Admissibility of Π_2-Inference Rules: interpolation, model completion, and contact algebras

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Aims of the Talk

In this talk we review some recent results of us concerning the decision problem of recognizing admissibility of some non-standard inference rules.

We begin by introducing our logical context and by illustrating some examples showing how such non-standard rules arise.

Modal Systems

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A *modal signature* Σ is a finite signature comprising Boolean operators $\wedge, \vee, \rightarrow, \leftrightarrow, \neg, \bot, \top$ as well as additional operators of any arity called the *modal* operators.

Out of Σ-symbols and out of a countable set of variables

$$x, y, z, \dots, p, q, r, \dots$$

one can build the set of propositional *Σ-formulas* (indicated with letters $F, G, \dots$ or $\varphi, \psi, \dots$). Notations such as $F(\underline{x})$ mean that the Σ-formula F contains at most the variables from the tuple $\underline{x}$; the notation $F(\underline{\varphi}/\underline{x})$ is used for substitutions.

Modal Systems

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A **modal system** $\mathcal{S}$ (over the modal signature Σ) is a set of Σ-formulas comprising tautologies, the distribution axioms for each *n*-ary modal operator $\Box$

$$\Box[\phi, \ldots, \psi \to \psi', \ldots, \phi] \to (\Box[\phi, \ldots, \psi, \ldots, \phi] \to \Box[\phi, \ldots, \psi', \ldots, \phi])$$

and closed under the rules of modus ponens (MP) (from ϕ and $\phi \to \psi$ infer ψ), uniform substitution (US) (from $F(\underline{x})$ infer $F(\underline{\psi}/\underline{x})$), and necessitation (N) (from ψ infer $\Box[\phi, \ldots, \psi, \ldots, \phi]$).

Modal Systems

We write $\vdash_{\mathcal{S}} \phi$ or $\mathcal{S} \vdash \phi$ to mean that $\phi \in \mathcal{S}$. If $\vdash_{\mathcal{S}} \phi \to \psi$ holds, we say that ψ is a *local* consequence of ϕ (modulo $\mathcal{S}$).

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We shall also need the *global* consequence relation $\phi \vdash_{\mathcal{S}} \psi$: this relation holds when ψ belongs to the smallest set of formulas containing $\mathcal{S}$ and ϕ that is closed under modus ponens and necessitation (notice that closure under uniform substitution here is not required).

Π_2-rules

Definition

An inference rule ρ is a Π_2-rule if it is of the form

$$\frac{F(\underline{\varphi}/\underline{x}, \underline{y}) \to \chi}{G(\underline{\varphi}/\underline{x}) \to \chi}$$

where $F(\underline{x}, \underline{y}), G(\underline{x})$ are propositional formulas.

We say that θ is obtained from ψ by an application of the rule ρ if

$$\psi = F(\underline{\varphi}/\underline{x}, \underline{y}) \to \chi \quad \text{and} \quad \theta = G(\underline{\varphi}/\underline{x}) \to \chi,$$

where $\underline{\varphi}$ is a tuple of formulas, χ is a formula, and $\underline{y}$ is a tuple of propositional letters not occurring in $\underline{\varphi}$ and χ.

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Let $\mathcal{S}$ be a propositional modal system. We say that the rule ρ is admissible in $\mathcal{S}$ if $\vdash_{\mathcal{S}+\rho} \varphi$ implies $\vdash_{\mathcal{S}} \varphi$ for each formula φ.

Π_2-rules

The fact that the rule ρ

$$\frac{F(\underline{\varphi}/\underline{x}, \underline{y}) \to \chi}{G(\underline{\varphi}/\underline{x}) \to \chi}$$

is admissible means that in the countably generated free (i.e. Lindenbaum) $\mathcal{S}$-algebra we have that the sentence

$$\forall \underline{x} \forall z \ \big(\ (\forall \underline{y} \, F(\underline{x}, \underline{y}) \leq z) \Rightarrow G(\underline{x}) \leq z \ \big)$$

is true (notice the universal quantifier in the antecedent).

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Remark. If in the antecedent F of ρ the variables $\underline{y}$ do not occur, the rule is admisible iff (trivially) we have $\vdash_{\mathcal{S}} G(\underline{x}) \to F(\underline{x})$. In this sense, our Π_2-rules have a special shape and their admissibility problem does not generalize the admissibility problem of standard rules.

Π_2-rules

The prototypical Π_2-rule is Gabbay's irreflexivity rule

$$\frac{y \wedge \Box\neg y \to \chi}{\top \to \chi}$$

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Let us also mention the *density* rule, which is admissible in various fuzzy systems (Metcalfe & Montagna, 2007).
This rule however does not fit our definitions (extending our results so as to encompass it seems to be an intersting challenging research direction).

De Vries Duality

Modal systems can be useful to model disparate phenomena and can be useful in various contexts, ranging from linguistics, to computer science, to mathematics.

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Definition

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Definition

An open subset U of a topological space is called regular open if $U = \text{int}(\text{cl}(U))$.

Let X be a compact Hausdorff space. The set $\text{RO}(X)$ of regular open subsets of X equipped with the well-inside relation $U \prec V$ iff $\text{cl}(U) \subseteq V$ forms a de Vries algebra.

De Vries Duality

Definition

A de Vries algebra is a complete boolean algebra equipped with a binary relation $\prec$ satisfying

(S1) $0 \prec 0$ and $1 \prec 1$;
(S2) $a \prec b, c$ implies $a \prec b \wedge c$;
(S3) $a, b \prec c$ implies $a \vee b \prec c$;
(S4) $a \leq b \prec c \leq d$ implies $a \prec d$;
(S5) $a \prec b$ implies $a \leq b$;
(S6) $a \prec b$ implies $\neg b \prec \neg a$;
(S7) $a \prec b$ implies there is c with $a \prec c \prec b$;
(S8) $a \neq 0$ implies there is $b \neq 0$ with $b \prec a$.

De Vries Duality

All the information carried by $(\mathrm{RO}(X), \prec)$ is enough to recover the compact Hausdorff space X up to homeomorphism.

Moreover, every de Vries algebra is isomorphic to one of the form $(\mathrm{RO}(X), \prec)$ for some compact Hausdorff space X.

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A connection of this framework to modal systems has been explored by [Balbiani, Tinchev, Vakarelov (2007)]. We follow the equivalent approach by [G. & N. Bezhanishvili, T. Santoli, Y. Venema (2019)].

Symmetric Strict Implication Algebras

Let $(B, \prec)$ be a de Vries algebra. We can turn $(B, \prec)$ into a boolean algebra with operators by replacing $\prec$ with a binary operator with values in $\{0, 1\}$ (the bottom and top of B).

$$a \rightsquigarrow b = \begin{cases} 1 & \text{if } a \prec b, \\ 0 & \text{otherwise.} \end{cases}$$

$\rightsquigarrow$ is the characteristic function of $\prec \subseteq B \times B$.

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Definition

Let $\mathcal{V}$ be the variety generated by de Vries algebras in the language of boolean algebras with a binary operator $\rightsquigarrow$. We call symmetric strict implication algebras the algebras of $\mathcal{V}$.

Definition (G. Bezhanishvili, N. Bezhanishvili, T. Santoli, Y. Venema (2019))

The symmetric strict implication calculus S^2IC is the system given by the axioms

1. $[\forall]\varphi \leftrightarrow (\top \rightsquigarrow \varphi)$,
2. $(\bot \rightsquigarrow \varphi) \wedge (\varphi \rightsquigarrow \top)$,
3. $[(\varphi \vee \psi) \rightsquigarrow \chi] \leftrightarrow [(\varphi \rightsquigarrow \chi) \wedge (\psi \rightsquigarrow \chi)]$,
4. $[\varphi \rightsquigarrow (\psi \wedge \chi)] \leftrightarrow [(\varphi \rightsquigarrow \psi) \wedge (\varphi \rightsquigarrow \chi)]$,
5. $(\varphi \rightsquigarrow \psi) \rightarrow (\varphi \rightarrow \psi)$,
6. $(\varphi \rightsquigarrow \psi) \leftrightarrow (\neg\psi \rightsquigarrow \neg\varphi)$,
7. $[\forall]\varphi \rightarrow [\forall][\forall]\varphi$,
8. $\neg[\forall]\varphi \rightarrow [\forall]\neg[\forall]\varphi$,
9. $(\varphi \rightsquigarrow \psi) \leftrightarrow [\forall](\varphi \rightsquigarrow \psi)$,

and modus ponens (for $\rightarrow$) and necessitation (for $[\forall]$).

Symmetric Strict Implication Algebras

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Axioms 2-4 are equivalent to the normality axioms for the binary connective $\neg x \rightsquigarrow y$.

Thus, in particular, S^2IC fits the conditions of being a modal system in our sense.

Symmetric Strict Implication Algebras

Theorem (G. Bezhanishvili, N. Bezhanishvili, T. Santoli, Y. Venema (2019))

$\vdash_{\mathsf{S^2IC}} \varphi$ *iff* $(B, \rightsquigarrow) \vDash \varphi$ *for every symmetric strict impl. algebra* $(B, \rightsquigarrow)$*.*

$\vdash_{\mathsf{S^2IC}} \varphi$ *iff* $(B, \prec) \vDash \varphi$ *for every de Vries algebra* $(B, \prec)$*.*

$\vdash_{\mathsf{S^2IC}} \varphi$ *iff* $(\mathrm{RO}(X), \prec) \vDash \varphi$ *for every compact Hausdorff space* X*.*

Analogous strong completeness results hold.

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$\vdash_{\mathsf{S^2IC}} \varphi$ *iff* $(\mathrm{RO}(X), \prec) \vDash \varphi$ *for every compact Hausdorff space* X.

Analogous strong completeness results hold.

Therefore, we can think of $\mathsf{S^2IC}$ as the modal calculus of compact Hausdorff spaces where propositional letters are interpreted as regular opens.

Contact Algebras

When a symmetric strict implication algebra is simple, $\rightsquigarrow$ becomes the characteristic function of a binary relation. Simple symmetric strict implication algebras correspond exactly to contact algebras.

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Definition

A contact algebra is a boolean algebra equipped with a binary relation $\prec$ satisfying the axioms:

(S1) $0 \prec 0$ and $1 \prec 1$;
(S2) $a \prec b, c$ implies $a \prec b \wedge c$;
(S3) $a, b \prec c$ implies $a \vee b \prec c$;
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Contact Algebras

The variety of symmetric strict implication algebras is a discriminator variety and hence it is generated by its simple algebras which correspond to contact algebras. Therefore,

$$\vdash_{\mathsf{S^2IC}} \varphi \ \text{ iff } \ (B, \prec) \vDash \varphi \text{ for every contact algebra } (B, \prec).$$

Since we also have (see above)

$$\vdash_{\mathsf{S^2IC}} \varphi \ \text{ iff } \ (B, \prec) \vDash \varphi \text{ for every de Vries algebra } (B, \prec).$$

we conclude that contact algebras and De Vries algebras (duals to compact Hausdorff spaces) are *indistinguishable* as far as the modal language of $\mathsf{S^2IC}$ is concerned.

De Vries vs Contact Algebras

However, De Vries algebras differ from contact algebras because they are assumed to be complete and to satify a couple of further axioms, namely (S7) and (S8).
Therefore, (S7) and (S8) are not expressible in S^2IC.

(S7) $a \prec b$ implies there is c with $a \prec c \prec b$;
(S8) $a \neq 0$ implies there is $b \neq 0$ with $b \prec a$.

These conditions involve the richer Π_2-fragment of the language (in particular they require an existential quantifier to be written).

What does this mean from the syntactic point of view?

De Vries vs Contact Algebras

Theorem

For each Π_2*-sentence* Φ *there is an inference rule* ρ *such that*

$$\vdash_{\mathsf{S^2IC}+\rho} \varphi \quad \textit{iff} \quad (B, \prec) \vDash \varphi$$

for every propositional formula φ *and for every contact algebra (equivalently: for every simple symmetric strict implication algebra)* $(B, \prec)$ *satisfying* Φ*.*

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The rules corresponding to (S7) and (S8) are

$$(\rho_7)\ \frac{(\varphi \rightsquigarrow p) \wedge (p \rightsquigarrow \psi) \to \chi}{(\varphi \rightsquigarrow \psi) \to \chi} \qquad (\rho_8)\ \frac{p \wedge (p \rightsquigarrow \varphi) \to \chi}{\varphi \to \chi}$$

Thus (S7) and (S8) (which are not expressible in $\mathsf{S^2IC}$) correspond to admissible Π_2-rules in $\mathsf{S^2IC}$.

Methods for Recognizing Admissibility

In the main part of the talk, we supply three methods for deciding admissibility of Π_2-rules; these methods involve

- conservative extensions,
- uniform (global) interpolants,
- model completions,

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- conservative extensions,
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respectively. In the last part of the talk, we turn to our main motivating case study, namely contact algebras.

Conservative Extensions

We say that $\varphi(\underline{x}) \wedge \psi(\underline{x}, \underline{y})$ is a conservative extension of $\varphi(\underline{x})$ in $\mathcal{S}$ if

$$\vdash_{\mathcal{S}} \varphi(\underline{x}) \wedge \psi(\underline{x}, \underline{y}) \to \chi(\underline{x}) \quad \text{implies} \quad \vdash_{\mathcal{S}} \varphi(\underline{x}) \to \chi(\underline{x})$$

for every formula $\chi(\underline{x})$.

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for every formula $\chi(\underline{x})$.

We say that $\mathcal{S}$ has the (local) *interpolation property* iff for every pair of Σ-formulas $\phi(\underline{x}, \underline{y}), \psi(\underline{y}, \underline{z})$ such that $\vdash_{\mathcal{S}} \phi \to \psi$ there is a formula $\theta(\underline{y})$ such that $\vdash_{\mathcal{S}} \phi \to \theta$ and $\vdash_{\mathcal{S}} \theta \to \psi$.

Similarly, we say that $\mathcal{S}$ has the *global interpolation property* iff for every pair of Σ-formulas $\phi(\underline{x}, \underline{y}), \psi(\underline{y}, \underline{z})$ such that $\phi \vdash_{\mathcal{S}} \psi$ there is a formula $\theta(\underline{y})$ such that $\phi \vdash_{\mathcal{S}} \theta$ and $\theta \vdash_{\mathcal{S}} \psi$.

In the context of our modal systems, the local interpolation property implies the global one, but the converse does not hold.

Conservative Extensions

Theorem

If $\mathcal{S}$ has the interpolation property, then a Π_2-rule ρ is admissible in $\mathcal{S}$ iff $G(\underline{x}) \wedge F(\underline{x}, \underline{y})$ is a conservative extension of $G(\underline{x})$ in $\mathcal{S}$.

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Therefore, if $\mathcal{S}$ has the interpolation property and conservativity is decidable in $\mathcal{S}$, then Π_2-rules are effectively recognizable in $\mathcal{S}$.

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Therefore, if $\mathcal{S}$ has the interpolation property and conservativity is decidable in $\mathcal{S}$, then Π_2-rules are effectively recognizable in $\mathcal{S}$.
Thus, well-known results (G., Lutz, Wolter, Zakharyaschev, AiML 2006) apply:

Corollary

The admissibility problem for Π_2-rules is

- coNEXPTIME-*complete in* K *and* S5*;*
- *in* EXPSPACE *and* coNEXPTIME-*hard in* S4.

Uniform Interpolants

The first method we supplied is probably the easiest to apply in concrete cases. We illlustrate however two other approaches, which are conceptually relevant and (especially the third one) more oriented to algebraic and model-theoretic methods - and less dependant on specific semantic algorithms from modal logic.

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We first need to recall what *uniform* interpolants are.

Uniform Interpolants

Definition

A *uniform local pre-interpolant* of a formula $\phi(\underline{x}, \underline{y})$ wrt the variables $\underline{x}$ is a formula $\exists^l_{\underline{x}}\phi$ such that: (i) in $\exists^l_{\underline{x}}\phi$ at most the variables $\underline{y}$ occur; (ii) for every formula $\psi(\underline{y}, \underline{z})$, we have

$$\vdash_{\mathcal{S}} \exists^l_{\underline{x}}\phi \to \psi \;\text{ iff }\; \vdash_{\mathcal{S}} \phi \to \psi \quad . \tag{1}$$

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$$\exists^g_{\underline{x}}\phi \vdash_{\mathcal{S}} \psi \;\text{ iff }\; \phi \vdash_{\mathcal{S}} \psi. \tag{2}$$

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If the local uniform pre-interpolant $\exists^l_{\underline{y}} F$ exists, then a Π_2-rule ρ of the form

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is admissible in $\mathcal{S}$ iff

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However, existence of uniform interpolants is a rare phenomenon; in addition, checking admissibility/conservativity by computing local uniform interpolants does not match optimal lower bounds, even in basic cases like the case of the system K.

Uniform Global Interpolants

Notice that there are cases where local uniform interpolants exist, but global do not and vice versa. Thus, it makes sense (at least in principle) to investigate cases where only global uniform interpolants are available. For the related results, we need to introduce *universal modalities*. We already met a universal modality, when axiomatizing symmetric strict implication algebras; the formal definition is below.

Uniform Global Interpolants

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An S5-modality $[\forall]$ is called a universal modality if

$$\vdash_{\mathcal{S}} \bigwedge_{i=1}^{n} [\forall](\varphi_i \leftrightarrow \psi_i) \to (\Box[\varphi_1, \ldots, \varphi_n] \leftrightarrow \Box[\psi_1, \ldots, \psi_n])$$

for every modality $\Box$ of $\mathcal{S}$.

Theorem

Suppose that $\mathcal{S}$ has uniform global pre-interpolants and a universal modality $[\forall]$. Then a Π_2-rule ρ is admissible in $\mathcal{S}$ iff

$$\vdash_{\mathcal{S}} [\forall]\forall^{g}_{\underline{y}}(F(\underline{x}, \underline{y}) \to z) \to (G(\underline{x}) \to z).$$

Model Completions

It is well-known (see the book G.-Zawadowski, Kluwer 2002) that, under suitable hypotheses (which are satisfied when there is a universal modality), existence of uniform global interpolants is equivalent to existence of a model completion for the theory axiomatizing $\mathcal{S}$-algebras.

Thus the hypotheses leading to our second method can be used in a model-theoretic environment.

Model Completions

To a Π_2-rule ρ

$$\frac{F(\underline{\varphi}/\underline{x}, \underline{y}) \to \chi}{G(\underline{\varphi}/\underline{x}) \to \chi}$$

we associate the first-order formula

$$\Pi(\rho) := \forall \underline{x}, z\Big(G(\underline{x}) \not\leq z \Rightarrow \exists \underline{y} : F(\underline{x}, \underline{y}) \not\leq z\Big).$$

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In the presence of a universal modality, an $\mathcal{S}$-algebra is simple iff

$$[\forall]x = \begin{cases} 1 & \text{if } x = 1, \\ 0 & \text{otherwise.} \end{cases}$$

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In the presence of a universal modality, an $\mathcal{S}$-algebra is simple iff

$$[\forall]x = \begin{cases} 1 & \text{if } x = 1, \\ 0 & \text{otherwise.} \end{cases}$$

In this case, since the the variety of $\mathcal{S}$-algebras is a discriminator variety, it is generated by the simple $\mathcal{S}$-algebras.

Model Completions

Theorem (G. Bezhanishvili, N. Bezhanishvili, T. Santoli, Y. Venema (2019))

Suppose that $\mathcal{S}$ has a universal modality. A Π_2-rule ρ is admissible in $\mathcal{S}$ iff for each simple $\mathcal{S}$-algebra $\mathcal{B}$ there is a simple $\mathcal{S}$-algebra $\mathcal{C}$ such that $\mathcal{B}$ is a subalgebra of $\mathcal{C}$ and $\mathcal{C} \models \Pi(\rho)$.

We shall exploit this theorem taking inspiration from model-theoretic algebra.

Model Completions

Recall that a *universal* first order theory T has a model completion iff there is a stronger theory $T^\star \supseteq T$ (in the same signature) such that (i) T and $T^\star$ prove the same quantifier-free formulae; (ii) $T^\star$ eliminates quantifiers.

Model Completions

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It turns out that the model completion of a universal first-order theory T, if it exists, is unique and it is the theory of the existentially closed models of T.

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It turns out that the model completion of a universal first-order theory T, if it exists, is unique and it is the theory of the existentially closed models of T.
The existence of a model-completion $T^\star$ of T implies that the class of the models of T has the amalgamation property (the latter turns out to be a necessary and sufficient condition for the existence of $T^\star$ in case T is locally finite and its language is finite).

Model Completions

The previous theorem (together with basic model theoretic facts) yields the following

Theorem

Suppose that $\mathcal{S}$ has a universal modality and let $T_{\mathcal{S}}$ be the first-order theory of the simple $\mathcal{S}$-algebras. If $T_{\mathcal{S}}$ has a model completion $T^{\star}_{\mathcal{S}}$, then a Π_2-rule ρ is admissible in $\mathcal{S}$ iff $T^{\star}_{\mathcal{S}} \models \Pi(\rho)$ where

$$\Pi(\rho) := \forall \underline{x}, z \Big(G(\underline{x}) \not\leq z \Rightarrow \exists \underline{y} : F(\underline{x}, \underline{y}) \not\leq z \Big).$$

Model Completions

Thus existence and decidability of $T^\star_{\mathcal{S}}$ yields the decidability of the admissibility problem for our rules.

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When $\mathcal{S}$ is decidable, locally tabular, amalgamable and has a universal modality, $T^\star_{\mathcal{S}}$ exists and we can exploit the above theorem by *enumerating open formulae* as follows. To compute the formula eliminating a quantifier $\exists y \psi(\underline{x}, y)$ in $T^\star_{\mathcal{S}}$, it is sufficient to take the conjunction of the (finitely many) universal formulae $\phi(\underline{x})$ which are $T_{\mathcal{S}}$-implied by $\psi(\underline{x}, y)$. The correctness of this procedure comes from general facts concerning model completions.

Model Completions

As an alternative, when $\mathcal{S}$ has a universal modality, is locally tabular, amalgamable and finite $\mathcal{S}$-algebras can be effectively recognized, one can go through *enumeration of finite algebras* as follows. To decide the $T^{\star}_{\mathcal{S}}$-validity of

$$\Pi(\rho) := \forall \underline{x}, z \Big(G(\underline{x}) \not\leq z \Rightarrow \exists \underline{y} : F(\underline{x}, \underline{y}) \not\leq z \Big).$$

one checks whether every finite $\mathcal{S}$-algebra generated by $\underline{x}, z$ and satisfying $G(\underline{x}) \not\leq z$ can be expanded to a finite $\mathcal{S}$-algebra generated by $\underline{x}, z, \underline{y}$ and satisfying $F(\underline{x}, \underline{y}) \not\leq z$. Again, this is justified by general model-theoretic facts.

Remark. It goes without saying that in principle the model completion can exist (and be decidable) even in case $\mathcal{S}$ is not locally tabular! In such cases we would nevertheless have a decision procedure.

The model completion $\mathrm{Con}^\star$

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Theorem

The theory of contact algebras Con *is locally finite and has the amalgamation property. Therefore, it admits a model completion* $\mathsf{Con}^\star$.

The model completion Con*

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Theorem

The theory of contact algebras Con *is locally finite and has the amalgamation property. Therefore, it admits a model completion* Con*.

Amalgamation can be established via duality: contact algebras are in fact dual to Stone spaces andowed with a closed, reflexive, symmetric relation. The duals of embeddings are continuous functions $f : (X_1, R_1) \to (X_2, R_2)$ satisfying the additional condition

$$\forall x, y \in X_2 \ [x R_2 y \ \Leftrightarrow \ \exists \tilde{x}, \tilde{y} \in X_1 \text{ s.t. } f(\tilde{x}) = x, \ f(\tilde{y}) = y \ \ \& \ \ \tilde{x} R_1 \tilde{y}].$$

The model completion Con*

We can consequently apply the above (bounded!) enumeration methods in order to check admissibility of Π_2-rules.
Given that the above duality trivializes in the case of finite algebras (topology is not needed), an enumeration of the involved finite algebras easily yields that the rules (ρ_7) and (ρ_8) we met at the beginning of the present talk, are in fact admissible.
As another example, consider the Π_2-rule

$$(\rho_9) \qquad \frac{(p \rightsquigarrow p) \wedge (\varphi \rightsquigarrow p) \wedge (p \rightsquigarrow \psi) \to \chi}{(\varphi \rightsquigarrow \psi) \to \chi}$$

corresponding to the Π_2-sentence

$$\Pi(\rho_9) \quad \forall x, y, z\, (x \rightsquigarrow y \not\leq z \to \exists u : (u \rightsquigarrow u) \wedge (x \rightsquigarrow u) \wedge (u \rightsquigarrow y) \not\leq z)$$

which holds in $(\mathrm{RO}(X), \prec)$ iff X is a Stone space.

The model completion $\mathsf{Con}^\star$

Using our enumeration methods, it is possible to show that this rule is admissible too. For the second method, it is sufficient to check that every finite algebra, generated by elements x, y, z satisfying $x \rightsquigarrow y \not\leq z$ can be embedded into a finite algebra, generated by an additional element u, satisfying $(u \rightsquigarrow u) \wedge (x \rightsquigarrow u) \wedge (u \rightsquigarrow y) \not\leq z$. This is easy to check via finite duality (in the finite case, topology is discrete, so it can be disregarded).

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Therefore, we obtain as a corollary that S^2IC is complete wrt Stone spaces.

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Therefore, we obtain as a corollary that S^{2}IC is complete wrt Stone spaces.

This fact was proved in [G. Bezhanishvili, N. Bezhanishvili, T. Santoli, Y. Venema (2019)].

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Theorem

The problem of recognizing the admissibility of a Π_2-rule in the symmetric strict implication calculus $\mathsf{S^2IC}$ *is co-*$\mathrm{NExpTime}$*-complete.*

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Theorem

The problem of recognizing the admissibility of a Π_2-rule in the symmetric strict implication calculus $\mathsf{S^2IC}$ *is co-*$\textsc{NExpTime}$*-complete.*

Notice that the above complexity bound is the same as for the modal systems K and $S5$.

The model completion $\mathrm{Con}^\star$

We finally consider the problem of axiomatizing $\mathrm{Con}^\star$:

Theorem

The model completion $\mathrm{Con}^\star$ *of the theory of contact algebras is finitely axiomatizable.*

An axiomatization is given by the axioms of contact algebras together with the following three sentences.

The model completion Con*

$$\forall a, b_1, b_2\ (a \neq 0\ \&\ (b_1 \vee b_2) \wedge a = 0\ \&\ a \prec a \vee b_1 \vee b_2 \Rightarrow$$
$$\exists a_1, a_2\ (a_1 \vee a_2 = a\ \&\ a_1 \wedge a_2 = 0\ \&\ a_1 \neq 0\ \&\ a_2 \neq 0\ \&\ a_1 \prec a_1 \vee b_1$$
$$\&\ a_2 \prec a_2 \vee b_2))$$

$$\forall a, b\ (a \wedge b = 0\ \&\ a \not\prec \neg b \Rightarrow \exists a_1, a_2\ (a_1 \vee a_2 = a\ \&\ a_1 \wedge a_2 = 0$$
$$\&\ a_1 \not\prec \neg b\ \&\ a_2 \not\prec \neg b\ \&\ a_1 \prec \neg a_2))$$

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Recall that model completions are always axiomatized by Π_2-sentences and that we can move back-and-forth between Π_2-sentences (in the first-order language of simple symmetric strict implication algebras aka contact algebras) and Π_2-rules in $\mathsf{S^2IC}$.

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It is not clear how to give a direct definition of what a basis of admissible Π_2-rules should be. In any case, any meaningful definition should be equivalent to the fact that the Π_2-rules of such a base, once translated to Π_2-sentences, should constitute an axiomatizazion of $\mathsf{Con}^\star$.

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If we read the above finite axiomatizability result in this way, we have shown that there is a finite base of admissible Π_2-rules for S^2IC.

THANK YOU!