Some facts and questions around the Kuznetsov problem

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The Question

Kuznetsov problem

Is every superintuitionistic logic topologically complete?

A. V. Kuznetsov asked this in 1974.

More precisely, an equivalent question is contained in his plenary lecture at the Vancouver ICM, which he could not deliver himself since Soviet officials did not let him go there.

In the meanwhile…

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(More precisely, Gerson was using neighborhood semantics.)

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where

$$\begin{aligned} J_1 &= \Box(p_0 \to ((\Diamond p_1) - p_1)), & J_2 &= \Box(-(p_0 \vee p_1) \to \Box(-(p_0 \vee p_1))), \\ J_3 &= \Box(B_1 \to ((\Diamond B_0) - (\Diamond C_0))), & J_4 &= \Box(C_1 \to ((\Diamond C_0) - (\Diamond B_0))), \\ J_5 &= \Box - (B_0 \wedge \Diamond B_1), & J_6 &= \Box - (C_0 \wedge \Diamond C_1) \end{aligned}$$

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and $A_m = (\Diamond B_{m+1} \wedge \Diamond C_{m+1}) - \Diamond B_{m+2}$,

where $B_0 = q_0$, $C_0 = r_0$, $B_1 = q_1$, $C_1 = r_1$ are sentence letters and

$$B_{m+2} = (\Diamond B_{m+1} \wedge \Diamond C_m) - \Diamond C_{m+1},$$

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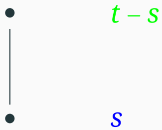

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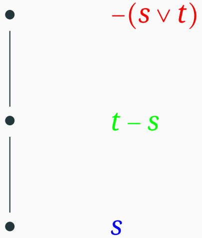

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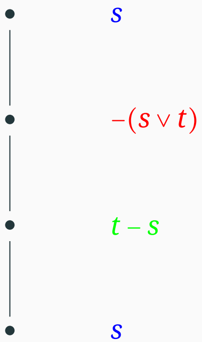

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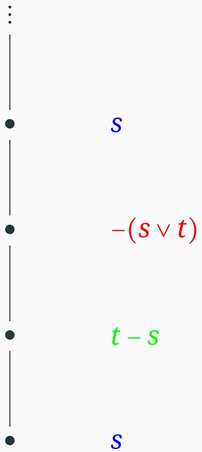

In a nutshell

In few words, $\mathbf{G} = E \to F$ allows, for any valuation $\mathscr{V}$ in a Kripke frame with $\mathscr{V}(E) \neq \varnothing$, to construct an infinite ascending chain; this then is incompatible with $\mathbf{H}$, so $-E$ holds in any L_{Fine}-Kripke frame.

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Gerson upgraded the infinite ascending chain to $W_0, W_1, W_2, \ldots$ in a neighborhood frame, with $W_i \cap W_j = \varnothing$ for $i \neq j$ and $W_n \subseteq \overline{W_{n+1}}$ (closure);

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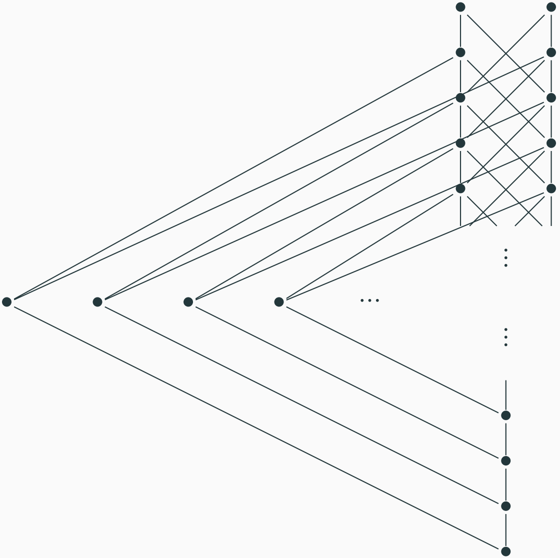
On a general frame, you can avoid having these infinite unions; and indeed Fine constructed an L_{Fine}-model with $\mathscr{V}(E) \neq \varnothing$ on the famous frame $\mathscr{F}$ of his name, thereby showing that $L_{\text{Fine}} \nvdash -E$.



Enter Shehtman

What Shehtman did in 1977:

he reduced the number of variables in **G** from six to two;

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For that, instead of interleave along a chain, he used interleave along an antichain! He achieved it using the Gabbay-de Jongh bounded branching formula $\mathbf{bb}_2$.

$\mathbf{bb}_2$

$\mathbf{bb}_2$ is

$$
\begin{aligned}
&[(x \to (y \lor z)) \to (y \lor z)] \\
\land&[(y \to (x \lor z)) \to (x \lor z)] \\
\land&[(z \to (x \lor y)) \to (x \lor y)] \\
&\to (x \lor y \lor z)
\end{aligned}
$$

expresses branching not exceeding two in finite Kripke structures.

$\mathbf{bb}_2$ is

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expresses branching not exceeding two in finite Kripke structures.

That is, no point shall have more than two immediate successors.

$\mathbf{bb}_2$

Topologically, $\mathbf{bb}_2$ says this: given three closed sets C_1, C_2, C_3 with $C_1 \cap C_2 = C_1 \cap C_3 = C_2 \cap C_3 = C$, if C is nowhere dense in each of the C_i then $C = \varnothing$.

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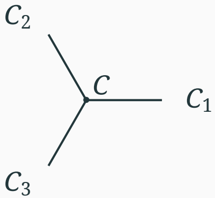
In other words, $\mathbf{bb}_2$ forbids such things:



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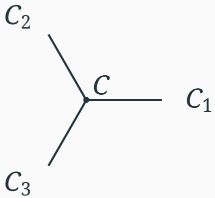

In the Kripke semantics, "nowhere dense" means "nowhere cofinal". Given lower sets $C \subset D$, we say that C is nowhere cofinal in D if $\downarrow(D - C) = D$.

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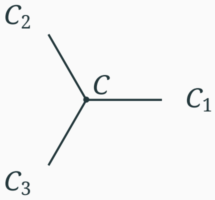

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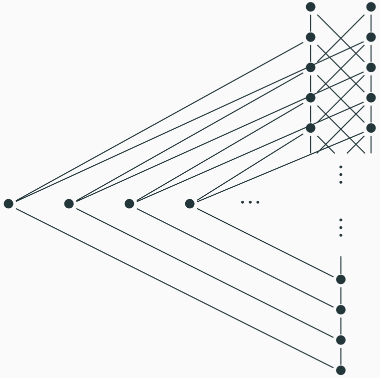
For Esakia spaces, this can be further reduced to $C \cap \max(D) = \varnothing$ (for clopen downsets $C \subset D$).

$\mathbf{bb}_2$ topologically

$\mathbf{bb}_2$ is a severe restriction on a topological space. It is less restrictive than hereditary extremal disconnectedness imposed by $(p \to q) \vee (q \to p)$ but still very strong.

Fine frame and Esakia semantics

In the complete Heyting algebra of all upper sets of the Fine frame $\mathscr{F}$, consider the subalgebra $\mathbb{A}$ generated by values of the variables p, q of **Ш**.



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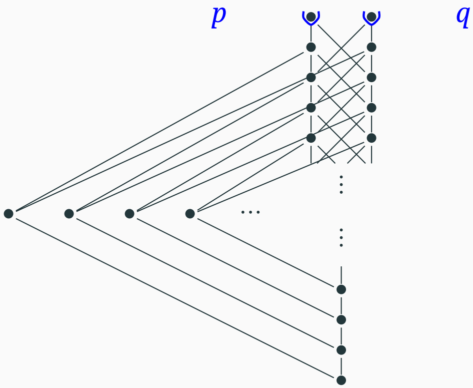

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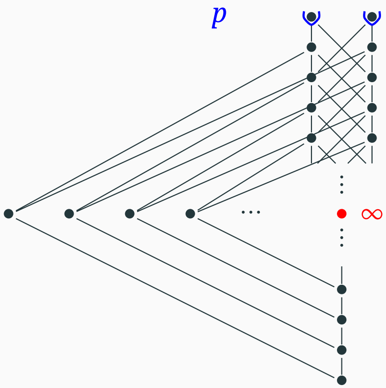

Our take on III

Let $m_i = \bot_i := p_i \wedge q_i$, $u_i = p_i \vee q_i$, and denote $\neg_i x := x \to \bot_i$.

Moreover let $a_i = \neg_i \neg_i u_i$ and $d_i = \neg_i u_i \vee \neg_i \neg_i u_i$.

Here $p_0 = p$, $q_0 = q$ and

$$p_{i+1} = q_i \vee \neg_i q_i, \qquad q_{i+1} = p_i \vee \neg_i p_i.$$

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Then, III is equivalent to both

$$(a_2 \to d_3) \to d_2$$

and to

$$(a_3 \to d_2) \wedge ((a_1 \to d_3) \to d_3).$$

Crucial: means $-\mathscr{V}(d_3)$ is nowhere {dense,cofinal,...} in $-\mathscr{V}(a_1)$!

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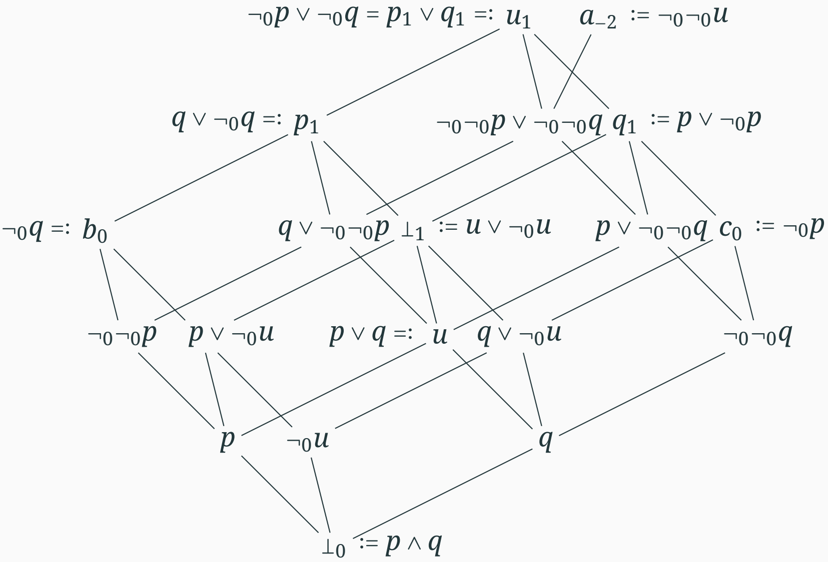
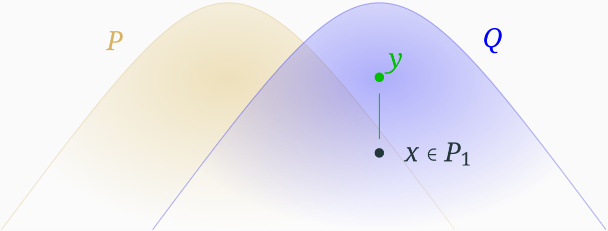


III in the Esakia semantics

$x \in P_1$:



$x \in P$ and there is a $y \geqslant x$ with $y \notin P$.

Here and in what follows for convenience we are using complementary clopen lower sets instead of clopen upper sets; $P := [\![p]\!] = -\mathscr{V}(p)$, $Q = [\![q]\!]$, etc.

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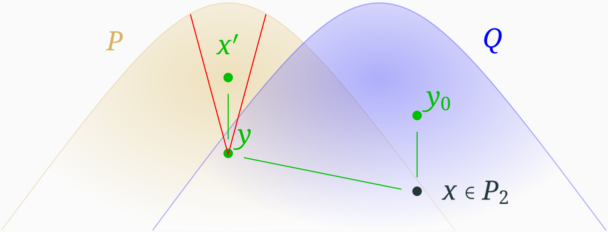

$x \in P_1$ and there is a $y \geqslant x$ with $y \in Q_1$ and $y \notin P_1$;

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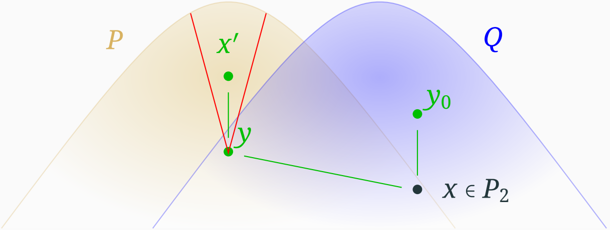
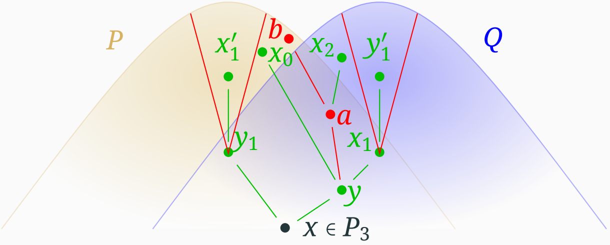


$x \in P_1$ and there is a $y \geqslant x$ with $y \in Q_1$ and $y \notin P_1$;

in detail: there is a $y_0 \geqslant x$ with $y_0 \notin P$ and there are $x' \geqslant y \geqslant x$ with $y \in Q$, $x' \notin Q$, and for no $c \geqslant y$ $c \notin P$.

III in the Esakia semantics

$x \in P_3$:



$x \in P_2$ and there is a $y \geqslant x$ with $y \in Q_2$ and $y \notin P_2$;

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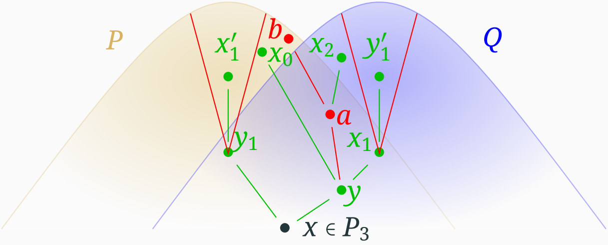
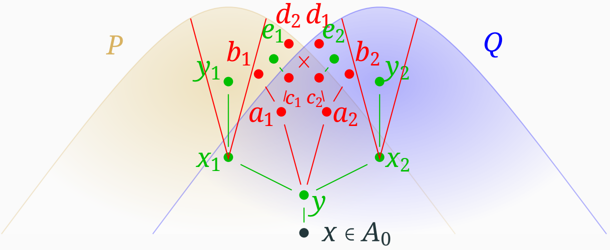


$x \in P_2$ and there is a $y \geqslant x$ with $y \in Q_2$ and $y \notin P_2$;

in detail: there are $x_1' \geqslant y_1 \geqslant x$ with $y_1 \in Q$, $x_1' \notin Q$, and for no $c \geqslant y_1$ $c \notin P$, and also there are $y_1' \geqslant x_1 \geqslant y \geqslant x$, $x_0 \geqslant y$ such that $x_0 \notin Q$, $x_1 \in P \cap Q$, $y_1' \notin P$, for no $c \geqslant x_1$ $c \notin Q$, and moreover for all $b \geqslant a \geqslant y$ with ...

III in the Esakia semantics

$x \in A_0$:



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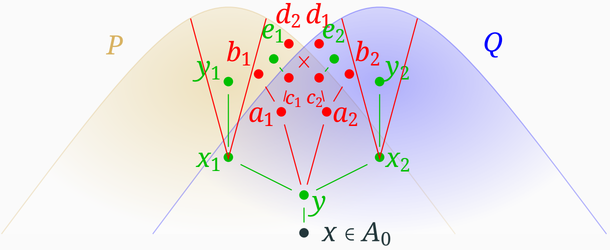

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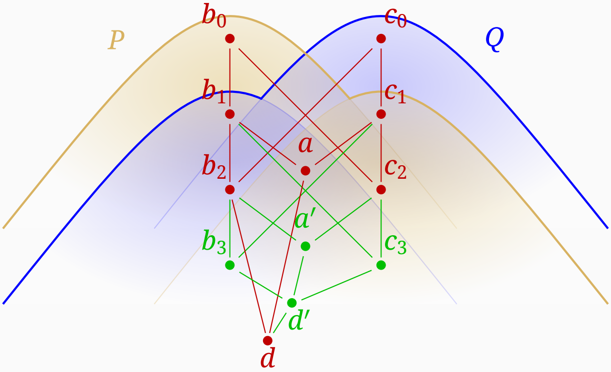
here we recall for the first time that we are in a Esakia space, hence this is equivalent to requiring that *there is a* $y \geqslant x$ *with* $y \in P_2 \cap Q_2 \cap \max(P_2 \cup Q_2)$

III in the Esakia semantics

III (well, almost...)



Algebraic semantics

For Heyting algebras, the natural question to consider is whether every variety of Heyting algebras is generated by *complete* Heyting algebras (Litak says Visser proposed to call this complete completeness).

This is weaker than topological completeness: a complete Heyting algebra is isomorphic to the algebra of all open sets of a topological space iff each of its elements is a join of primes.

(p prime means $x \wedge y = p \Rightarrow x = p \vee y = p$.)

“Staying closer to Kripke”

Every Heyting algebra of all upper sets of a Kripke frame is in fact bi-Heyting (its opposite is Heyting too).

Bi-Heyting algebras provide algebraic semantics for Heyting-Brouwer logics.

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In fact, we showed there are continuum many such, adapting a result of Litak from the Kripke context.

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In fact, we showed there are continuum many such, adapting a result of Litak from the Kripke context.

This also implies that there are continuum many varieties of Heyting algebras not generated by complete bi-Heyting algebras.

“Staying closer to Kripke”

And also that there are continuum many superintuitionistic logics incomplete with respect to complete bi-Heyting algebras.

The latter has been also proved by Guillaume Massas using completely different approach related to the semantics of the Propositional Lax Logic and the Dragalin semantics.

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In the Fairtlough-Mendler/Dragalin semantics, one can efficiently represent any complete Heyting algebra by (some) upper sets of a Kripke frame in such a way that arbitrary infinite intersections are preserved!

“Staying closer to Kripke”

The tradeoff is that even the binary unions are not preserved.

This comes very close to our main obstacle.

Using III we obtain closed sets $A_0, A_1, A_2, \ldots$ with $D_i = A_i \cap A_{i+1} = A_i \cap A_{i+2} = \ldots$ nowhere dense in A_i.

This allows us to violate $\mathbf{bb}_2$ using $C_0 = A_0 \cup A_3 \cup A_6 \cup \cdots$, $C_1 = A_1 \cup A_4 \cup A_7 \cup \cdots$, $C_2 = A_2 \cup A_5 \cup A_8 \cup \cdots$, with $C = \bigcup D_i$.

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This allows us to violate $\mathbf{bb}_2$ using $C_0 = A_0 \cup A_3 \cup A_6 \cup \cdots$, $C_1 = A_1 \cup A_4 \cup A_7 \cup \cdots$, $C_2 = A_2 \cup A_5 \cup A_8 \cup \cdots$, with $C = \bigcup D_i$.

However to achieve $C_i \cap C_j = C$ we need infinite joins of closed sets to distribute over binary meets, which implies bi-Heytingness.

“Staying closer to Kripke”

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In the Dragalin/Fairtlough-Mendler semantics, infinite joins are just infinite unions! But binary meets go out of control, unfortunately.

“Staying closer to Kripke”

Still another sense in which one may “stay closer to Kripke” is to consider Scott topologies on directed-complete partial orders.

However for a continuous poset this topology is bi-Heyting (in fact, topologies of continuous posets are precisely completely distributive lattices).

(Is a characterization of dcpo’s with bi-Heyting (not necessarily completely distributive) Scott topology known?)

So to get something new in this direction one has to deal with really ugly dcpo’s.

A couple of questions about quantifier eliminability

When trying to simplify the Shehtman axiom, this question arose:

Given a pair of elements a, b of a Heyting algebra such that $a = x \vee \neg x$ and $b = y \vee \neg y$ for some x, y with $x \wedge y = \varnothing$. Can one write an equivalent condition characterizing such a, b, without mentioning x, y?

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For the well known (much) simpler case, note that a has form $x \vee \neg x$ for some x if and only if $\neg a = \bot$ (the bottom element of the algebra).

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In Esakia semantics, $a = x \vee \neg x \Leftrightarrow \neg a = \perp$ means that the clopen upper set a contains the whole maximum.

Whereas $a = \neg x \vee \neg\neg x$ means that a contains the “fat maximum” – all points which see a unique point in the maximum.

A couple of (hopefully) easier questions

Another possible simplification – throw in some (intuitionistic) modalities.

For example, the Kuznetsov-Muravitsky calculus **KM** extends the language of **IPC** with a modality Δ;

intended interpretation of Δ in the topological semantics is

$$\llbracket \Delta\varphi \rrbracket = -\delta - \llbracket \varphi \rrbracket\,,$$

where δS is the set of limit points of the set S.

A couple of (hopefully) easier questions

This increases expressive power sufficiently to talk, e. g., about scatteredness:

$$(\Delta\varphi \to \varphi) \to \varphi.$$

The question about topological (in)completeness of extensions of **KM** might be not so difficult to answer.

A couple of (hopefully) easier questions

Not directly related to Kuznetsov but “similar in spirit”: is every variety of Heyting algebras generated by bi-Heyting algebras?

Thank you!