Interpolation Meets Cyclic Proofs

Bahareh Afshari

ILLC, University of Amsterdam
FLoV, University of Gothenburg

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- Syntactic approach via sequent calculus: (complete) sequent calculus that admits elimination of cuts.
- Craig interpolation is often (but not always) provable via induction over the cut-free derivations.
- There is an intimate connection between interpolation and the existence of sequent calculi. (Iemhoff; Kuznets; …)
- In this talk, I demonstrate

  uniform interpolation ⇒ complete proof systems

  connection is deeper, extending to most expressive logics of all, i.e. fixed point logic.

Uniform interpolation

Definition

Craig interpolation: for every valid implication $A \longrightarrow C$ there is a formula B (the interpolant) in the common vocabulary of A and C s.t.

$$A \longrightarrow B \text{ is valid} \quad \& \quad B \longrightarrow C \text{ is valid}$$

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Modal logics are known to widely enjoy interpolation:

K, T, GL, S4, S5, ...

Also true for modal μ-calculus (D'Agostino and Hollenberg 2000) (A., Leigh and Menedez TABLEAUX 2021).

Systems for fixed point logic

Gentzen sequents: $\Gamma \Rightarrow \Delta$

Axioms:

$$p \Rightarrow p \qquad p, \neg p \Rightarrow \emptyset \qquad \emptyset \Rightarrow p, \neg p \qquad \bot \Rightarrow \emptyset \qquad \emptyset \Rightarrow \top$$

Logical rules: $\vee^l, \vee^r, \wedge^l, \wedge^r$
Modality rules:

$$\frac{\Gamma, A \Rightarrow \Delta}{\Pi, \Box\Gamma, \Diamond A \Rightarrow \Diamond\Delta, \Pi'}\,\mathrm{mod}^l \qquad \frac{\Gamma \Rightarrow \Delta, A}{\Pi, \Box\Gamma \Rightarrow \Diamond\Delta, \Box A, \Pi'}\,\mathrm{mod}^r$$

Regeneration rules:

$$\frac{\Gamma \Rightarrow \Delta, A(\mu x A)}{\Gamma \Rightarrow \Delta, \mu x A}\,\mu^r \qquad \frac{\Gamma, A(\nu x A) \Rightarrow \Delta}{\Gamma, \nu x A \Rightarrow \Delta}\,\nu^l$$

Analogously, μ^l and ν^r.

Interpolating cycles

A cyclic proof is a finite tree built from these rules s.t. every leaf is either an axiom, or a successful repeat:

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A successful repeat on the left/right are interpolated by μ/ν formula:

$$\begin{array}{c}\Gamma \overset{y}{\Rightarrow} \Delta \\ \vdots \\ \Gamma \overset{I(y)}{\Rightarrow} \Delta \\ \vdots\end{array}$$

1. Leaves $\Gamma \Rightarrow y$ and $y \Rightarrow \Delta$ with y fresh.
2. Inductively we build $I(y)$ and proofs:

$$\begin{array}{c}\Gamma \Rightarrow y \\ \vdots \\ \Gamma \Rightarrow I(y)\end{array} \qquad \begin{array}{c}y \Rightarrow \Delta \\ \vdots \\ I(y) \Rightarrow \Delta\end{array}$$

3. The interpolant is either $\mu y I$ or $\nu y I$.

More complex interpolants

A repeated sequent may have been used for multiple non-axiomatic leaves:

$$\Gamma \overset{y'}{\Rightarrow} \Delta$$

$$\Gamma \overset{y}{\Rightarrow} \Delta \qquad \vdots$$

$$\vdots \qquad\qquad \vdots$$

$$\text{- - - - - - - - - - - -}$$

$$\Gamma \overset{I(y,y')}{\Rightarrow} \Delta$$

$$\vdots$$

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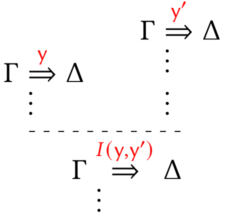

System of equations:

$$y =_\mu I(y, y')$$
$$y' =_\nu I(y, y')$$
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The proof induces a priority ('subsumption') ordering $y \sqsubset y' \sqsubset \ldots$ which determines the quantifier order: $\mu y \nu y' I$ or $\nu y' \mu y I$

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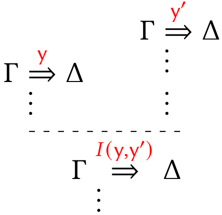

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The order is important to ensure that, after interpolating, every infinite path has an infinitely progressing thread.

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- Run proof search on $A \Rightarrow$?
- We rely on the schematic property of sequent calculus
- Three general cases to check
  - non-axiomatic leaves (previous slide)
  - disjunctions on the left, conjunctions on the right
  - modalities

$$\frac{\Gamma, A \overset{I_0}{\Rightarrow} \Delta \qquad \Gamma, B \overset{I_1}{\Rightarrow} \Delta}{\Gamma, A \vee B \overset{I_0 \vee I_1}{\Rightarrow} \Delta} \vee_l \qquad \frac{\Gamma \overset{I}{\Rightarrow} \Delta, A \qquad \Gamma \overset{I}{\Rightarrow} \Delta, B}{\Gamma \overset{I}{\Rightarrow} \Delta, A \wedge B} \wedge_r$$

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In the case of modal μ-calculus the interpolant can be chosen to be conjunctive formulas. This together with 'nice' properties of cyclic proofs allows one to derive completeness for Kozen's proof system!

Conjunctive formulas

The operator $(a)\Gamma$ is the dual operator to the ‘disjunctive’ modality used frequently in μ-calculus. It is representable by the standard modalities:

$$(a)\{C_1, \ldots, C_n\} = [a]C_1 \vee \cdots \vee [a]C_n \vee \langle a\rangle(C_1 \wedge \cdots \wedge C_n).$$

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Definition (Conjunctive formula)

The conjunctive formulas are formed freely from variables, atoms $\top$ and $\bot$, the two quantifiers, arbitrary conjunctions, and disjunctions of specific form

$$(a_0)\Gamma_0 \vee \cdots \vee (a_k)\Gamma_k$$

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where $a_0, \dots, a_k$ are pairwise distinct action labels.

$(a_0)\Gamma_0 \vee \cdots \vee (a_k)\Gamma_k$ is valid iff there is a valid formula in $\Gamma_0 \cup \cdots \cup \Gamma_k$.

Strong consequence

Δ is a strong L-consequence of Γ, written $\Gamma \leq_L \Delta$, if for every L-sequent Π and proof $\pi \vdash \Pi, \Gamma$ there exists a proof $\pi' \vdash \Pi, \Delta$ such that every path of Π-ancestors through π' is witnessed as a path of Π-ancestors of π.

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Properties of strong consequence:

- If $A \leq B$ then $\Gamma, A \leq \Gamma, B$.
- $A \wedge B \leq A$ and $A \wedge B \leq B$.
- If $\Gamma \subseteq \Gamma'$ then $\Gamma \leq \Gamma'$.
- $A \leq B$ implies $\vdash A \rightarrow B$.
- $\sigma \mathrm{x} A \leq A(\sigma \mathrm{x} A) \leq \sigma \mathrm{x} A$.
- If $A \leq B$ then for all $C(\mathrm{x})$, $C(A) \leq C(B)$.
- If $A \leq B(A)$ then $A \leq \nu \mathrm{x} B(\mathrm{x})$.
- If $A(\mathrm{x}) \leq B(\mathrm{x})$ then $\sigma \mathrm{x} A \leq \sigma \mathrm{x} B$.

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Theorem (A., Leigh and Menendez Turata, WIP)

Given A *and* V *a vocabulary, there exists a conjunctive* $(V \cap V(A))$*-formula* A^* *such that* $A^* \leq A \leq_V A^*$*.*

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Suppose A is valid. Since A^* is equivalent to A we have $\vdash A^*$. As A^* is conjunctive $Koz \vdash A^*$. Together with $Koz \vdash A^* \longrightarrow A$, we conclude $Koz \vdash A$.