, as far as the residuated lattice approach is concerned, we shall generalize the *Kalman lattices* of [2] and the *Nelson conucleus algebras* of [3] by simultaneously dropping the requirements of (i) involutivity of the negation, (ii) commutativity of the monoid operation (thus we shall work with *two* residuated implications, the left and right residuals of the monoid operation), and (iii) integrality of the factor algebras employed in the twist construction. With regards to the other approach, we shall generalize the *Rasiowa-type algebras* of [3] (i) by allowing the negation to be non-involutive and (ii) by not postulating the existence of a neutral element for the semigroup operation; as in the preceding case, here too we shall be dealing with two ("weak") implications.

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[1]As the twist construction shows, the impact of adding the extra constant is substantial, and in consequence both the twist representation and the term equivalence result are much more straightforward for eN4-lattices than for N4-lattices (see [20]).

[2]What is even worse, is that not even Nelson algebras can be viewed as a subvariety of eN4-lattices: this is because the definition of eN4-lattices implies (essentially for technical reasons) that the interpretation of e must be not only the neutral element of the monoid operation, but also a fixpoint of the negation, a requirement that no (non-trivial) Nelson algebra can satisfy.

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Intuitionistic modal algebras and twist representations

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A *modal Heyting algebra* is obtained by enriching a Heyting algebra $\langle H; \wedge, \vee, \rightarrow, 0, 1\rangle$ with a unary modal operator $\Box$ satisfying the following identity:

$$x \rightarrow \Box y = \Box x \rightarrow \Box y.$$

Such an operator is also known in the literature as a *nucleus*, or a *multiplicative closure operator*. Many natural constructions give rise to nuclei. For instance, having fixed an element $a \in H$ of a Heyting algebra, we can obtain a nucleus by setting either $\Box x := a \rightarrow x$ or $\Box x := a \vee x$, or $\Box x := (x \rightarrow a) \rightarrow a$. So, in particular, the identity map, the constant map $x \mapsto 1$ and the double negation map also define nuclei (see [8, 1] for further examples).

The class of modal Heyting algebras (and some of its subreducts) has been studied since the 1970s, usually within the framework of topology and sheaf theory [8, 9, 3, 2]. A more recent paper [5] proposed a logic based on modal Heyting algebras (called *Lax Logic*) as a tool in the formal verification of computer hardware. Even more recently, another connection between modal Heyting algebras and logic emerged within the study of the algebraic semantics of *quasi-Nelson logic* [16, 15]. The latter may be viewed as a common generalization of both intuitionistic logic and *Nelson's constructive logic with strong negation* [10] obtained by deleting the double negation law.

As shown in [15, 12, 11], there exists a formal relation between the algebraic counterpart of quasi-Nelson logic and the class of modal Heyting algebras which parallels the well-known connection between *Nelson algebras* and Heyting algebras (see e.g. [17]). This relation – which, as we shall see, concerns the algebras in the full language as well as some of their subreducts – provides, in our view, further motivation for the study of modal Heyting algebras from a logical as well as an algebraic point of view. It is interesting to note that, with the notable exception of [1], studies of this kind are scant in the literature – perhaps owing to the mainly topological interest in this class of algebras? The purpose of the present contribution is to fill in this gap, at least partly, and at the same time to draw attention to certain subreducts of modal Heyting algebras whose interest is motivated by recent developments in the theory of quasi-Nelson logic.

Since a modal Heyting algebra is usually presented in the language $\{\wedge, \vee, \rightarrow, \Box, 0, 1\}$, fragments that appear to be of natural interest (from a logico-algebraic perspective) are, for instance, the implication-free one $\{\wedge, \vee, \Box\}$ – perhaps enriched with the lattice bounds 0 and 1 – and the implicational one $\{\rightarrow, \Box\}$. The former, whose models are distributive lattices enriched with a modal operator, is in fact the main object of [1], while the latter – whose models are *Hilbert algebras*, the algebraic counterpart of the purely implicational fragment of intuitionistic logic, expanded with a modal operator – was studied, mainly from a topological perspective, as far back as in [8], and as recently as in [4]. Other less obvious but, in our opinion, also interesting classes of algebras emerged in the course of our recent investigations on quasi-Nelson logic

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and its algebraic counterpart, the variety of *quasi-Nelson algebras*. An interest in these classes of algebras, however, can also be motivated within the limits of the traditional framework of modal Heyting algebras, as explained below.

A well-known fact on modal Heyting algebras [8, Thm. 2.12] is that, for every such algebra $\mathbf{H} = \langle H; \wedge, \vee, \to, \Box, 0, 1\rangle$, the set $H_\Box := \{a \in H : a = \Box a\}$ of fixpoints of the $\Box$ operator can itself be endowed with a modal Heyting algebra structure by defining, for every n-ary algebraic operation $f \in \{\wedge, \vee, \to, \Box, 0, 1\}$, the operation $f_\Box$ given, for all $a_1, \ldots, a_n \in H_\Box$, by $f_\Box(a_1, \ldots, a_n) := \Box f(a_1, \ldots, a_n)$.

Denoting this algebra by $\mathbf{H}_\Box$, we observe that, the universe $H_\Box$ can equivalently be defined as the nucleus image $\{\Box a : a \in H\}$ of $\mathbf{H}$. While $\mathbf{H}_\Box$ is indeed a modal Heyting algebra, it is a very special one on which the $\Box$ operator is the identity map. This very fact, in turn, is essential in ensuring that $\mathbf{H}_\Box$ has a Heyting algebra reduct; for instance we have, for all $a, b \in H_\Box$,

$$a \wedge_\Box b = \Box(a \wedge b) = \Box a \wedge \Box b = a \wedge b$$

guaranteeing that $\wedge_\Box$ is a meet semilattice operation on $H_\Box$. A similar reasoning applies to the other operations, although the join $\vee_\Box$ (computed in $\mathbf{H}_\Box$) does not coincide with the join $\vee$ (computed in $\mathbf{H}$), i.e. $\mathbf{H}_\Box$ is not a subalgebra of $\mathbf{H}$. This construction is easily seen to be a generalization of Glivenko's result relating Heyting and Boolean algebras (the latter corresponding to the case where $\Box x = \neg\neg x$).

Thus, although nothing prevents one from considering each operation $f_\Box$ as defined on the whole universe H, in general $\wedge_\Box$ and $\vee_\Box$ will not be semilattice operations on H, and $\to_\Box$ will not be a Heyting (i.e. a relative pseudo-complement) implication on H (on the other hand, we always have $\Box_\Box = \Box$ and $1_\Box = 1$). By definition, these new operations will be generalizations of the intuitionistic ones, which can be retrieved by requiring $\Box$ to be the identity map on H. In this respect natural questions to ask are, in our opinion, (1) which properties each generalized operation $f_\Box$ retains, and (2) whether some particular choice of $f_\Box$ has any independent interest that may justify further study.

A first answer to the latter question may be sought within the theory of quasi-Nelson logic. Indeed, as shown in the papers [15, 12, 11, 13], some of the above-defined operations of type $f_\Box$ naturally arise within the study of fragments of the quasi-Nelson language. From this standpoint, it is also interesting to observe that the classes of algebras one obtains through the *twist representation* (see below) combine the original Heyting operations with the new ones. Thus, for instance, one of the classes of algebras arising in this way retains the original meet semilattice operation (and the lattice bounds) while replacing the Heyting implication with a generalized counterpart: that is, we are looking at the $\{\wedge, \to_\Box, 0, 1\}$-subreducts of modal Heyting algebras. We stress that these new algebras are not the result of an arbitrary choice of operations, but arise as twist factors in the representation of subreducts of quasi-Nelson algebras, as we now proceed to explain.

A *quasi-Nelson algebra* may be defined as a commutative integral bounded residuated lattice (see e.g. [6] for formal definitions of these terms) $\mathbf{A} = \langle A; \sqcap, \sqcup, *, \Rightarrow, \bot\rangle$ that (upon letting $\sim x := x \Rightarrow \bot$) satisfies the *Nelson identity*: $(x \Rightarrow (x \Rightarrow y)) \sqcap (\sim y \Rightarrow (\sim y \Rightarrow \sim x)) = x \Rightarrow y$.

Quasi-Nelson algebras arise as the algebraic counterpart of quasi-Nelson logic, which can be viewed either as a generalization (i.e. a weakening) common to Nelson's constructive logic with strong negation and to intuitionistic logic, or as the extension (i.e. strengthening) of the well-known substructural logic FL_{ew} (the *Full Lambek Calculus with Exchange and Weakening*) by the *Nelson axiom*:

$$((x \Rightarrow (x \Rightarrow y)) \sqcap (\sim y \Rightarrow (\sim y \Rightarrow \sim x))) \Rightarrow (x \Rightarrow y).$$

We refer to [16] for further details on quasi-Nelson logic, as well as for other equivalent characterizations of the variety of quasi-Nelson algebras (which can e.g. also be obtained as the class of $(0,1)$-*congruence orderable* commutative integral bounded residuated lattices).

Formally, every Heyting algebra may be viewed as a quasi-Nelson algebra (on which $\wedge = *$, $\vee = \sqcup$, $\rightarrow = \Rightarrow$ and $0 = \bot$) and, as noted earlier, the double negation map defines a modal operator on every Heyting algebra $\mathbf{H}$. If we replace $\mathbf{H}$ by a quasi-Nelson algebra $\mathbf{A}$, then the double negation map need not define a nucleus on $\mathbf{A}$, but can be used to obtain one on a special quotient $H(\mathbf{A})$, which is the (Heyting) algebra canonically associated to each quasi-Nelson algebra $\mathbf{A}$ via the twist construction.

Given a quasi-Nelson algebra $\mathbf{A}$, consider the map given, for all $a \in A$, by $a \mapsto a * a$. The kernel θ of this map is a congruence of the reduct $\langle A; \sqcap, \sqcup, * \rangle$ which is also compatible with the double negation operation and with the *weak implication* $\Rightarrow^2$ given by $x \Rightarrow^2 y := x \Rightarrow (x \Rightarrow y)$. Letting $\Box(x/\theta) := {\sim}{\sim} x/\theta$, we thus have a quotient algebra $H(\mathbf{A}) = \langle A/\theta; \sqcap, \sqcup, \Rightarrow^2, \Box, \bot \rangle$, which is a modal Heyting algebra (where $* = \sqcap$). Moreover, $\mathbf{A}$ embeds into a *twist-algebra* over $H(\mathbf{A})$, defined as follows.

Given a a modal Heyting algebra $\mathbf{H} = \langle H; \wedge, \vee, \rightarrow, \Box, 0, 1 \rangle$, define the algebra $\mathbf{H}^{\bowtie} = \langle H^{\bowtie}; \sqcap, \sqcup, *, \Rightarrow, \bot \rangle$ with universe $H^{\bowtie} := \{\langle a_1, a_2 \rangle \in H \times H_{\Box} : a_1 \wedge a_2 = 0\}$ and operations given, for all $\langle a_1, a_2 \rangle, \langle b_1, b_2 \rangle \in H \times H$, by:

$$\begin{aligned}
\bot &:= \langle 0, 1 \rangle \\
\langle a_1, a_2 \rangle * \langle b_1, b_2 \rangle &= \langle a_1 \wedge b_1, \Box((a_1 \rightarrow b_2) \wedge (b_1 \rightarrow a_2)) \rangle \\
\langle a_1, a_2 \rangle \sqcap \langle b_1, b_2 \rangle &:= \langle a_1 \wedge b_1, \Box(a_2 \vee b_2) \rangle \\
\langle a_1, a_2 \rangle \sqcup \langle b_1, b_2 \rangle &:= \langle a_1 \vee b_1, \Box(a_2 \wedge b_2) \rangle \\
\langle a_1, a_2 \rangle \Rightarrow \langle b_1, b_2 \rangle &:= \langle (a_1 \rightarrow b_1) \wedge (b_2 \rightarrow a_2), \Box(a_1 \wedge b_2) \rangle.
\end{aligned}$$

A *quasi-Nelson twist-algebra over* $\mathbf{H}$ is any subalgebra $\mathbf{A} \leq \mathbf{H}^{\bowtie}$ satisfying $\pi_1[A] = H$.

The *twist representation theorem* says that every quasi-Nelson algebra $\mathbf{A}$ embeds into the twist-algebra $(H(\mathbf{A}))^{\bowtie}$ through the map given by $a \mapsto \langle a/\theta, {\sim} a/\theta \rangle$ [16].

The previous definition suggests that certain term operations of the language of modal Heyting algebras may be of particular interest in the study of fragments of the quasi-Nelson language. Consider, for instance, the monoid operation $(*)$. In order to define it, on a quasi-Nelson algebra $\mathbf{A} \leq \mathbf{H}^{\bowtie}$, we need two operations on $\mathbf{H}$: the semilattice operation $\wedge$ (for the first component) and, for the second component, an implication-like operation (let us denote it by $\rightharpoonup$) which can be given by $x \rightharpoonup y := x \rightarrow \Box y$. The latter claim may not be obvious, but using the properties of the twist construction and the modal operation, it is not hard to verify the following equalities:

$$\begin{aligned}
\Box((a_1 \rightarrow b_2) \wedge (b_1 \rightarrow a_2)) &= \Box((a_1 \rightarrow \Box b_2) \wedge (b_1 \rightarrow \Box a_2)) \\
&= \Box(a_1 \rightarrow \Box b_2) \wedge \Box(b_1 \rightarrow \Box a_2) \\
&= (a_1 \rightarrow \Box b_2) \wedge (b_1 \rightarrow \Box a_2) \\
&= (a_1 \rightharpoonup b_2) \wedge (b_1 \rightharpoonup a_2).
\end{aligned}$$

These observations led to the introduction of the class of algebras dubbed $\rightharpoonup$-*semilattices* in [13], where it is shown in particular that the $\{*, {\sim}\}$-subreducts of quasi-Nelson algebras are precisely the algebras representable as twist-algebras over $\rightharpoonup$-semilattices. Similar considerations motivated the introduction of other term operations of the language of modal Heyting algebras, such as the following: $x \odot y := \Box(x \wedge y)$ and $x \oplus y := \Box(x \vee y)$. As shown in [13], the corresponding classes of modal algebras allow us to establish twist representations for (respectively) the classes of $\{\Rightarrow^2, \sim\}$-subreducts and of $\{\wedge, *, \Rightarrow, \sim\}$-subreducts of quasi-Nelson algebras. Other subreducts may be obtained by adding a modal operator to more traditional classes of intuitionistic algebras, such as implicative semilattices (corresponding to the $\{*, \Rightarrow, \sim\}$-subreducts of quasi-Nelson algebras), distributive lattices (corresponding to the $\{\wedge, \vee, \sim\}$-subreducts studied in [14]) and pseudo-complemented lattices (corresponding to the "two-negations" subreducts studied in [12]).

The previous considerations suggest the above-mentioned classes of modal algebras as mathematical objects that may be of interest both in themselves and in relation to the study of non-classical logics, in particular Nelson's logics[1]. The aim of the present contribution is to improve our understanding of these classes of algebras from an algebraic as well as a topological point of view.

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[1] Beyond the Nelson realm, (0-free subreducts of prelinear) modal Heyting algebras also feature in the twist-type representation introduced in [7] for *Sugihara monoids*, a variety of algebras related to relevance logics.

Algebras of Counterfactual Conditionals

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A *counterfactual conditional* (or simply a counterfactual) is a conditional statement of the form "If [antecedent] were the case, then [consequent] would be the case", where the antecedent is usually assumed to be false. Counterfactuals have been studied in different fields: for instance in the philosophy of language and in linguistics (e.g. [2] and [12]), in artificial intelligence (e.g. [3]), and philosophy (e.g. [8]). The logical analysis of counterfactuals is rooted in the work of Lewis [9, 7] and Stalnaker [13] who have introduced what has become the standard semantics for counterfactual conditionals based on particular Kripke models equipped with a similarity relation among the possible worlds.

In Lewis' language, a counterfactual is formalized as a formula of the kind "$\varphi \Box\!\!\rightarrow \psi$" which is intended to mean that if φ were the case, then ψ would be the case. Lewis [9] has introduced different logics of counterfactuals arising from his semantics; these logics have been studied from a proof-theoretic perspective by, for instance, Negri and Sbardolini [10] and Lellman and Pattinson [6].

Although the research on counterfactuals and their logic has been prolific, a deep and coherent algebraic investigation of Lewis' logic of counterfactuals is, to the best of the authors' knowledge, still missing. Some steps towards this direction can be found in the work of Nute [11] and Weiss [14], however their approach only shows that (some of the) Lewis logics of counterfactuals are weakly complete with respect to some algebraic structures obtained from Boolean algebras by introducing a binary operator $\star$ that stands for the counterfactual conditional connective.

In the present work, we start filling this gap by providing an equivalent algebraic semantics, in the sense of Blok-Pigozzi [1], for the logics of *global* consequence associated to Lewis' systems **C0**, **C1** and **C2** introduced in [7]. These systems correspond to the systems V, VC, and VCS in [9]. It is worth mentioning that the system **C1** is, in Lewis' own opinion, the "correct logic of counterfactuals conditionals as we ordinarily understand them" (see [7, p.80]).

More precisely, in analogy with modal logics, we start by observing that, to each of Lewis' systems **Ci** (with $0 \leq \mathbf{i} \leq 2$), we can associate two logics, that is, the logic of *global* consequence and the logic of *local* consequence, which differ depending on how one specifies the rule of deduction within conditionals (DWC in [7, p. 80]). Then, for each system **Ci**, we define an associated class of Boolean algebras equipped with a binary operator $\Box\!\!\rightarrow$ that stands for the counterfactual connective. We show that each of these classes of can be axiomatized by means of equations, and is therefore a variety. We then prove completeness for $\mathbf{Ci}_l$ and $\mathbf{Ci}_g$ with respect to their associated class. In particular, it turns out that $\mathbf{Ci}_l$ is the logic preserving degrees of truth of the class of **Ci**-algebras, and that **Ci**-algebras provide an equivalent algebraic semantics for $\mathbf{Ci}_g$ with $\tau = \{x \approx 1\}$ and $\Delta = \{x \to y, y \to x\}$ witnessing the algebraizability of $\mathbf{Ci}_g$. As a consequence of algebraizability, we obtain that all axiomatic extensions of $\mathbf{Ci}_g$ are also algebraizable.

It is worth noticing that said structures do not belong to the framework of *Boolean algebras with operators*, as studied for instance by Jipsen in [5], since $\Box\!\!\rightarrow$ is not additive, in the sense that it does not preserve the Boolean disjunction on the left. Nonetheless, such algebras are well-behaved from the point of view of their structure theory, indeed they are ideal-determined with respect to 1 in the sense of [4]. Thus, congruences are characterized by their 1-blocks, which turn out to be particular lattice filters respecting a further condition involving $\Box\!\!\rightarrow$. Notably, we characterize the structure theory of these algebras in order to study the subdirectly irreducible and directly indecomposable algebras of such class via the description of *congruence elements*, that is, elements whose upset is a congruence filter.

We observe that, while the class of **Ci**-algebras are also an algebraic semantics with respect to $\mathbf{Ci}_l$, it is not the case that they provide an equivalent algebraic semantics for it. Indeed, it can be seen that the congruence filters of the **Ci**-algebras do not correspond to the deductive filters induced by the logic $\mathbf{Ci}_l$. This shows an interesting parallel with the modal logic case, where the class of modal algebras provide an equivalent algebraic semantics for the logic of global consequence $\mathbf{K}_g$, whereas the logic of local consequence $\mathbf{K}_l$, although being complete with respect to the class of modal algebras, is not algebraizable.

Now, it is important to stress that both the global and local consequences of **Ci** admit a possible worlds semantics: a Lewis' model for a **Ci**-logic consists in a tuple $\langle W, \mathcal{S}, v\rangle$ where $\mathcal{S} : W \to \wp(\wp(W))$ and $\mathcal{S}(w)$ is nested, i.e. for each $S, T \in \mathcal{S}(w)$, either $S \subseteq T$ or $T \subseteq S$. Depending on stronger constraints imposed on $\mathcal{S}$, Lewis defines models for each **Ci** system (e.g., **C1**-models are those in which $\mathcal{S}(w)$ is centered, i.e. $\{w\} \in \mathcal{S}(w)$). We provide a completeness result for local and global consequences with respect to Lewis' models, again in parallel with the case of modal logic, in which global an local consequence of **K** corresponds, respectively, to the global and local logical consequence relation over Kripke frames. Given these results, we then aim at studying the duality relations between our algebras of counterfactuals and Lewis' models.

Finally, using our framework, we analyse the connections between $\mathbf{Ci}_l$ and $\mathbf{Ci}_g$. In particular: we observe that $\mathbf{Ci}_l$ and $\mathbf{Ci}_g$ share the same theorems, but differ with respect to the logical consequences; while $\mathbf{Ci}_l$ has the deduction theorem with respect to the classical implication, $\mathbf{Ci}_g$ does not; we investigate whether global and local consequence can be characterized in terms of each other, as in the case of modal logic.

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Modal Algebraic Models of Counterfactuals

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The aim of the present work is to put forward an algebraic approach to counterfactual conditionals (or simply *counterfactuals* from now on) based on Boolean Algebras of Conditionals as defined in [3]. A Boolean algebra of Conditionals (BAC), $\mathcal{C}(\mathbf{A})$, is a Boolean algebra obtained starting from any Boolean algebra $\mathbf{A} = \langle A, \wedge, \vee, \neg, \top, \bot \rangle$ and taking a certain quotient of the free Boolean algebra generated by the pairs $(a, b) \in A \times A$ with $b \neq \bot$. Each basic element in a BAC is identified with $(a \mid b)$ which is intended to represent the conditional event "a given b", where b is the *antecedent* and a the *consequent*. The framework of BACs offers an innovative and privileged perspective on conditionals events: as it is shown in [3], BACs are a valuable tool to analyze the algebraic properties of conditionals events, their logic and their relation with probability measures.

The framework of BACs, although promising, is not yet fully developed in all its potentialities. Our goal is to extend BACs in order to account for *counterfactual* conditional events. More precisely, we consider a normal modal operator $\Box$ on a BAC so defining *modal* Boolean Algebras of Conditionals $\langle \mathcal{C}(\mathbf{A}), \Box \rangle$ that we name *Lewis algebras*. We investigate the properties of these new structures and the resulting logic of counterfactuals. Our idea is motivated by the fact that, although counterfactuals are not captured by BACs, a normal modal operator $\Box$, when combined with the algebraic properties of conditional events, logically behaves very similarly to the counterfactual conditional operator $\Box\!\!\rightarrow$ in David Lewis's semantics for counterfactuals (see [5] and [6]) so as to interpret a Lewis' counterfactual $b \Box\!\!\rightarrow a$ as $\Box(a \mid b)$ in a modal BAC. By doing so, already basic properties of counterfactuals can be proved to hold in our modal framework. For instance, $\Box(a \mid b) \wedge \Box(c \mid b) = \Box(a \wedge c \mid b)$ holds in every modal BAC and analogously $((b \Box\!\!\rightarrow a) \wedge (b \Box\!\!\rightarrow c)) \leftrightarrow (b \Box\!\!\rightarrow (a \wedge c))$ is valid in Lewis' semantics for counterfactuals.

Starting from this construction, we analyze the properties of the dual Kripke frame of Lewis algebras, in the sense of Jónsson-Tarski (see [7]). In particular, for a Lewis algebra $\langle \mathcal{C}(\mathbf{A}), \Box \rangle$, we discuss what conditions should be imposed on $\Box$, in order to characterize Lewis' different logics for counterfactuals, and what properties these conditions imply on the dual frame. In particular, we show that if a Lewis algebra $\langle \mathcal{C}(\mathbf{A}), \Box \rangle$, satisfies the following identities:

(1) $\Box(a \mid \top) = (a \mid \top)$

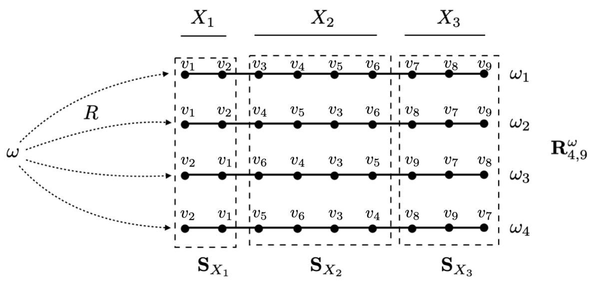
(2) $\Box(a \mid a \vee b) \vee \Box(b \mid a \vee b) \vee (\Box(c \mid a \vee b) \to \Box((c \mid a) \wedge (c \mid b))) = 1$

then, the resulting modal logic of conditionals corresponds to a slightly stronger logic than the system C1, that Lewis himself claims to be the "correct logic of counterfactual conditionals" (see [5, p. 80]).

Slightly more formally, for every Boolean algebra $\mathbf{A}$, we call *Lewis algebra* any modal BAC $\mathcal{L}(\mathbf{A}) = \langle \mathcal{C}(\mathbf{A}), \Box \rangle$ satisfying (1) and (2). The dual frame $\langle \mathrm{at}(\mathcal{C}(\mathbf{A})), R \rangle$ of $\mathcal{L}(\mathbf{A})$ will be called a *Lewis Frame* and we denote it by $\mathrm{F}_{\mathcal{L}(\mathbf{A})}$. We then show how the above (1) and (2) characterize specific properties of Lewis frames. In particular, if $\mathbf{A}$ is a finite Boolean algebra with atoms

$v_1, \ldots, v_n$, a Lewis frame takes the form $\mathcal{F}_{\mathcal{L}(\mathbf{A})} = \langle \mathrm{at}(\mathcal{C}(\mathbf{A})), \mathrm{R}\rangle$ where $\mathrm{at}(\mathcal{C}(\mathbf{A}))$ denotes the finite set of atoms of the BAC $\mathcal{C}(\mathbf{A})$. In such a case, $\mathrm{F}_{\mathcal{L}(\mathbf{A})}$ validates (1) iff R is serial and each $\omega \in \mathrm{at}(\mathcal{C}(\mathbf{A}))$ only accesses to worlds that have the same initial element as ω. As for the latter, recall that the atoms of a BAC, $\mathcal{C}(\mathbf{A})$, can be identified with strings of maximal length of atoms in $\mathbf{A}$, i.e. $\omega = \langle v_1, v_2, \ldots, v_n\rangle$.

Thus, (1) characterizes the following properties of $\mathrm{F}_{\mathcal{L}(\mathbf{A})}$: (*i*) for all $\omega = \langle v_1, v_2, \ldots, v_n\rangle \in \mathrm{at}(\mathcal{C}(\mathbf{A}))$ there is a $\omega' = \langle v'_1, v'_2, \ldots, v'_n\rangle$ such that $\omega R \omega'$; and (*ii*) if $\omega = \langle v_1, v_2, \ldots, v_n\rangle$ and $\omega' = \langle v'_1, v'_2, \ldots, v'_n\rangle$ are such that $\omega R\omega'$, then $v_1 = v'_1$. Condition (2) characterizes a property on Lewis frames that we call *sphericity*. This property defines, for each $\omega \in \mathrm{at}(\mathcal{C}(\mathbf{A}))$, a certain composition of the set $R[\omega] = \{\omega' \in \mathrm{at}(\mathcal{C}(\mathbf{A})) \mid \omega \mathrm{R} \omega'\}$ of accessible worlds from ω. More precisely, one can easily display the elements of $R[\omega]$ in a finite matrix (see the figure below for an example). This will be called the *matrix generated by* $R[\omega]$, and it will be denoted by $\mathbf{R}^{\omega}_{k,n}$. If $\mathrm{F}_{\mathcal{L}(\mathbf{A})}$ satisfies (2) then, for all $\omega \in \mathrm{at}(\mathcal{C}(\mathbf{A}))$, $\mathbf{R}^{\omega}_{k,n}$ can be partitioned into submatrices such that they do not share any element with each other and each of them contains the same elements in all its rows and its columns. Although sphericity has an intricate formulation, it is easier to grasp with a graphical example:



The matrix $\mathbf{R}^{\omega}_{k,n}$ is induced by the sphericity condition as it can be partioned into disjoint cells $\mathbf{S}_{X_1}, \mathbf{S}_{X_2}$ and $\mathbf{S}_{X_3}$ and each of them contains the same elements in its columns and its rows. Hence, we get that a Lewis frames validates (2) iff it satisfies sphericity.

The semantic conditions (with respect to a Lewis frames) for a counterfactual of the form $\Box(\varphi \mid \psi)$ correspond to the usual modal Kripke-semantic conditions: $\Box(\varphi \mid \psi)$ is true at ω iff $(\varphi \mid \psi)$ is true at all the ω' such that $\omega R \omega'$[1], and the semantic conditions for Boolean combinations of formulas are the usual classical ones. Given the characterization of the class of Lewis frames, we show how to go back and forth from Lewis frames to sphere models for counterfactuals[2]. In particular, we show that each Lewis frame $\mathcal{F}_{\mathcal{L}(\mathbf{A})}$ corresponds to a sphere model $\mathcal{M}$ satisfying exactly the same counterfactuals formulas as $\mathcal{F}_{\mathcal{L}(\mathbf{A})}$, and viceversa. This correspondence between the two semantic frameworks allows us to prove soundness and completeness of the logic $\mathsf{C1}^+ = \mathsf{C1} + \Box(\varphi \mid \psi) \to \Diamond(\varphi \mid \psi)$ (for all ψ such that $\psi \not\leftrightarrow \bot$) with respect to Lewis frames and Lewis algebra.

The above results represent a step towards an algebraic approach to counterfactual conditionals. Although the research on the semantics of counterfactuals has been prolific (see for instance [1], [2] and [4]), an algebraic framework to analyze counterfactual conditionals is, to the best of the authors' knowledge, still missing. In the present work, we have tried to start filling this gap. Finally, we will see how Lewis algebras can contribute to understanding the uncertain quantifications of counterfactuals by analyzing how a belief function P behaves on a Lewis algebra, so as to represent the uncertainty of a counterfactual as $P(\Box(a \mid b))$.

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[1]For an analysis of the truth conditions of a conditional $(\varphi \mid \psi)$ with respect to a $\omega = \langle v_1, \ldots, v_n\rangle$, see [3].

[2]See [6] for more details the sphere-based semantics for counterfactuals.

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Strong standard completeness for S5-modal Łukasiewicz [2]

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Abstract

We study the S5-modal expansion of the logic based on the Łukasiewicz t-norm. We exhibit an infinitary propositional calculus and show that it is strongly complete with respect to this logic. These results are derived from properties of monadic MV-algebras: functional representations of simple and finitely subdirectly irreducible algebras.

In [3] Hájek introduced an S5-modal expansion of any axiomatic extension $\mathcal{C}$ of his Basic Logic which is equivalent to the one-variable monadic fragment of the first-order extension $\mathcal{C}\forall$ of $\mathcal{C}$. We present next a slight generalization of his definition. Let $Prop$ be a countably infinite set of propositional variables, and let Fm be the set of formulas built from $Prop$ in the language of Basic Logic expanded with two unary connectives $\Box$ and $\Diamond$. Consider a class $\mathbb{C}$ of totally ordered BL-algebras. To interpret the formulas in Fm, consider triples $\mathbf{K} := \langle X, e, \mathbf{A}\rangle$ where X is a non-empty set, $\mathbf{A} \in \mathbb{C}$, and $e\colon X \times Prop \to A$ is a function. The *truth value* $\|\varphi\|_{\mathbf{K},x}$ of a formula φ in $\mathbf{K}$ at a point $x \in X$ is defined by recursion. For propositional variables $p \in Prop$ put $\|p\|_{\mathbf{K},x} := e(x, p)$. The definition of the truth value is then extended for the logical connectives in the language of Basic Logic in the usual way, and for the new unary connectives by

$$\|\Box\psi\|_{\mathbf{K},x} := \inf_{x'\in X} \|\psi\|_{\mathbf{K},x'}, \quad \text{and} \quad \|\Diamond\psi\|_{\mathbf{K},x} := \sup_{x'\in X} \|\psi\|_{\mathbf{K},x'}.$$

Note that the infima and suprema above may not exist in general in $\mathbf{A}$; hence, we restrict our attention to *safe* structures, that is, structures $\mathbf{K}$ for which $\|\varphi\|_{\mathbf{K},x}$ is defined for every $\varphi \in Fm$ at every point x. Given $\Gamma \subseteq Fm$, we say that a safe structure $\mathbf{K}$ is a *model* of Γ if $\|\varphi\|_{\mathbf{K},x} = 1$ for every $x \in X$ and $\varphi \in \Gamma$. For a set of formulas $\Gamma \cup \{\varphi\}$ we write $\Gamma \vDash_{\mathrm{S5}(\mathbb{C})} \varphi$ if every model of Γ is also a model of φ. The logic just defined depends on the class $\mathbb{C}$ and is denoted by $\mathrm{S5}(\mathbb{C})$. In case $\mathbb{C}$ is the class of totally ordered $\mathcal{C}$-algebras corresponding to an axiomatic extension $\mathcal{C}$ of Basic Logic we get the original definition given by Hájek; this logic was denoted by $\mathrm{S5}(\mathcal{C})$ in [3], but we reserve this notation for a related logic defined by means of an axiomatic system (see below).

We are interested in expansions of the infinite-valued Łukasiewicz logic, which we denote by $\mathcal{L}$. Recall that the equivalent algebraic semantics of $\mathcal{L}$ is the variety $\mathbb{MV}$ of MV-algebras. We write $\mathbb{MV}_{\mathrm{to}}$ for the class of totally ordered MV-algebras. Thus, $\mathrm{S5}(\mathbb{MV}_{\mathrm{to}})$ is the S5-modal expansion of $\mathcal{L}$ defined by Hájek. Consider now the logic $\mathrm{S5}(\mathcal{L})$ on the same language as $\mathrm{S5}(\mathbb{MV}_{\mathrm{to}})$ defined by the following axiomatic system:

- Axioms:

  Instantiations of axiom-schemata of $\mathcal{L}$

  $\Box\varphi \to \varphi$

[*] Speaker.

$$\varphi \to \Diamond\varphi$$
$$\Box(\nu \to \varphi) \to (\nu \to \Box\varphi)$$
$$\Box(\varphi \to \nu) \to (\Diamond\varphi \to \nu)$$
$$\Box(\varphi \vee \nu) \to (\Box\varphi \vee \nu)$$
$$\Diamond(\varphi * \varphi) \equiv (\Diamond\varphi) * (\Diamond\varphi)$$

where φ is any formula, ν is any propositional combination of formulas beginning with $\Box$ or $\Diamond$, and $\alpha \equiv \beta$ abbreviates $(\alpha \to \beta) \wedge (\beta \to \alpha)$.

- Rules of inference:

Modus Ponens: $\frac{\varphi, \varphi \to \psi}{\psi}$

Necessitation: $\frac{\varphi}{\Box\varphi}$

In [1] the authors show a strong completeness theorem stating that $\mathrm{S5}(\mathbb{MV}_{\mathrm{to}}) = \mathrm{S5}(\mathcal{L})$. In this article we study the logic $\mathrm{S5}([0,1]_{\mathrm{L}})$ where $[0,1]_{\mathrm{L}}$ is the standard Łukasiewicz t-norm on the unit real interval, of course $\mathrm{S5}([0,1]_{\mathrm{L}})$ is a shorthand for $\mathrm{S5}(\{[0,1]_{\mathrm{L}}\})$. Note that $\mathrm{S5}([0,1]_{\mathrm{L}})$ is not finitary since it is a conservative expansion of the logic of $[0,1]_{\mathrm{L}}$, which is not finitary. Thus, a strong completeness theorem for $\mathrm{S5}(\mathcal{L})$ with respect to $\mathrm{S5}([0,1]_{\mathrm{L}})$ is not possible.

However, adding one infinitary rule to the axiomatic system defining $\mathrm{S5}(\mathcal{L})$ is enough to obtain a logic $(\mathrm{S5}(\mathcal{L})_\infty)$ strongly complete with respect to $\mathrm{S5}([0,1]_{\mathrm{L}})$. This had already been shown for the propositional and first-order cases in [5]. We follow the ideas in [4] and provide an adequate algebraic representation for simple algebras needed to obtain the monadic completeness theorem. The infinitary rule in question is:

$$\frac{\Box\phi \vee (\Box\alpha \to (\Box\beta)^n) \text{ for every } n \in \mathbb{N}}{\Box\phi \vee (\Box\alpha \to \Box\alpha * \Box\beta)}.$$

We use algebraic methods to prove the completeness results stated in the previous paragraphs. The representation theorems and properties that we prove here for monadic MV-algebras are also interesting in their own right since they improve our understanding of these structures.

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Relevant Reasoners in a Classical World

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In this paper we provide a framework for epistemic logic based on relevant modal logic aimed at avoiding the *logical omniscience* problem. In particular, we will be interested in the following instances of the problem, where $\Box$ models belief and $n \geq 0$:

$$\frac{\varphi_1 \land \cdots \land \varphi_n \to \psi}{\Box\varphi_1 \land \cdots \land \Box\varphi_n \to \Box\psi} \qquad \textit{Conjunctive Regularity} \qquad \text{(C-Reg)}$$

$$\frac{\varphi_1 \to (\ldots(\varphi_n \to \psi)\ldots)}{\Box\varphi_1 \to (\ldots(\Box\varphi_n \to \Box\psi)\ldots)} \qquad \textit{Implicative Regularity} \qquad \text{(I-Reg)}$$

Famously, standard relational semantics for normal modal logics validate both. On the other hand, relational semantics for *relevant* modal logic [3] avoids (I-Reg) and the special case of (C-Reg) for $n = 0$, Necessitation. The difference between (C-Reg) and (I-Reg) expresses the assumption that while beliefs of agents are represented as "automatically" closed under conjunction introduction, they are not seen as closed under implication elimination. As Sequoiah-Grayson [7] points out, this can be understood as meaning that while agents are assumed to automatically *aggregate* their beliefs, they are not assumed to automatically *combine* them.

In the relational semantics for relevant modal logics, validity is defined in terms of a set of *logical states*, but the failure of (I-Reg) is made possible by allowing the modal accessibility relation to reach out of the set of logical states. This is a feature the relevant semantics has in common with the so-called *non-normal states* approaches to the logical omniscience problem [4, 5, 8]. In these approaches, however, the set of normal states consists of classical possible worlds. The logic generated by these semantics extends classical propositional logic with epistemic modalities that are not closed under inference rules of classical propositional logic.

It makes sense to assume, though, that epistemic modalities are closed under *some* logic. More specifically, the requirement of a *relevant* connection between a piece of information and a conclusion agents draw on its basis makes some form of relevant logic a natural candidate. It has been argued, for instance, that processing an input φ in a context yields 'a contextual implication, a conclusion [ψ] deducible from the input and the context together, but from neither input nor context alone' [9]. As noted in [1], such informational interpretation of relevance is embodied in Routley and Meyer's relational semantics for relevant logics, in particular in the ternary relation interpreting implication.

In classical epistemic logic, it is sufficient for $\Box\varphi_1 \land \ldots \land \Box\varphi_n \to \Box\psi$ to be valid that ψ is classically implied by $\varphi_1 \land \ldots \land \varphi_n$. On the relevant criterion, the classical validity of the salient implication should not be sufficient. However, its *relevant* validity should.

In this presentation, we outline a framework for relevant epistemic logic based on these ideas. Our framework models agents as *relevant reasoners in a classical world*: the agent reasons in accordance with a relevant modal logic, but the propositional fragment of our logic is classical.

More specifically, we consider a wide range of relevant modal logics, extending the following system BM.C, based on the basic system considered in [3]:

(a1)	$p \to p$	(a7)	$q \to (p \lor q)$
(a2)	$\neg(p \land q) \to (\neg p \lor \neg q)$	(a8)	$((p \to q) \land (p \to r)) \to (p \to (q \land r))$
(a3)	$(\neg p \land \neg q) \to \neg(p \lor q)$	(a9)	$((p \to r) \land (q \to r)) \to ((p \lor q) \to r)$
(a4)	$(p \land q) \to p$	(a10)	$(p \land (q \lor r)) \to ((p \land q) \lor (p \land r))$
(a5)	$(p \land q) \to q$	(a11)	$(\Box p \land \Box q) \to \Box(p \land q)$
(a6)	$p \to (p \lor q)$	(a12)	$(\Box_L p \land \Box_L q) \to \Box_L(p \land q)$

plus the rules of Uniform substitution (US) and Modus ponens (MP) and

$$(\text{Adj})\ \frac{\varphi \qquad \psi}{\varphi \land \psi} \qquad\qquad (\text{Aff})\ \frac{\varphi' \to \varphi \qquad \psi \to \psi'}{(\varphi \to \psi) \to (\varphi' \to \psi')}$$

$$(\text{Con})\ \frac{\varphi \to \psi}{\neg\psi \to \neg\varphi} \qquad\qquad (\Box_L\text{-Mon})\ \frac{\varphi \to \psi}{\Box_L\varphi \to \Box_L\psi}$$

$$(\Box\text{-Mon})\ \frac{\varphi \to \psi}{\Box\varphi \to \Box\psi}$$

Then, for each relevant modal logic L, extending BM.C with axioms/rules corresponding to stronger propositional and modal properties of the agent, we develop a "classical" modal logic CL. The key feature of our framework, connecting L and CL, is the *relevant reasoning (meta)rule*

$$\frac{\vdash_{\mathsf{L}} \varphi_1 \land \ldots \land \varphi_n \to \psi}{\vdash_{\mathsf{CL}} \Box\varphi_1 \land \ldots \land \Box\varphi_n \to \Box\psi} \tag{RR}$$

for $n \geq 1$. In order to obtain closure under (RR), we introduce the auxiliary modal operator $\Box_L$ "expressing" provability in L in the sense that

$$\vdash_{\mathsf{L}} \varphi \iff \vdash_{\mathsf{CL}} \Box_L\varphi \tag{LCL}$$

Each L is closed under (C-Reg) and we will prove that CL proves $\varphi \to \psi$ if it proves $\Box_L(\varphi \to \psi)$. Hence, closure under (RR).

In order to ensure that the propositional fragment of CL is classical propositional logic, CPC, we modify the standard relational semantics of relevant modal logic. In our semantics based on so-called W-models, validity in a model is defined as satisfaction throughout a set of designated states that, as far as propositional connectives are concerned, behave like classical possible worlds.

We stress that while (RR) is satisfied, the standard logical omniscience problem is avoided in our framework since CL is generally *not* closed under (C-Reg) nor under Necessitation. This follows from the fact that while validity is defined as satisfaction in all standard states (in our case, possible worlds), the epistemic accessibility relation Q may connect standard states with non-standard states.

Definition 1. *A* bounded *frame is a relevant modal frame* $(S, \leq, R, *, Q, Q_L)$ *where* $R \subseteq S^3$ *is downward (upward) monotone in its first and second (third) argument,* $* : S \to S$ *is anti-monotonic and* $Q, Q_L \subseteq S^2$ *are downward (upward) monotone in their first (second) argument. Moreover,* $(S, \leq)$ *is a bounded poset, i.e. there are elements* $0, 1 \in S$ *such that for all* $s \in S$ $0 \leq s \leq 1$*, such that for all* $s, t \in S$*, the following are satisfied* ($Q_{(L)} \in \{Q, Q_L\}$)*:*

$$1^* = 0 \text{ and } 0^* = 1 \tag{1}$$

$$Q_{(L)}00 \tag{2}$$
$$Q_{(L)}1s \Rightarrow s = 1 \tag{3}$$
$$R010 \tag{4}$$
$$R1st \Rightarrow (s = 0 \text{ or } t = 1) \tag{5}$$

Relevant modal frames are a variant of frames as defined by Fuhrmann [3], but with a binary relation Q_L instead of the set of logical states L. The definition of bounded frames is taken from Seki [6].

Definition 2. *A W-*frame *is a structure $\boldsymbol{F} = (F, W)$ where F is a bounded frame, $W \subseteq S$ is a set of possible worlds, i.e. the following conditions are satisfied:*

$$w^* = w \tag{6}$$
$$Rwww \tag{7}$$
$$Rwst \Rightarrow (s = 0 \text{ or } w \leq t) \tag{8}$$
$$Rwst \Rightarrow (t = 1 \text{ or } s \leq w^*) \tag{9}$$
$$(\forall w \in W)(\forall s, t, u)(Q_L wu \ \& \ Rust \Rightarrow s \leq t) \tag{10}$$
$$(\forall s)(\exists w \in W)(\exists u)(Q_L wu \ \& \ Russ) \tag{11}$$

*A W-*model *based on $\boldsymbol{F}$ is $\boldsymbol{M} = (\boldsymbol{F}, V)$ where $V : Pr \to S(\uparrow)$, the set of upward closed subsets of S, such that $1 \in V(p)$ for all p and $0 \notin V(p)$ for all $p \in Pr$.*

Conditions (10)-(11) enable W-frames to simulate validity in relevant modal models. In W-frames, the set of states $Q_L(W) = \{u \mid \exists w(w \in W \ \& \ Q_L wu)\}$ "plays the role" of the set of logical states. For each W-frame F, we define the following operations on 2^S:

$$X \wedge^F Y = X \cap Y \qquad X \vee^F Y = X \cup Y$$
$$X \circ^F Y = \{u \mid \exists s, t(s \in X \ \& \ t \in Y \ \& \ Rstu)\}$$
$$X \to^F Y = \{s \mid \{s\} \circ^F X \subseteq Y\} \qquad \neg^F X = \{s \mid s^* \notin X\}$$
$$\Box^F X = \{s \mid \forall t(Qst \Rightarrow t \in X)\} \qquad \Box^F_L X = \{s \mid \forall t(Q_L st \Rightarrow t \in X)\}$$

and, for each W-model $\boldsymbol{M}$, the $\boldsymbol{M}$- *interpretation* $[\![\,]\!]_{\boldsymbol{M}}$ as a function $[\![\,]\!]_{\boldsymbol{M}} : Fm_{\mathcal{L}} \to S(\uparrow)$ such that $[\![p]\!]_{\boldsymbol{M}} = V(p)$ and

$$[\![c(\varphi_1, \ldots, \varphi_n)]\!]_{\boldsymbol{M}} = c^F\left([\![\varphi_1]\!]_{\boldsymbol{M}}, \ldots, [\![\varphi_n]\!]_{\boldsymbol{M}}\right)$$

for all $c \in \{\wedge, \vee, \to, \neg, \Box, \Box_L\}$. Crucially, a formula φ is valid in a class of W-frames iff it is valid in each W-model based on a W-frame belonging to the class, i.e. iff $W \subseteq [\![\varphi]\!]_{\boldsymbol{M}}$.

Given that validity is defined with respect to a special subset of situations, representing possible worlds, for all W-models $\boldsymbol{M}$ propositional formulas behave classically when interpreted at worlds $w \in W$. That is, we can prove that:

- $(\boldsymbol{M}, w) \models \neg\varphi$ iff $(\boldsymbol{M}, w) \not\models \varphi$
- $(\boldsymbol{M}, w) \models \varphi \to \psi$ iff $(\boldsymbol{M}, w) \not\models \varphi$ or $(\boldsymbol{M}, w) \models \psi$.

Definition 3. *For all relevant modal logics* L*, we define* CL *as the axiom system comprising*

1. CPC *with (MP) and (US) where substitutions are functions from Pr to $Fm_{\mathcal{L}}$;*

2. *for all axioms* φ *of* L*, an axiom* $\Box_L\varphi$*, and for all inference rules* $\dfrac{\varphi_1\dots\varphi_n}{\psi}$ *of* L*, the rule* $\dfrac{\Box_L\varphi\dots\Box_L\varphi_n}{\Box_L\psi}$*;*

3. *The Bridge Rule (BR)* $\dfrac{\Box_L(\varphi\to\psi)}{\varphi\to\psi}$.

The fact that each CL is closed under (RR) for all $n>0$ is established as follows. If $\vdash_{\mathsf{L}} \bigwedge_{i\leq n}\varphi_i\to\psi$, then $\vdash_{\mathsf{L}} \bigwedge_{i\leq n}\Box\varphi_i\to\Box\psi$ using monotonicity and regularity of $\Box$ in L, and so $\vdash_{\mathsf{CL}} \Box_L\big(\bigwedge_{i\leq n}\Box\varphi_i\to\Box\psi\big)$ by (LCL). But then $\vdash_{\mathsf{CL}} \bigwedge_{i\leq n}\Box\varphi_i\to\Box\psi$ follows using (BR).

Our main technical result is a general completeness theorem for CL with respect to W-models.

Theorem 1. *For any logic* L *and* W*-model* $\boldsymbol{M}$*,* $\vdash_{\mathsf{CL}}\varphi\Leftrightarrow W\subseteq[\![\varphi]\!]_{\boldsymbol{M}}$*.*

After proving the completeness theorem, we will discuss the following generalization of our framework. Each logic L considered in [3] contains the axiom (C) and is closed under (C-Reg). In contrast to [7], a case can be made against conjunctive regularity for $\wedge$, arguing that agents' beliefs tend to come in non-interacting clusters, or frames of mind [2], and therefore belief aggregation is not automatic. A natural generalisation of the present framework explores a neighborhood semantics for the epistemic modality, where crucially the collection of sets in the neighborhood of a state need not be closed under intersection. A relevant modal logic L based on neighborhood semantics then would have the congruence rule

$$\frac{\varphi\leftrightarrow\psi}{\Box\varphi\leftrightarrow\Box\psi}\,, \tag{Con}$$

as its only distinctively modal principle. We will outline how our completeness result generalizes to the neighborhood setting.

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Proof Theory for Intuitionistic Temporal Logic

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Topological dynamics is a branch of dynamical systems theory which studies the asymptotic behaviour of continuous functions on topological spaces. A (topological) dynamic system is a topological space $\mathcal{X} = (X, \tau)$ equipped with a continuous function $S : X \longrightarrow X$. Based on Tarski's observations that modal logic can be evaluated in topological spaces [10], Artemov et al. introduced in 1997 a temporal logic which extends modal logic by the next operator $\bigcirc$ to reason about topological dynamic systems [1]. From a temporal point of view the continuous function S can be regarded as a time-function which maps points of the topological space from one time moment to the next. The next operator is therefore used to reason about the behaviour of S. The work of Artemov et al. was later continued by Kremer and Mints [6] by extending their system with the temporal operators $\Diamond$ called eventually and $\Box$ called henceforth. The resulting system is called Dynamic Topological Logic (DTL). The addition of $\Diamond$ and $\Box$ substantially increases the expressive power of DTL and allows one to formulate interesting properties of dynamical systems. The project to build a logic to reason about topological dynamics however suffered a setback when Konev et al. proved that DTL is not decidable [5]. As a consequence of this result the focus of the project has shifted from DTL to an intuitionistic variant of DTL called Intuitionistic Temporal Logic (ITL). This focus shift is motivated by the observation that intuitionistic logic has better computational properties than classical logic and so it is hoped that ITL is decidable. Indeed, first results about ITL are promising: In 2018, Fernández-Duque established decidability of a fragment of ITL called $\mathsf{ITL}_\Diamond$ which only contains the next and the eventually operator [4]. Importantly, henceforth and eventually are not interdefinable in ITL (in contrast to DTL) as the base logic of ITL is intuitionistic. The proof of decidability relies on model theoretic techniques, in particular on the construction of so-called quasi models. Later, Boudou et al. proved completeness of this fragment with respect to the class of topological dynamic systems [2] by using similar techniques.

While the semantical aspects of ITL have been studied quite extensively in recent years, there is little known about the proof theory of ITL. Our long term goal is to fill this gap and provide a satisfying proof theory for intuitionistic temporal logic. For a start, we aim to investigate the proof theory of $\mathsf{ITL}_\Diamond$. Our project roughly consists of three main steps:

1. Define a sound and complete cyclic proof system for $\mathsf{ITL}_\Diamond$.
2. Establish cut-elimination either syntactically or by an indirect argument.
3. Use the cut-free system to obtain a syntactic decidability proof and invesitigate the complexity of the validity problem.

*Speaker.

At the point of writing this abstract we have completed step 1 and we are currently investigating the second step. In the following we describe in more detail each step.

For step 1 we define a cyclic proof system called $\mathsf{ITL}^c_\diamond$ which is based on a standard multi-conclusion sequent calculus for intuitionistic logic presented in [7]. This calculus is extended by rules for the next operator and the eventually operator. In particular, the rules for $\diamond$ are standard unfolding rules, which replace the formula $\diamond A$ by its equivalent unfolding $A \vee \bigcirc\diamond A$. The rules for $\diamond$ together with the cycle mechanism characterize the formula $\diamond A$ as the least fixed point of the function $X \mapsto A \vee \bigcirc X$. Importantly, as the logic $\mathsf{ITL}_\diamond$ is not only sound and complete with respect to topological dynamic models but also with respect to the class of dynamic Kripke frames [2], the topological semantics does not play a role in the presented calculus. As henceforth is not definable in our language, there does not exist any form of fixed point alternation in $\mathsf{ITL}_\diamond$. This implies that characterizing successful repetitions in a cyclic proof is a much easier task than for other fixed point logics such as the modal mu-calculus. In particular, we do not require a focus mechanism for our system. Soundness of $\mathsf{ITL}^c_\diamond$ is established by a minimal counter model approach which is common in the literature (see for example [9]). For completeness we consider a Hilbert style proof system for $\mathsf{ITL}_\diamond$ which is proven to be complete with respect to the class of topological dynamic systems in [2] and show how to embed it into the cyclic calculus $\mathsf{ITL}^c_\diamond$. As a consequence of this technique we do not obtain cut-free completeness, as the cut-rule is needed to derive the modus ponens rule. An important goal of our work is therefore to also establish cut free completeness, which brings us to step 2.

For step 2 we plan to establish a cut-elimination result by providing a syntactic cut-elimination procedure inspired by the continuous cut-elimination procedure of Savateev and Shamkanov in [8]. To that end we define a non-wellfounded proof system called $\mathsf{ITL}^n_\diamond$ for $\mathsf{ITL}_\diamond$. We first show how to unfold a cyclic proof into a non-wellfounded proof and vice versa, how to prune a non-wellfounded proof into a cyclic one. By doing so we establish soundness and completeness of the non-wellfounded system. Then a procedure is described which, given a non-wellfounded proof in which the cut-rule is applied, pushes the occurrence of the cut-rule upwards. By applying the procedure infinitely many times, we create an infinite sequence of non-wellfounded proofs which has the property that in each proof the first appearence of cut occurs above the first appearence of cut in the previous proof. By taking the limit of this construction, we obtain a cut-free non-wellfounded proof which can be pruned back into a cyclic proof. The main difficulty in this approach lies in showing that the limit of this sequence is a tree which satisfies the global trace conditions required for soundness of the system. In case such a cut-elimination procedure does not work for $\mathsf{ITL}^c_\diamond$ we would consider establishing cut-elimination indirectly by giving a completeness proof without cut via a standard proof search argument.

Finally, for step 3, we plan to establish decidabilty of $\mathsf{ITL}_\diamond$ by translating the non-wellfounded calculus $\mathsf{ITL}^n_\diamond$ minus cut into a parity game called proof search game. This proof search game is played by two players called Prover and Refuter. It is Prover's goal to show that a given sequent is derivable in $\mathsf{ITL}^n_\diamond$ and Refuter's goal to show the opposite, i.e. the sequent is not derivable. The positions of the game include all sequents that can be built from formulas in the Fischer-Ladner closure of the given sequent as well as every possible rule application including only such sequents. Whenever a match is in a position which is a sequent, it is Prover's turn and she can choose which rule to apply to that sequent. Next, it is Refuter's turn who can choose at which premise of the rule instance chosen by Prover the game continues. Therefore, a match in the proof search game corresponds to a finite or infinite path of a $\mathsf{ITL}^n_\diamond$-pre-proof.

Consequently, the strategy tree of Prover corresponds to some $\mathsf{ITL}^n_\diamond$-pre-proof. Observe that this corresponding pre-proof is analytic, as the game only consists of sequents that occur in the Fischer-Ladner closure of the endsequent. We establish the result that a sequent is provable in $\mathsf{ITL}^n_\diamond$ if and only if Prover has a positional winning strategy in the corresponding game. We then use a result proven by Calude et al. in [3] to establish the existence of an algorithm deciding for each sequent whether Prover has a positional winning strategy in the corresponding game and so whether the sequent is provable in $\mathsf{ITL}^n_\diamond$. The aforementioned result also establishes a first complexity bound for the validity problem of $\mathsf{ITL}_\diamond$. However, it is unclear whether such an approach would give us an optimal complexity bound. We plan to investigate this question and to give alternative decision procedures.

Our work is a continuation of the project to develop logics for reasoning about topological dynamics with good computational properties. We hope to provide a first insight into the proof theory of intuitionistic temporal logics and lay a foundation to investigate more complicated logics, in particular the logic ITL based on the full language with next, eventually and henceforth. The work on cut elimination is especially interesting, as surprisingly little can be found about this topic for cyclic proofs in general and we are interested in filling this gap. Furthermore, we hope to provide a new proof of decidability of $\mathsf{ITL}_\diamond$ which, in contrast to [4], relies entirely on syntactic arguments.

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Framing Faultiness Kripke Style

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Epistemic analysis has been used in distributed systems as a potent tool [1, 4] for studying agents' uncertainty about the global state of the system, including the global time in asynchronous systems. It is based on the *runs and systems framework* that views global states of a distributed system as possible worlds in a Kripke model. The importance of this methodology is underscored by the broadly applicable *Knowledge of Preconditions Principle* [8], formulated recently by Moses, which states that in all models of distributed systems, if φ is a necessary condition for agent i to perform an action, then agent i knowing that φ holds, written $K_i\varphi$, is also a necessary condition for this agent to perform this action. The agent's complete reliance on its local state as the source of information about the system naturally induces an equivalence relation on the global states, resulting in agents' knowledge being described by the multimodal epistemic logic $\mathsf{S5}_n$.

This epistemic analysis via the runs and systems framework was recently [5, 6] extended to *fault-tolerant systems* with so-called *byzantine* agents [7]. (Fully) byzantine agents are the worst-case faulty agents to participate in a distributed system: not only can they arbitrarily deviate from their respective protocols, but their perception of their own actions and the events they observe can be corrupted, possibly unbeknownst to them, resulting in false memories. Whether byzantine agents are actually present in a system or not, the very possibility of their presence has drastic and debilitating effects on the epistemic state of all agents, due to their inability to rule out the so-called *Brain-in-a-Vat Scenario* [9]. In a distributed system, a brain-in-a-vat agent is a faulty agent with completely corrupted perceptions that provide no reliable information about the system [6]. It has been shown that agents' inability to rule out being a brain in a vat precludes them from knowing many basic facts, including their own correctness/faultiness, in both asynchronous [6] and synchronous [10] distributed systems.

The extended runs and systems framework was used in [3] to analyze the *Firing Rebels with Relay* (FRR) problem, a simplified version of the *consistent broadcasting* primitive [11], which has been used as a pivotal building block in distributed algorithms, e.g., for byzantine fault-tolerant clock synchronization, synchronous consensus, etc. Instead of knowledge (unattainable due to the brain-in-a-vat scenario), the analysis of FRR hinges on a weaker epistemic notion called *hope*, which was initially defined as $H_i\varphi := \mathit{correct}_i \to K_i(\mathit{correct}_i \to \varphi)$ and axiomatized in [2] with the help of designated atoms $\mathit{correct}_i$, representing agent i's correctness, as an extension of $\mathsf{K45}_n$ with special axioms regarding atoms $\mathit{correct}_i$.

It turns out that defining faultiness $\mathit{faulty}_i := \neg\mathit{correct}_i$ as inconsistent hopes, i.e.,

$$\mathit{correct}_i \quad := \quad \neg H_i\bot,$$

makes it possible to deal away with designated atoms $\mathit{correct}_i$ and, hence, to avoid the dependency of accessibility relations $\mathcal{H}_i$ for hope modalities H_i on the valuation function in Kripke models for the logic of hope. In this formulation, the logic of hope becomes $\mathsf{KB4}_n$, the logic of the class $\mathcal{KB4}_n$ of transitive and symmetric frames and is axiomatized according to Fig. 1.

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$$
\begin{array}{llll}
 & P: \quad \text{all propositional tautologies} & & \\
K^H\!: & H_i(\varphi \to \psi) \land H_i\varphi \to H_i & B^H : & \varphi \to H_i\neg H_i\neg\varphi \\
4^H : & H_i\varphi \to H_iH_i\varphi & & \\
MP\!: & \dfrac{\varphi \quad \varphi \to \psi}{} & Nec^H\!: & \dfrac{\varphi}{H_i\varphi}
\end{array}
$$

Figure 1: Axiom system $\mathscr{H}$ for the logic of hope

Theorem 1 (Folklore)**.** *Logic $\mathscr{H}$ is sound and complete with respect to class $\mathcal{KB4}_n$.*

We demonstrate the utility of this reformulation of the logic of hope by encoding a standard limitation on the number of faulty agents in a fault-tolerant distributed system as a *frame-characterizable* property in logic $\mathscr{H}$. It is typical to formulate distributed protocols under the assumption that at most f of the n agents can become faulty ($0 \leq f < n$). This is a natural restriction given that clearly no outcome of agents' protocols can be guaranteed if, e.g., all agents can ignore these protocols. We can encode such requirements by an additional axiom

$$Byz_f \qquad := \qquad \bigvee_{\substack{G\subseteq\mathcal{A}\\|G|=n-f}} \ \bigwedge_{i\in G} \neg H_i\bot.$$

Remark 2. $Byz_0 = \bigwedge_{i\in\mathcal{A}} \neg H_i\bot$ simply states that all n agents are correct.

Proposition 3. *Axiom Byz_f is characterized by the* all-but-f-seriality *property of frames*

$$(\forall w \in W)(\exists G \subseteq \mathcal{A})\Big(|G| = n - f \land (\forall i \in G)\mathcal{H}_i(w) \neq \varnothing\Big),$$

where $\mathcal{H}_i(u) := \{y \in W \mid u\mathcal{H}_iy\}$. In other words, each world must have outgoing arrows for all but f agents.

Definition 4. *Class $\mathcal{KB4}_n^{n-f}$* consists of all frames from $\mathcal{KB4}_n$ that are all-but-f-serial.

Corollary 5. *$\mathscr{H} + Byz_f$ is sound and complete with respect to $\mathcal{KB4}_n^{n-f}$.*

While hope alone is sufficient to restrict the number of faulty agents, we argue that the proper language for reasoning about agents' uncertainty in distributed systems with fully byzantine agents should include both hope H_i and knowledge K_i modalities for all agents. Thus, on the Kripke side, one needs to add accessibility relations $\mathcal{K}_i$ for the K_i modalities. In this language, the connection between knowledge and hope of agent i is represented by the (almost) frame characterizable axiom KH (each direction of equivalence (1) is characterized separately):

$$H_i\varphi \quad \leftrightarrow \quad \big(\neg H_i\bot \to K_i(\neg H_i\bot \to \varphi)\big). \tag{1}$$

Proposition 6. *On the class of frames with shift serial $\mathcal{H}_i$, i.e., with outgoing $\mathcal{H}_i$-arrows whenever there are incoming ones, the right-to-left direction of (1) is characterized by frame property $\mathcal{H}$in$\mathcal{K}$ stating that $\mathcal{H}_i \subseteq \mathcal{K}_i$.*

Proposition 7. *The left-to-right direction of (1) is characterized by frame property* one$\mathcal{H}$ *stating that*

$$(\forall w, v \in W) \qquad \big(\mathcal{H}_i(w) \neq \varnothing \land \mathcal{H}_i(v) \neq \varnothing \land w\mathcal{K}_iv \quad \Longrightarrow \quad w\mathcal{H}_iv\big).$$

It turns out that the $\mathsf{KB4}_n$ properties of hope can be derived in the combined logic $\mathscr{KH}$ of hope and knowledge that is obtained by extending $\mathsf{S5}_n$ for knowledge modalities with the connection axiom KH from (1) and the *necessary consistency* axiom $H^\dagger := H_i \neg H_i \bot$ for hope ($H^\dagger$ is characterized by shift seriality).

Theorem 8. *Logic $\mathscr{KH}$ is sound and complete with respect to class $\mathcal{KH}$ of models where every $\mathcal{K}_i$ is an equivalence relation, every $\mathcal{H}_i$ is shift serial, and properties $\mathcal{H}\mathrm{in}\mathcal{K}$ and $\mathrm{one}\mathcal{H}$ are satisfied.*

Proposition 9. *In class $\mathcal{KH}$, each accessibility relation $\mathcal{H}_i$ is symmetric and transitive. Hence, $\mathcal{H}_i$ are partial equivalence relations, so that property $\mathrm{one}\mathcal{H}$ can be described as "no $\mathcal{K}_i$-equivalence class contains more than one $\mathcal{H}_i$-partial-equivalence class."*

Corollary 10 (In fault-free systems, hope is knowledge)**.** $\mathscr{KH} + Byz_0 \quad \vdash \quad H_i\varphi \leftrightarrow K_i\varphi.$

We now use the language of hope and knowledge to formalize the consequences of the *brain-in-a-vat scenario.* These consequences were first established in [6] via a semantic analysis of runs and systems models:

- $iByz := \neg K_i \neg H_i \bot$, i.e., agents cannot reliably establish their own correctness;
- $BiV := H_i\bot \to \neg K_i H_j \bot \land \neg K_i \neg H_j \bot$ for $i \neq j$, i.e., a faulty agent lacks any reliable information about other agents, such as whether another agent is correct or faulty.

From these two principles, we can derive by purely syntactic means that no agent knows whether other agents are correct or faulty, as proved in [6] by semantic methods:

Proposition 11. $\mathscr{KH} + iByz + BiV \quad \vdash \quad \neg K_i \neg H_j \bot \land \neg K_i H_j \bot \qquad$ *for all $i \neq j$.*

Proposition 12. *Axiom $iByz$ is characterized by the* i-may-aseriality *frame property requiring $(\forall w \in W)\big(\exists w' \in \mathcal{K}_i(w)\big) \quad \mathcal{H}_i(w') = \varnothing$, stating that each world has a $\mathcal{K}_i$-indistinguishable world with no $\mathcal{H}_i$-outgoing arrows. Axiom BiV for $i \neq j$ is characterized by the* BiValence *frame property requiring*

$$(\forall w \in W) \qquad \Big(\mathcal{H}_i(w) = \varnothing \quad \Longrightarrow \quad \big(\exists w', w'' \in \mathcal{K}_i(w)\big)\big(\mathcal{H}_j(w') \neq \varnothing \land \mathcal{H}_j(w'') = \varnothing\big)\Big).$$

We can also easily derive by purely modal means that the brain-in-a-vat scenario is not compatible with fault-free systems: $\mathscr{KH} + Byz_0 \quad \vdash \quad \neg iByz \qquad$ for each $i \in \mathcal{A}$.

Another interesting special case is $f = 1$. On the one hand, half of BiV becomes derivable and, hence, redundant. If any agent and no more than one can be faulty, then agents cannot establish the faultiness of other agents: $\mathscr{KH} + Byz_1 + iByz \quad \vdash \quad \neg K_i H_j \bot \qquad$ for all $i \neq j$.

On the other hand, the other half of BiV leads to undesirable consequences. For $f = 1$, the inability of faulty agents to establish correctness of others would lead to the inability of any agent to establish own faultiness: $\mathscr{KH} + Byz_1 + (H_i\bot \to \neg K_i \neg H_j \bot) \quad \vdash \quad \neg K_i H_i \bot \qquad$ for all $i \neq j$.

Remark 13. Intuitively, if an agent establishes its own faultiness, which does not run afoul of $iByz$ and can be used, e.g., for self-repairing agents, then it will thereby establish the correctness of all other agents. Prohibiting this by the respective half of BiV should be avoided, while the other half is derivable anyway. We, therefore, propose to use $\mathscr{KH} + Byz_f + BiV + iByz$ for $f \geq 2$ or $\mathscr{KH} + Byz_1 + iByz$ for $f = 1$.

Theorem 14. *Axiom system $\mathscr{KHC}$ for common knowledge and common hope consisting of all the axioms of $\mathscr{KH}$ plus the following axioms and inference rules:*

$$Mix^H := C_G^H\varphi \to E_G^H(\varphi \land C_G^H\varphi) \qquad Ind^H: \ \textit{from } \psi \to E_G^H(\varphi \land \psi), \textit{ infer } \psi \to C_G^H\varphi$$
$$Mix^K := C_G^K\varphi \to E_G^K(\varphi \land C_G^K\varphi) \qquad Ind^K: \ \textit{from } \psi \to E_G^K(\varphi \land \psi), \textit{ infer } \psi \to C_G^K\varphi$$

is sound and complete with respect to class $\mathcal{KH}$.

In summary, we provided a description of epistemic views and uncertainties of agents in fault-tolerant distributed systems with fully byzantine agents by means of a multimodal logic with two types of modalities, hope and knowledge (including common hope and common knowledge), proved completeness, and showed how system specifications and properties of such agents can be represented by frame-characterizable properties. This analysis yielded new insights, for instance, into the distinctions between the case of fault-tolerant systems with at most one vs. several byzantine agents. This distinction was already observed in [6] but the newly provided axiomatic representation explains which of the general properties of byzantine agents are violated when all but one agents are correct.

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Local Modal Product Logic is decidable

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Expansions with modal operators of many-valued logics have been proposed and studied in the literature following two main approaches. In this work, we contribute to the one introduced by Fitting [7, 8] and Hajek [9], which is based on considering a semantical definition of these logics which enriches the Kripke semantics with evaluations over corresponding many-valued algebras. In the literature, special attention has been devoted to modal expansion of the fuzzy logics associated with the basic continuous t-norms: Łukasiewicz modal logics [10], Gödel modal logics [3, 4, 12] and Product modal logics [15]. Most of these studies focus on the logics arising from the semantics with classical Kripke frames (namely, where the accessibility relation between worlds of the Kripke models is still a binary relation, and it is not valued over the algebra), and where the variables at the worlds of the model are the only elements evaluated over the corresponding algebras. In the following, we will refer by minimal modal fuzzy logics to those defined in this fashion, with $\Box$ ($\Box$-fragment), $\Diamond$ ($\Diamond$-fragment) or both modal operators (bi-modal logic). In general, the operators $\Box$ and $\Diamond$ are not interdefinable, and the logic with the two modal operators is possibly strictly weaker than the addition of the corresponding two mono-modal fragments [12]. Further, two logical consequences naturally arise from the same semantics: the local (where truth of the premises implies truth of the consequence world-wise) and global (where truth of the premises in the whole model implies the same for the consequence).

In different works, several of the decision problems concerning minimal modal fuzzy logics have been closed. Due to their very different characteristics, the studies in each case exploit particularities of each of the logics, relying little on general tools. It is known that the minimal local modal logics expanding Gödel logic are decidable [1, 2], that global modal Łukasiewicz[1] and bi-modal Product logics are undecidable [14] and that local modal Łukasiewicz logic is decidable [13]. Further, it is known that the analogous of the previous local logics for many-valued Kripke frames are decidable, which can be found in the same publications and, for the product case, it follows from the results in [5].

Two main questions concerning decidability of the previous minimal logics remain open: it is not known whether global Gödel modal logics are decidable or not, and it was also not known whether local bi-modal Product logic was decidable. Remarkably enough, the approach used in [3, 4] cannot be used to solve the first question, and the approach from [5] also does not serve as inspiration for proving decidability of local bi-modal Product logic for models with crisp accessibility relation.

In this work, we answer positively to the second open problem, and show that local bi-modal Product logic is decidable. Let us formally introduce the logic and sketch the ideas that allow us to conclude the decidability result stated before. Let $\mathcal{V}$ be a countable set of variables, and Fm the set of formulas over $\mathcal{V}$ with the algebraic language $\langle \odot/2, \rightarrow /2, \bot/0, \Box/1, \Diamond/1\rangle$. The interpretations of the symbols $\odot, \rightarrow$ and $\bot$ in a product algebra $\mathbf{A}$ is the natural one corresponding to its algebraic operations. For the definition of Product algebra and Product logic, see for instance [9].

[1]In which $\Box$ and $\Diamond$ are inter-definable, and so minimal fragments concerning modal operators all collapse.

Definition 1.1. Let $\mathbf{A}$ be a Product algebra. A (crisp) A-Kripke model $\mathfrak{M}$ is a structure $\langle W, R, e\rangle$ where W is a non-empty set, R is a binary relation over W and $e\colon W \times \mathcal{V} \to A$. A Kripke model uniquely determines an $\mathbf{A}$-Kripke model by extending the evaluation[2] to $e\colon W \times Fm \to A$ as follows:

$$e(v,\bot) \coloneqq 0^{\mathbf{A}}, \qquad e(v,\varphi\star\psi) \coloneqq e(v,\varphi)\star^{\mathbf{A}} e(v,\psi) \qquad \text{for } \star\in\{\odot,\to\}$$
$$e(v,\Box\varphi) \coloneqq \bigwedge_{Rvw} e(w,\varphi), \qquad e(v,\Diamond\varphi) \coloneqq \bigvee_{Rvw} e(w,\varphi)$$

We let $\mathbb{K}_\Pi$ denote the class of all $[0,1]_\Pi$-Kripke models, for $[0,1]_\Pi$ the standard Product algebra. For a set of formulas $\Gamma\cup\{\varphi\}\subseteq Fm$, we will write $\Gamma\vdash_{\mathbb{K}_\Pi}\varphi$ whenever for every $\mathfrak{M}\in\mathbb{K}_\Pi$ and every $v\in W$, if $e(v,\gamma)=1$ for each $\gamma\in\Gamma$ then $e(v,\varphi)=1$ as well. This entailment relation is what we referred to in the introduction as the local bi-modal Product logic. For convenience, for a model $\mathfrak{M}$ we write $\Gamma\not\vdash_{\mathfrak{M}}\varphi$ whenever there is some $v\in W$ for which $e(v,\Gamma)\subseteq\{1\}$ and $e(v,\varphi)<1$.

The main result we will present in the conference is the following.

Theorem 2.1. *For a finite set of formulas $\Gamma\cup\{\varphi\}\subset_\omega Fm$, the problem of determining whether $\Gamma\vdash_{\mathbb{K}_\Pi}\varphi$ is decidable. Consequently, the set of theorems of the logic $\vdash_{\mathbb{K}_\Pi}$ is recursive.*

In the rest of the abstract, we will sketch the ideas that allow to prove the previous result.

A formula formula $\heartsuit\varphi$ starting with a modality $\heartsuit\in\{\Box,\Diamond\}$ is said to be witnessed in a world v of a model $\mathfrak{M}$ if there is a world w with Rvw and such that $e(v,\heartsuit\varphi)=e(w,\varphi)$. A model is witnessed if every formula is witnessed at every world of the model. It is known that modal product logic, as predicate product logic, is not complete with respect to witnessed models, which contrasts with the Łukasiewicz case. Nevertheless, it is complete with respect to so-called quasi-witnessed models. These are models where unwitnessing situations are rather limited: for each world v in the model and each formula $\heartsuit\varphi$ starting with a modality, either the formula is witnessed in v or the formula is of the form $\Box\varphi$ and $e(v,\Box\varphi)=0$.

In this work, in order to prove decidability of $\vdash_{\mathbb{K}_\Pi}$, we rely in a more specific result, of which quasi-witnessed completeness is a corollary. In [11] it is proven that predicates (and so, modal) product logic is complete with respect to models valued over a particular product algebra.

Definition 2.2. The ***lexicographic sum*** $\mathbb{R}^{\mathbb{Q}}=\langle\mathbb{R}^{\mathbb{Q}},+,\leqslant\rangle$ is the ordered abelian group of functions $f\colon\mathbb{Q}\to\mathbb{R}$ whose support is well-ordered (namely, such that $\{q\in\mathbb{Q}\colon f(q)\neq 0\}$ is a well-ordered subset of $\mathbb{Q}$). Addition is defined component-wise and the ordering on $\mathbb{R}^{\mathbb{Q}}$ is lexicographic.

The transformation $\mathfrak{B}$ introduced in [6] can be applied to the previous l-group, obtaining a product chain. Let us denote $(\mathbb{R}^{\mathbb{Q}})^-=\{a\in\mathbb{R}^{\mathbb{Q}}\colon a\leqslant\mathbf{0}\}$, where $\mathbf{0}$ stands for the neutral element of the group, namely, the constant function 0. Then, $\mathfrak{B}(\mathbb{R}^{\mathbb{Q}})$ is the product algebra with universe $(\mathbb{R}^{\mathbb{Q}})^-\cup\{\bot\}$ where the order is inherited from $\mathbb{R}^{\mathbb{Q}}$ (thus $\mathbf{0}$ is the maximum element) and $\bot$ is the minimum element, and the operations of the algebra are defined by letting

$$x\odot y=\begin{cases}x+y & \text{if } x,y\in(\mathbb{R}^{\mathbb{Q}})^-,\\ \bot & \text{otherwise}\end{cases}\qquad x\to y=\begin{cases}\mathbf{0}\wedge(y-x) & \text{if } x,y\in(\mathbb{R}^{\mathbb{Q}})^-,\\ \mathbf{0} & \text{if } x=\bot\\ \bot & \text{if } y=\bot \text{ and } x\in(\mathbb{R}^{\mathbb{Q}})^-\end{cases}$$

Theorem 2.9 from [11] is stated for predicate product logic (over all product chains), and so, we can restrict it naturally to the modal standard product logic as follows:

[2] If the corresponding infima/suprema do not exist, does values are undefined.

Corollary 2.3 (From Theorem 2.9, [11]). *Let $\Gamma \cup \{\varphi\}$ a set of modal formulas such that $\Gamma \not\models_{\mathbb{K}_\Pi} \varphi$. Then there is a countable, quasi-witnessed $\mathfrak{B}(\mathbb{R}^{\mathbb{Q}})$-Kripke structure $\mathfrak{M}$ such that $\Gamma \not\models_{\mathfrak{M}} \varphi$.*

Models over the above algebra are not only quasi-witnessed, but, when analyzed paying attention to a finite set of formulas Σ[3], they satisfy more specific conditions. For if we have a formula $\Box\varphi \in \Sigma$ unwitnessed in a world v in a model, it holds that there is some world $v_{\Box\varphi}$[4] with $Rvv_{\Box\varphi}$ and such that there is a negative rational number q for which

$$\begin{aligned}
&e(v_{\Box\varphi}, \varphi)[q] < 0,\\
&e(v_{\Box\varphi}, \varphi)[p] = 0 \text{ for all } p < q, \text{ and}\\
&e(v_{\Box\varphi}, \psi)[p] = 0 \text{ for all } p \leqslant q \text{ and all } \Box\psi \in \Sigma \text{ such that } e(v, \Box\psi) > 0.
\end{aligned}$$

For each $\psi, \Box\varphi \in \Sigma$, $v \in W$ and $\Box\varphi$ unwitnessed in v, consider the element of the algebra $\alpha(v_{\Box\varphi}, \psi)$ defined by $\bot$ if $e(v_{\Box\varphi}, \psi) = \bot$ and, in other case,

$$\alpha(v_{\Box\varphi}, \psi)[p] := \begin{cases} 0 & \text{if } p > q \\ e(v_{\Box\varphi}, \psi)[p] & \text{otherwise} \end{cases}$$

Observe that $\bot < \alpha(v_{\Box\varphi}, \varphi) < \top$, and $\alpha(v_{\Box\varphi}, \psi) = \top$ for each $\Box\psi \in \Sigma$ such that $e(v, \Box\psi) > \bot$.

By iterating the previous idea, it is possible to extend any model with additional worlds in such a way that, for every unwitnessed formula $\Box\varphi$ at a world v, we have two special successors of v, $v_{\Box\varphi}$ and $v'_{\Box\varphi}$, such that for each formula $\psi \in \Sigma$ there is a value $\alpha_{\langle v, \Box\varphi\rangle}(\psi)$ for which:

$$\begin{aligned}
&\bot < \alpha_{\langle v, \Box\varphi\rangle}(\varphi) < \top\\
&\alpha_{\langle v, \Box\varphi\rangle}(\psi) = \top \text{ for each } \Box\psi \in \Sigma \text{ with } e(v, \Box\psi) > \bot\\
&\alpha_{\langle v, \Box\varphi\rangle}(\psi) = \bot \text{ iff } e(v_{\Box\varphi}, \psi) = \bot,\\
&\text{if } e(v_{\Box\varphi}, \psi_1) \leqslant e(v_{\Box\varphi}, \psi_2) \text{ then } \alpha_{\langle v, \Box\varphi\rangle}(\psi_1) \leqslant \alpha_{\langle v, \Box\varphi\rangle}(\psi_2) \text{ for each } \psi_1, \psi_2 \in \Sigma,\\
&e(v'_{\Box\varphi}, \psi) = e(v_{\Box\varphi}, \psi) + \alpha_{\langle v, \Box\varphi\rangle}(\psi)
\end{aligned}$$

Restricting this extended model $\mathfrak{M}^+$ to the worlds witnessing the witnessed formulas from Σ and to the pairs above whenever an unwitnessed formula appears, we obtain a finite model $\mathfrak{M}'$ with the above characteristics. Observe that the values themselves are not relevant, and only the information concerning which successor is the witness of each formula, and the above information for what concerns the unwitnessed formulas. All this information can be faithfully encoded with a derivation in the propositional product logic with Δ, let say $\Gamma_{\mathfrak{M}'} \vdash_\Pi \varphi_{\mathfrak{M}'}$, whose involved formulas are constructively defined from the original ones and the model. Since all the possible combinations of unwitnessed formulas form a finite set, the set of possible models from which we start can be also taken as finite, $\{\mathfrak{M}_1 \dots, \mathfrak{M}_n\}$ (where n is bounded exponentially by the modal depth of Σ), which allows to claim the following:

Proposition 3.1. *If $\Gamma \not\models_{\mathbb{K}_\Pi} \varphi$ there is some $\mathfrak{M}_i$ in $\{\mathfrak{M}_1 \dots, \mathfrak{M}_n\}$ such that $\Gamma_{\mathfrak{M}'_i} \not\models_{\Pi_\Delta} \varphi_{\mathfrak{M}'_i}$.*

The key part is that, from the previous propositional condition we can build back a $[0,1]_\Pi$-Kripke model (which will be infinite whenever any of the variables associated to the values $\alpha_{\langle v, \Box\varphi\rangle}(\psi)$ is different from 1) with the desired property, namely:

[3] Since we will be addressing the decidability question for entailments of a formula φ from a finite set of premises Γ, in practice Σ will be the closure under subformulas of the set $\Gamma \cup \{\varphi\}$.

[4] In fact, infinitely many ones.

Proposition 3.2. *If there is some $\mathfrak{M}_i$ in $\{\mathfrak{M}_1 \ldots, \mathfrak{M}_n\}$ such that $\Gamma_{\mathfrak{M}'_i} \not\vdash_{\Pi_\Delta} \varphi_{\mathfrak{M}'_i}$, then we can construct an $[0,1]_\Pi$-Kripke model $\mathfrak{N}$ from $\Gamma_{\mathfrak{M}'_i} \cup \{\varphi'_{\mathfrak{M}_i}\}$ such that $\Gamma \not\vdash_{\mathfrak{N}_i} \varphi$.*

This construction is done by defining, for each variable associated to a value $\alpha_{\langle v, \Box\varphi\rangle}(\psi)$ that is below 1, an infinite set of points v_i in each of which the value of each formula $\psi \in \Sigma$ is sent to the value taken by ψ in v [5] multiplied by $\alpha_{\langle v, \Box\varphi\rangle}(\psi)^i$. This allows us to replicate, in a regular way, the behavior of the unwitnessed formulas, without affecting the others.

Since the logic $\vdash_{\Pi_\Delta}$ is decidable, and the previous constructions are recursive, this allows to immediately conclude the decidability of $\vdash_{\mathbb{K}_\Pi}$.

Furthermore, this approach proves that the modal logic arising from the Kripke models evaluated over all Product algebras and that arising from the models evaluated over the standard Product algebra coincide.

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[5] This comment refers to the fact that in $\Gamma_{\mathfrak{M}'_i} \cup \{\varphi'_{\mathfrak{M}_i}\}$, the propositional variables are of the form x_v, for x in $\mathcal{V}$ or a formula begining by a modality. Thus, we can talk about the value of a formula in a world even if we are in fact working with a propositional homomorphism.

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One-sorted Program Algebras*

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A *Kleene algebra* [6] is an idempotent semiring $(K, \cdot, +, 1, 0)$, and hence a semilattice with partial order $x \leq y \iff x + y = y$, which is expanded with an operation ${}^* : K \to K$ such that

$$1 + xx^* \leq x^* \qquad y + xz \leq z \implies x^*y \leq z$$
$$1 + x^*x \leq x^* \qquad y + zx \leq z \implies yx^* \leq z.$$

A standard example of a Kleene algebra is a relational Kleene algebra where K is a set of binary relations over some set S, $\cdot$ is relational composition, $+$ is set union, * is reflexive transitive closure, 1 is identity on S and 0 is the empty set; another standard example is the Kleene algebra of regular languages over some finite alphabet.

A *Kleene algebra with tests* $\mathbf{K} = (K, B, \cdot, +, {}^*, 1, 0, {}^-)$, a.k.a. a *KAT*, is a two-sorted structure where $(K, \cdot, +, {}^*, 1, 0)$ is a Kleene algebra, and $B \subseteq K$ with $(B, \cdot, +, {}^-, 1, 0)$ a Boolean algebra [7]. The inference rules of Propositional Hoare logic (PHL) can be derived in (the equational theory of) KAT [8], i.e., it is a simple algebraic framework for verifying properties of propositional while programs. KAT is PSPACE-complete [2], has computationally attractive fragments [12], and its extensions have been applied beyond while programs, for instance in network programming languages [1].

Every Kleene algebra is a KAT; take $B = \{0, 1\}$ and define $\bar{0} = 1$ and $\bar{1} = 0$. A standard example of a KAT is a relational Kleene algebra (*rKAT*) expanded with a Boolean subalgebra of the *negative cone*, i.e. the elements $x \leq 1$, which in the relational case are subsets of the identity relation. The equational theories of KAT and rKAT coincide [9].

For various reasons, a one-sorted alternative to KAT may be desirable. For instance, "one-sorted domain semirings are easier to formalise in interactive proof assistants and apply in program verification and correctness" [4, p. 576]. A one-sorted alternative called *Kleene algebra with antidomain* was introduced in [3].

A *domain* operation [3] on a semiring $\mathbf{A}$ is any $d : K \to K$ such that

$$d(x) \leq 1 \qquad x \leq d(x)x$$
$$d(0) = 0 \qquad d(x + y) = d(x) + d(y)$$
$$d(xy) = d(xd(y)). \tag{1}$$

On a relational Kleene algebra one can define the *relational domain* operation d as follows

$$d(R) := \{(s, s) \mid \exists u.(s, u) \in R\}.$$

Then d satisfies the domain axioms above, and in fact the equational theory of domain semirings coincide with the equational theory of relation algebras in the signature $(\cdot, +, 0, 1, d)$ [10], but

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not the quasi-equation theory, and not necessarily in the signature which includes the Kleene-*. Informally, $d(R)$ represents the set of states in which the program associated with R has a terminating computation.

If $\mathbf{K}$ is a Kleene algebra with domain operation d, then $d(K) := \{y \mid \exists x.y = d(x)\}$ is a bounded distributive lattice contained in the negative cone in which $\cdot$ is the meet operation [3]. (It is an open problem to determine under which conditions $d(K)$ is a Heyting algebra.) In order to obtain a Boolean algebra from the distributive lattice $d(\mathbf{K})$, one has to make sure that each test $d(x)$ is complemented in $d(K)$, that is, for each $d(x)$ there is $y \in d(K)$ such that $d(x)y = 0$ and $d(x) + y = 1$. An elegant solution to this problem presented in [3] consists in expanding Kleene algebras with a single unary operation a (*antidomain*) that allows to define a domain operation d and has properties entailing that $a(x)$ is a complement of $d(x)$.

A *Kleene algebra with antidomain*, *KAA*, is a Kleene algebra expanded with an operation $a : K \to K$ such that

$$\begin{aligned} a(x)x &= 0 \\ a(xy) &\leq a(x\,a(a(y)) \\ a(x) + a(a(x)) &= 1 \end{aligned}$$

If one defines $d(x) := a(a(x))$, then d is a domain operation, and $a(x)$ is a complement of $d(x)$, so that $d(\mathbf{K}) = (d(K), \cdot, +, 1, 0)$ is a Boolean algebra [3]. On a relational Kleene algebra $a(R) = \{(w, w) \mid \neg\exists v. (w, v) \in R)\}$ is an antidomain operation and $a(a(R)) = d(R)$.

It is known that KAA is decidable in EXPTIME [11], and KAA can be used to create modal operators that invert the sequential composition rule of PHL. Such inversions are derivable from KAA but not KAT [13]. However, KAA has certain features that may be undesirable depending on the application. First, if $\mathbf{K}$ is a KAA, $d(\mathbf{K})$ is necessarily the maximal Boolean subalgebra of the negative cone of $\mathbf{K}$; see Thm. 8.5 in [3]. In a sense, then, every "proposition" is considered a test, contrary to some of the intuitions expressed in [7]. These intuitions also collide with the approach of taking KAT as KA with a Boolean negative cone [4, 5]. Second, not every Kleene algebra expands to a KAA, not even every finite one; see Prop. 5.3 in [3]. This is in contrast to the fact that every Kleene algebra expands to a KAT. This feature is caused by (1) (called *locality*) and the authors of [3] express interest in variants of d not satisfying (1).

In this talk we generalize KAA to a framework we'll call *one-sorted Kleene algebra with tests*. A *KAt* is a Kleene algebra expanded by two unary operations t and t' such that

$$\begin{aligned} t(0) &= 0 & t(1) &= 1 \\ t(t(x) + t(y)) &= t(x) + t(y) & t(t(x)t(y)) &= t(x)\,t(y) \\ t(x)t(x) &= t(x) & t(x) &\leq 1 \\ 1 &\leq t'(t(x)) + t(x) & t'(t(x))\,t(x) &\leq 0 \\ t'(t(x)) &= t(t'(t(x))). \end{aligned}$$

Already KAt has most of the desired features of KAA: every KAt contains a Boolean subalgebra of tests (obtained as the image of t, where t' is complementation on test elements), and the equational theory of KAT embeds into the equational theory of KAt. In addition, every Kleene algebra expands into a KAt (ensuring that it is a conservative expansion), and the subalgebra of tests in KAt is not necessarily the maximal Boolean subalgebra of the negative cone. We then consider various extensions of KAt with axioms known from KAA to show which properties of the domain operator are still consistent with the desired features of KAt. For example, the equational theory of KAT embeds into a class K of KAt's provided one of the following sufficient conditions hold: (1) every KAT 'expands' to a member of K, or (the more restrictive) (2) every rKAT 'expands' to a member of K. We say that a KAT $\mathbf{K} = (K, B, \cdot, +, ^*, 1, 0, ^-)$ *expands* into a KAt $\mathbf{A} = (K, \cdot, +, ^*, 1, 0, t, t')$ iff $B = t(K)$. The variety of KAt's satisfy (1) while the variety of KAA's only satisfies (2). In addition, we consider a variant of the KAt framework where test complementation is defined using a residual of Kleene algebra multiplication.

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Modules with Fusion and Implication based over Distributive Lattices

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Abstract

In this paper we introduce the class of **DLFI**-modules, i.e. *Modules with Fusion and Implication based over Distributive Lattices.* We extend the well known duality between distributive lattices and Priestley spaces, in order to exhibit a bi-lattice Priestley-like duality for **DLFI**-modules. We prove that the category of **DLFI**-modules is dually equivalent to the category of Urquhart spaces.

Bounded lattices with additional operators occur often as algebraic models of Non-Classical Logics. This is the case of Boolean algebras which are the algebraic semantics of classical logic, Heyting algebras which model intuisionistic logic, BL-algebras which correspond to algebraic semantics of basic propositional logics [5], MTL-algebras which are the algebraic semantics of the basic fuzzy logic of left-continuous t-norms [3], Modal algebras which model propositional modal logics [2], to name a few. In all these cases, the operations $\vee$ and $\wedge$ model logical disjunction and conjunction, and the additional operations are usually interpretations of other logical connectives such as the modal connectives for necessity ($\Box$) or possibility ($\Diamond$), or various types of implication. All these operations has as a common property: *The preservation of some part of the lattice structure*, i.e.

$$\Box(1) = 1, \qquad \Box(x \wedge y) = \Box(x) \wedge \Box(y), \tag{1}$$

$$\Diamond(0) = 0, \qquad \Diamond(x \vee y) = \Diamond(x) \vee \Diamond(y). \tag{2}$$

In some sense, the aforementioned may suggest that these ideas can be treated as a more general phenomenon which can be studied by employing tools of universal algebra. Some papers in which this approach is used are [4] and [6]. Nevertheless, in an independent way, a more concrete treatment of the preservation of the lattice structure by two additional connectives in a distributive lattice led to the introduction of a new class of algebras, the class of Distributive Lattices with Fusion and Implication [1], which encompasses all the algebraic structures mentioned before. In this paper a Priestley-like duality is developed, extending the duality obtained in [7] for algebras of Relevant logics.

The aim of this paper is to exhibit a bi-lattice Priestley-like duality between the class of *Modules with Fusion and Implication based over Distributive Lattices* or **DLFI**-modules, for short, and Urquhart spaces. The results we obtain generalize the ones obtained by Celani in [1].

Definition 1. Let **A**, **B** be two bounded distributive lattices. We shall say that a structure $\langle \mathbf{A}, \mathbf{B}, f \rangle$ is a ***DLFI**-module*, if $f\colon A \times B \to A$ is a function such that for every $x, y \in A$ and every $b, c \in B$ the following conditions hold:

(F1) $f(x \vee y, b) = f(x, b) \vee f(y, b)$,

(F2) $f(x, b \vee c) = f(x, b) \vee f(x, c)$,

(F3) $f(0^{\mathbf{A}}, b) = 0^{\mathbf{A}}$,

(F4) $f(x, 0^{\mathbf{B}}) = 0^{\mathbf{A}}$.

We shall say that a structure $\langle \mathbf{A}, \mathbf{B}, i \rangle$ is a ***DLI**-module*, if $i \colon B \times A \to A$ is a function such that for every $x, y \in A$ and every $b, c \in B$ the following conditions hold:

(I1) $i(b, x \wedge y) = i(b, x) \wedge i(b, y)$,

(I2) $i(b \vee c, x) = i(b, x) \wedge i(c, x)$,

(I3) $i(b, 1^{\mathbf{A}}) = 1^{\mathbf{A}}$.

Finally, we shall say that a structure $\langle \mathbf{A}, \mathbf{B}, f, i \rangle$ is a ***DLFI**-module*, if $\langle \mathbf{A}, \mathbf{B}, f \rangle$ is a **DLF**-module and $\langle \mathbf{A}, \mathbf{B}, i \rangle$ is a **DLI**-module.

We stress that, when considering some particular cases, the notion of **DLFI**-modules collapses into important examples of structures well known in the litterature. For instance, in the case of $B = A$, if we regard the functions $x \circ y = f(x, y)$ and $x \to y = i(x, y)$, it follows that the structure $\langle \mathbf{A}, \circ, \to \rangle$ is a bounded distributive lattice with fusion and implication [1]. Other example of interest arise when we consider the case $B = \{1\}$ and we regard the functions $\Diamond(x) = f(x, 1)$ and $\Box(x) = i(1, x)$. It is easy to see that $\langle \mathbf{A}, \Diamond, \Box \rangle$ is a bounded distributive lattice with a possibility modal operator and a necessity modal operator [2].

Let $\langle \mathbf{A}, \mathbf{B}, f, i \rangle$, $\langle \hat{\mathbf{A}}, \hat{\mathbf{B}}, \hat{f}, \hat{i} \rangle$ be two **DLFI**-modules. We shall say that a pair $(\alpha, \gamma) : \langle \mathbf{A}, \mathbf{B}, f, i \rangle \to \langle \hat{\mathbf{A}}, \hat{\mathbf{B}}, \hat{f}, \hat{i} \rangle$ is a **DLFI**-homomorphism, if $\alpha \colon A \to \hat{A}$ and $\gamma \colon B \to \hat{B}$ are homomorphisms between bounded distributive lattices and the following diagrams commute:

$$\begin{array}{ccc} A \times B & \xrightarrow{f} & A \\ \downarrow{\scriptstyle \alpha\times\gamma} & & \downarrow{\scriptstyle \alpha} \\ \hat{A} \times \hat{B} & \xrightarrow[\hat{f}]{} & \hat{A} \end{array} \qquad \begin{array}{ccc} B \times A & \xrightarrow{i} & A \\ \downarrow{\scriptstyle \gamma\times\alpha} & & \downarrow{\scriptstyle \alpha} \\ \hat{B} \times \hat{A} & \xrightarrow[\hat{i}]{} & \hat{A} \end{array}$$

We write $\mathcal{DLFI}$ for the category of **DLFI**-modules and homomorphisms.

Definition 3. A *FI-frame* is a structure $\mathbb{F} = (X, Y, \leq_X, \leq_Y, R, T)$ such that $(X, \leq_X)$, $(Y, \leq_Y)$ are posets, R is a subset of $X \times Y \times X$ and T is a subset of $Y \times X^2$ satisfying:

(C1) If $(x, y, z) \in R$, $x' \leq_X x$, $y' \leq_Y y$ and $z \leq_X z'$, then $(x', y', z') \in R$.

(C2) If $(x, y, z) \in T$, $x' \leq_Y x$, $y' \leq_X y$ and $z \leq_X z'$, then $(x', y', z') \in T$.

Let $\mathbb{F} = (X, Y, \leq_X, \leq_Y, R, T)$ and $\mathbb{G} = (Z, W, \leq_Z, \leq_W, S, L)$ be two FI-frames. A map $\mathbb{F} \to \mathbb{G}$ between FI-frames is a pair (f, g), where $f : X \to Z$ and $g : Y \to W$ are morphisms of posets satisfying the following conditions:

(P1) If $(x, y, z) \in R$, then $(f(x), g(y), f(z)) \in S$.

(P2) If $(x, y, z) \in T$, then $(g(x), f(y), f(z)) \in L$.

(M1) If $(x', y', f(z)) \in S$, then there exist $x \in X$ and $y \in Y$ such that $(x, y, z) \in R$, $x' \leq_Z f(x)$ and $y' \leq_W g(y)$.

(M2) If $(x', f(y), z') \in L$, then there exist $x \in Y$ and $z \in X$ such that $(x, y, z) \in T$, $x' \leq_W g(x)$ and $f(z) \leq_Z z'$.

If $(X, \leq)$ is an ordered set, then we write $\mathcal{P}_i(X)$ for the increasing sets of X. Now, let $\mathbb{F} = (X, Y, \leq_X, \leq_Y, R, T)$ be an FI-frame. Let us to consider the following sets, for every $U \in \mathcal{P}_i(Y)$ and $V \in \mathcal{P}_i(X)$:

$$f(V, U) = \{z \in X \mid (x, y, z) \in R \text{ for some } (x, y) \in V \times U\}, \tag{3}$$
$$i(U, V) = \{y \in X \mid \text{for every } x \in Y, z \in X, ((x, y, z) \in T \text{ and } x \in U) \text{ implies } z \in V\}. \tag{4}$$

If $(X, \leq, \mathcal{T})$ is a Priestley space we write $\mathcal{C}(X)$ for the set of clopen increasing sets and ϵ_X for order isomorphism from X onto $\mathcal{X}(\mathcal{C}(X))$ defined by $\epsilon_X(y) = \{U \in \mathcal{C}(X) \mid x \in U\}$.

Definition 4. An *Urquhart space* is a structure $\mathbb{F} = (X, Y, \leq_X, \leq_Y, \mathcal{T}_X, \mathcal{T}_Y, R, T)$ such that $(X, \leq_X, \mathcal{T}_X)$ and $(Y, \leq_Y, \mathcal{T}_Y)$ are Priestley spaces, $R \subseteq X \times Y \times X$, $T \subseteq Y \times X^2$, for every $U \in \mathcal{C}(Y), V \in \mathcal{C}(X)$, $f(V, U), i(U, V) \in \mathcal{C}(X)$, and for every $x \in Y$ and $y, z \in X$:

i) If $f(\epsilon_X(x), \epsilon_Y(y)) \subseteq \epsilon_X(z)$, then $(x, y, z) \in R$.

ii) If $i(\epsilon_Y(x), \epsilon_X(y)) \subseteq \epsilon_X(z)$, then $(x, y, z) \in T$.

If $\mathbb{F} = (X, Y, \leq_X, \leq_Y, \mathcal{T}_X, \mathcal{T}_Y, R, T)$ and $\mathbb{G} = (Z, W, \leq_Z, \leq_W, \mathcal{T}_Z, \mathcal{T}_W, S, L)$ are Urquhart spaces, then a U-map $\mathbb{F} \to \mathbb{G}$ is a pair (f, g) such that $f : X \to Z$ and $g : Y \to W$ are monotonous, continuous and satisfying the conditions (P1), (P2), (M1) and (M2). We write $\mathcal{US}$ for the category of Urquhart spaces and U-maps.

Let $\langle \mathbf{A}, \mathbf{B}, f, i \rangle$ be a **DLFI**-module. We define the ternary relations $R_{\mathbf{A}} \subseteq \mathcal{X}(A) \times \mathcal{X}(B) \times \mathcal{X}(A)$ and $T_{\mathbf{A}} \subseteq \mathcal{X}(B) \times \mathcal{X}(A) \times \mathcal{X}(A)$ by

$$(Q, R, P) \in R_{\mathbf{A}} \iff f(Q, R) \subseteq P,$$

and

$$(R, P, Q) \in T_{\mathbf{A}} \iff i(R, P) \subseteq Q.$$

We claim that

$$(\mathcal{X}(A), \mathcal{X}(B), \subseteq_{\mathcal{X}(A)}, \subseteq_{\mathcal{X}(B)}, \mathcal{T}_A, \mathcal{T}_B, R_{\mathbf{A}}, T_{\mathbf{A}})$$

is an Urquhart space. On the other hand, if $\mathbb{F} = (X, Y, \leq_X, \leq_Y, \mathcal{T}_X, \mathcal{T}_Y, R, T)$ is an Urquhart space, then we also claim that

$$\langle \mathcal{C}(X), \mathcal{C}(Y), f, i \rangle$$

(with f and i as defined in (3) and (4)) is a **DLFI**-module. From these facts we will prove that such correspondences extend into morphisms in the categories $\mathcal{US}$ and $\mathcal{DLFI}$, respectively.

This allows us to state the main result of this paper.

Theorem 1. The categories $\mathcal{US}$ and $\mathcal{DLFI}$ are dually equivalent.

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A focused linear nested system for multi-modalities

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Abstract

Linear logic (LL) have been used as a logical framework for establishing sufficient conditions for cut-admissibility of object logics (OL). However, some logical systems cannot be adequately encoded in LL, the most symptomatic cases being sequent systems for modal logics. In this extended abstract[1], we present a focus linear-nested sequent (LNS) for MMLL (a variant of linear logic with subexponentials), and show that it is possible to establish a cut-admissibility criterion for LNS systems for substructural multi-modal logics.

Introduction. Analytic calculi consist solely of rules that compose the formulas to be proved in a stepwise manner. The best known formalism for proposing analytic proof systems is Gentzen's *sequent calculus.*Unfortunately, sequent systems are not expressive enough for constructing analytic calculi for many modal logics. As a result, many formalisms extending sequent systems have been proposed over the last 30 years, including *hypersequent calculi* ([Avr96]), *nested calculi* ([Brü09]) and *labeled calculi* ([Sim94]).

We study cut-admissibility under the *linear nested system* formalism – LNS ([Lel15]), where a single sequent is replaced with a list of sequents, and the inference rules govern the transfer of formulas between the different sequents. We lift to LNS the method developed by [MP13]. More precisely, we proposed a cut-free focused system for a logic (MMLL) that extends linear logic (LL) [Gir87] with subexponentials featuring different modal behaviors. We also encode different object-level logical systems as theories in MMLL. The proposed encodings are adequate at the highest level and, more interesting, we show that cuts at the object-level can be eliminated by cuts at the MMLL level. Hence, by proving an easy to verify criterion called *cut-coherence*, we obtain for free cut-admissibility results for many modal and substructural logics.

Linear nested systems. A linear nested sequents (LNS) is a finite list of sequents that matches the *history* of a backward proof search in an ordinary sequent calculus [Lel15]. For instance, the modal rules for the axiom K are defined as follows:

$$\frac{\mathcal{G} /\!\!/ \Gamma \vdash \Delta /\!\!/ \cdot \vdash F}{\mathcal{G} /\!\!/ \Gamma \vdash \Delta, \Box F}\ \Box_R \qquad \frac{\mathcal{G} /\!\!/ \Gamma \vdash \Delta /\!\!/ \Gamma', F \vdash \Delta'}{\mathcal{G} /\!\!/ \Gamma, \Box F \vdash \Delta /\!\!/ \Gamma' \vdash \Delta'}\ \Box_L$$

Reading bottom up, while in $\Box_R$ a new nesting/component is created and F is moved there, in $\Box_L$ *exactly one* boxed formula is moved into an existing nesting, losing its modality. Components in a LNS have a tight connection to *worlds* in Kripke-like semantics, so that LNS is an adequate framework for describing alethic modalities. Moreover, information is fragmented

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[1] A full version of this paper, that extends [OPX20], is already under evaluation in *Mathematical Structures in Computer Science.*

Axioms: $\mathsf{K}\ \Box(F \supset G) \supset (\Box F \supset \Box G)$ $\mathsf{D}\ \neg(\Box F \wedge \Box\neg F)$ $\mathsf{T}\ \Box F \supset F$ $\mathsf{4}\ \Box F \supset \Box\Box F$

$$\frac{\Gamma \vdash \Delta /\!\!/ \Sigma, F \vdash \Pi}{\Gamma, \Box F \vdash \Delta /\!\!/ \Sigma \vdash \Pi}\ \Box_L \quad \frac{\Gamma \vdash \Delta /\!\!/ \cdot \vdash F}{\mathcal{G} /\!\!/ \Gamma \vdash \Delta, \Box F}\ \Box_R \quad \frac{\Gamma \vdash \Delta /\!\!/ \cdot \vdash \cdot}{\mathcal{G} /\!\!/ \Gamma \vdash \Delta}\ \mathsf{d} \quad \frac{\mathcal{G} /\!\!/ \Gamma, F \vdash \Delta}{\mathcal{G} /\!\!/ \Gamma, \Box F \vdash \Delta}\ \mathsf{t} \quad \frac{\Gamma \vdash \Delta /\!\!/ \Sigma, \Box F \vdash \Pi}{\Gamma, \Box F \vdash \Delta /\!\!/ \Sigma \vdash \Pi}\ \mathsf{4}$$

Figure 1: Some modal axioms and their linear nested sequent rules.

into components and rules act locally on formulas and are usually context independent. Hence, the movement of formulas on derivations can be better predicted and controlled.

In this work, besides intuitionistic and classical logics, we are interested in reasoning about linear nested systems for some notable extensions of the normal modal logic K. Fig. 1 presents some modal axioms and the respective linear nested rules. Let $\mathcal{A} = \{\mathsf{T}, \mathsf{4}, \mathsf{D}\}$. Extensions of the logic K are represented by KR, where $\mathsf{R} \subseteq \mathcal{A}$. For instance, $\mathsf{S4} = \mathsf{KT4}$.

Modalities can be combined, giving rise to multi-modal logics. *Simply dependent multi-modal logics* are characterized by a triple $(N, \preccurlyeq, F)$, where N is a denumerable set, $(N, \preccurlyeq)$ is a partial order, and F is a mapping from N to the set $\mathfrak{L}$ of extensions of modal logic K with axioms from the set $\mathcal{A}$. The *logic described by* $(N, \preccurlyeq, F)$ has modalities $\Box_i$ for every $i \in N$, with axioms for the modality i given by the logic $F(i)$ and interaction axioms $\Box_j A \supset \Box_i A$ for every $i, j \in N$ with $i \preccurlyeq j$.

Linear logic with multi-modalities. Classical linear logic (LL, [Gir87]) is a resource conscious logic, in the sense that formulas are consumed when used during proofs, unless marked with the exponential ? (whose dual is !). Formulas marked with ? behave *classically*, *i.e.*, they can be contracted and weakened during proofs. LL connectives include the additive conjunction & and disjunction $\oplus$ and their multiplicative versions $\otimes$ and $⅋$, together with their units.

$\mathsf{LNS}_{\mathsf{LL}}$ ([LOP17]) is an end-active, linear nested system for linear logic. In this system, the promotion rule is split into the following local rules:

$$\frac{\vdash \Gamma /\!\!/ \vdash F}{\mathcal{E} /\!\!/ \vdash \Gamma, !F}\ ! \qquad \frac{\vdash \Gamma /\!\!/ \vdash \Delta, ?F}{\vdash \Gamma, ?F /\!\!/ \vdash \Delta}\ ?$$

Observe that no checking must be done in the context in order to apply the ? rule: The only checking is in the ! rule, where $\mathcal{E}$ should be the empty sequent or an empty list of components. Note the similarities between the LNS rules ! and $\Box_R$; and ? and $\mathsf{4}$ in Fig. 1. Indeed, in ([LOP17]) such similarities were exploited in order to propose extensions of $\mathsf{LNS}_{\mathsf{LL}}$ with multi-modalities, called *subexponentials*, allowing for different modal behaviors.

Similar to modal connectives, exponentials in LL are not *canonical* ([DJS93]), in the sense that if $i \neq j$ then $!^i F \not\equiv !^j F$ and $?^i F \not\equiv ?^j F$. Intuitively, this means that we can mark the exponentials with *labels* taken from a set $\mathcal{S}$ organized in a pre-order $\preceq$, obtaining (possibly infinitely-many) exponentials ($!^i, ?^i$ for $i \in \mathcal{S}$). Also as in multi-modal systems, the pre-order determines the provability relation: $!^b F$ *implies* $!^a F$ iff $a \preceq b$.

In ([LOP17]) we extended the concept of *simply dependent multimodal logics* to the substructural case, where subexponentials consider not only the structural axioms for contraction ($\mathsf{C}:$ $!^i(F) \multimap !^i F \otimes !^i F$) and weakening ($\mathsf{W}:\ !^i F \multimap 1$) but also the subexponential version of axioms $\{\mathsf{K}, \mathsf{4}, \mathsf{D}, \mathsf{T}\}$: $\mathsf{K}:\ !^i(F \multimap G) \multimap !^i F \multimap !^i G$ $\quad \mathsf{D}:\ !^i F \multimap ?^i F$ $\quad \mathsf{T}:\ !^i F \multimap F$ $\quad \mathsf{4};\ !^i F \multimap !^i !^i F$

This means that $?^i$ can behave classically or not, but also with exponential behaviors different from those in LL. Hence, by assigning different modal axioms one obtains, in a modular way, a class of different substructural modal logics. For instance, subexponentials assuming T allow for dereliction and those assuming $\mathsf{4}$ are persistent (while those assuming only K are not). In fact, substructural KD can be seen as a fragment of elementary linear logic ELL.

Our main goal is to show how this new class of subexponentials can be applied to the problem of characterizing cut-admissibility of object-level logical systems. The first step is to

Structural rules:	$\mathsf{pos}_i : \lfloor A \rfloor^\perp \otimes (?^i \lfloor A \rfloor)$	$\mathsf{neg}_i : \lceil A \rceil^\perp \otimes (?^i \lceil A \rceil)$
Intuitionistic implication:	$\supset_L: \lfloor A \supset B \rfloor^\perp \otimes (\lceil A \rceil \otimes \lfloor B \rfloor)$	$\supset_R: \lceil A \supset B \rceil^\perp \otimes !^{t4}(\lfloor A \rfloor \parr \lceil B \rceil)$
Modal rules:	$\square_{Li} : \lfloor \square A \rfloor^\perp \otimes ?^i \lfloor A \rfloor$	$\square_{Ri} : \lceil \square A \rceil^\perp \otimes !^i \lceil A \rceil$

Figure 2: Encoding of structural, intuitionistic implication and modal rules.

propose a focused [And92] system for the logic. Below the modal rules of the system:

$$\frac{\vdash \Theta; \cdot \Uparrow \cdot /\!/^i \vdash \cdot; \cdot \Uparrow F}{\vdash \Theta; \cdot \Downarrow !^i F} \; !^i \qquad \frac{\vdash \Upsilon; \cdot \Uparrow L}{\vdash \Theta^u; \cdot \Uparrow \cdot /\!/^i \vdash \Upsilon; \cdot \Uparrow L} \; \mathsf{R_r} \qquad \frac{\vdash \Theta; \cdot \Uparrow \cdot /\!/^i \vdash \cdot; \cdot \Uparrow \cdot}{\vdash \Theta; \cdot \Uparrow \cdot} \; \mathsf{D_d} \qquad \frac{\vdash \Theta^u; \cdot \Uparrow F}{\vdash \Theta^u; \cdot \Downarrow !^{\mathsf{c}} F} \; !^{\mathsf{c}}$$

$$\frac{\vdash \Theta; \Gamma \Uparrow \cdot /\!/^i \vdash \Upsilon, j{+} : F; \cdot \Uparrow L}{\vdash \Theta, j : F; \Gamma \Uparrow \cdot /\!/^i \vdash \Upsilon; \cdot \Uparrow L} \; ?^i{}_{\mathsf{4}} \qquad \frac{\vdash \Theta; \cdot \Uparrow \cdot /\!/^i \vdash \Upsilon; \cdot \Uparrow L, F}{\vdash \Theta, j : F; \cdot \Uparrow \cdot /\!/^i \vdash \Upsilon; \cdot \Uparrow L} \; ?^i{}_{\mathsf{kl}} \qquad \frac{\vdash \Theta; \cdot \Uparrow \cdot /\!/^i \vdash \Upsilon, \mathsf{c} : F; \cdot \Uparrow L}{\vdash \Theta, j : F; \cdot \Uparrow \cdot /\!/^i \vdash \Upsilon; \cdot \Uparrow L} \; ?^i{}_{\mathsf{ku}}$$

These rules have some interesting characteristics that ease the use of the system and its formalization in Coq (https://github.com/meta-logic/MMLL). Consider a subexponential j. When j features the axiom $\mathsf{4}$, the rules $?^i{}_{\mathsf{kl}}$ (linear K) and $?^i{}_{\mathsf{ku}}$ (unbounded K) cannot be applied. Dually, if the subexponential does not feature $\mathsf{4}$, the rule $?^i{}_{\mathsf{4}}$ is not enabled and the use of $?^i{}_{\mathsf{kl}}$ (resp. $?^i{}_{\mathsf{ku}}$) is only possible if j is linear (resp. unbounded). The rules have also a better control of contraction, thus avoiding the need of guessing the number of times a formula must be copied to the next component. Note that the rule $?^i{}_{\mathsf{4}}$ moves the formula F stored in the context j to the context $j+$ (a unbounded version of j featuring T). This has two immediate effects: The formula F can be copied to yet another component (once it is created) reflecting the behavior of the modal rule $\mathsf{4}$ (persistence); moreover, since the axiom T is present in j, the formula F can be also used in the last component by applying the decision rule. In other words, the rule $?^i_{\mathsf{4}}$ embeds both the behavior of K (moving formulas between components) and also $\mathsf{4}$ (by keeping the modality of the formula). On the other hand, the behavior of K, without $\mathsf{4}$, is specified by the rules $?^i{}_{\mathsf{kl}}$ and $?^i{}_{\mathsf{ku}}$. In the first case, j is linear and then F is not contracted. In the second case, F is placed in the context c, an unbounded subexponential not related to any other subexponential. Hence, F cannot be moved to other components.

We have proved cut-elimination for this system by using five different cut-rules that are mutually eliminated. Such procedure have been mechanized in Coq.

Object logics. We have shown that different LNS systems can be specified as MMLL theories. The encoding of the OL's inference rules is modular and it allows for the specification of multi-modal logics in a uniform way. We have proved that the resulting specifications are *adequate*: an OL sequent S is provable iff the encoding of S is also provable in MMLL.

Roughly, OL *formulas* are specified using the meta-level (MMLL) predicates $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$, that identify the occurrence of such formulas on the left and on the right side of the sequent respectively. Hence, OL sequents of the form $B_1, \ldots, B_n \vdash C_1, \ldots, C_m$, $n, m \geq 0$, are specified as the multiset of atomic MMLL formulas $\lfloor B_1 \rfloor, \ldots, \lfloor B_n \rfloor, \lceil C_1 \rceil, \ldots, \lceil C_m \rceil$.

Inference rules of the OL are specified as rewriting clauses that replace the principal formula in the conclusion of the rule by the active formulas in the premises. The LL connectives indicate how these OL formulas are connected: contexts are copied (&) or split ($\otimes$), in different inference rules ($\oplus$) or in the same sequent ($\parr$). Here some examples for the classical logic connectives:

$\wedge_L : \lfloor A \wedge B \rfloor^\perp \otimes (\lfloor A \rfloor \oplus \lfloor B \rfloor)$ $\quad$ $\wedge_R : \lceil A \wedge B \rceil^\perp \otimes (\lceil A \rceil \,\&\, \lceil B \rceil)$ $\quad$ $\mathtt{f}_L : \lfloor \mathtt{f} \rfloor^\perp \otimes \top$
$\rightarrow_L: \lfloor A \rightarrow B \rfloor^\perp \otimes (\lceil A \rceil \otimes \lfloor B \rfloor)$ $\quad$ $\rightarrow_R: \lceil A \rightarrow B \rceil^\perp \otimes (\lfloor A \rfloor \parr \lceil B \rceil)$ $\quad$ $\mathsf{init} : \lfloor A \rfloor^\perp \otimes \lceil A \rceil^\perp$

In the intuitionistic system $\mathsf{LNS_I}$ [LOP17], the rule $\supset_R$ creates a new component while `lift` moves formulas across components. Such behavior can be specified with the unbounded subexponential $t4$ featuring K, T and $\mathsf{4}$ as in Fig. 2. This figure also shows the (parameterized) clauses specifying the rules for box. As expected, the modalities of the subexponential i are determined by the modal behavior of the encoded modality $\square_i$.

It is worth noticing the *modularity* of the encodings: all the modal systems have *exactly* the same encoding, only differing on the meta-level modality. This is a direct consequence of locality, granted by LNS. This also opens the possibility of being able to adequately encode a larger class of modal systems. For instance, if we are considering a (modal) substructural logic where formulas not necessarily behave classically, it suffices to remove the clauses pos and/or neg accordingly.

Cut-elimination for object logics. We showed that an easy-to-check criterium, called *cut-coherence* implies that cuts at the object-level can be eliminated by cuts at the meta-level.

Consider the (multiplicative) OL cut rule specified as the clause $\mathsf{cut} = \exists F.(\lfloor F\rfloor \otimes \lceil F\rceil)$. Cut-coherence is the property that allows us to show the duality, at the meta-level, of the predicates $\lfloor F\rfloor$ and $\lceil F\rceil$. More precisely, let $\mathcal{C}$ be the set of connectives of the OL $\mathcal{L}$. The *encoding* of $\mathcal{L}$ as an MMLL theory is a pair of functions $\mathtt{B}\lfloor\cdot\rfloor$ and $\mathtt{B}\lceil\cdot\rceil$ from $\mathcal{C}$ to MMLL of the form $\mathtt{E}\lfloor\star\rfloor = \exists F_1,...,F_n.(\lfloor\star(F_1,...,F_n)\rfloor^\perp \otimes \mathtt{B}\lfloor\star\rfloor)$ $\mathtt{E}\lceil\star\rceil = \exists F_1,...,F_n.(\lceil\star(F_1,...,F_n)\rceil^\perp \otimes \mathtt{B}\lceil\star\rceil)$. We say that the resulting MMLL theory is *cut-coherent* if, for each connective $\star \in \mathcal{C}$, and $F = \star(F_1,...,F_n)$, the following sequent is provable $\vdash \omega : \mathsf{cut}; \Uparrow \forall F_1,...,F_n.((\mathtt{B}\lfloor\star\rfloor)^\perp \parr (\mathtt{B}\lceil\star\rceil)^\perp)$.

All the encodings we have proposed for substructural modal logics based on multiplicative-additive linear logic with different modalities extending K are cut-coherent. Then, for all these encodings, the following result can be applied.

Theorem: Cut-coherence. Let $\mathcal{T}_\mathcal{L}$ be the theory of a given OL $\mathcal{L}$, and let Ψ be a multiset and Θ a subexponential context containing only atoms of the form $\lceil\cdot\rceil$ and $\lfloor\cdot\rfloor$. The sequent $\vdash \omega : \{\mathcal{T}_\mathcal{L}, \mathsf{cut}\}, \Theta; \Psi \Uparrow \cdot$ is provable iff $\vdash \omega : \mathcal{T}_\mathcal{L}, \Theta; \Psi \Uparrow \cdot$ is provable.

As future work, it would be interesting to analyze the case of non-normal modal logics ([LP19]), as well as to explore the failure cases.

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