The general algebraic framework for Mathematical Fuzzy Logic

Petr Cintula[1] Carles Noguera[2]

[1]Institute of Computer Science
Czech Academy of Sciences

[2]Department of Information Engineering and Mathematics
University of Siena

Three stages of development of an area of logic

Chagrov (*K voprosu ob obratnoi matematike modal’noi logiki*,
Online Journal Logical Studies, 2001)

distinguishes three stages in the development of a field in logic:

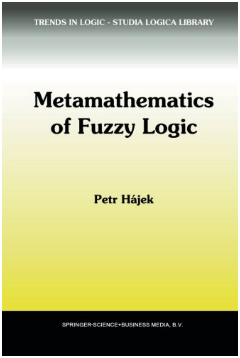
1. Emerging of the area
2. Development of particular logics and introduction of new ones
3. Universal methods

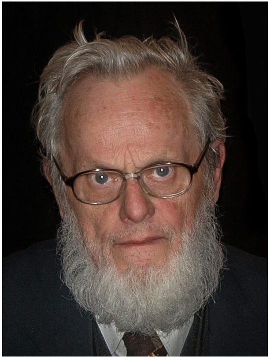
Three stages of development of MFL

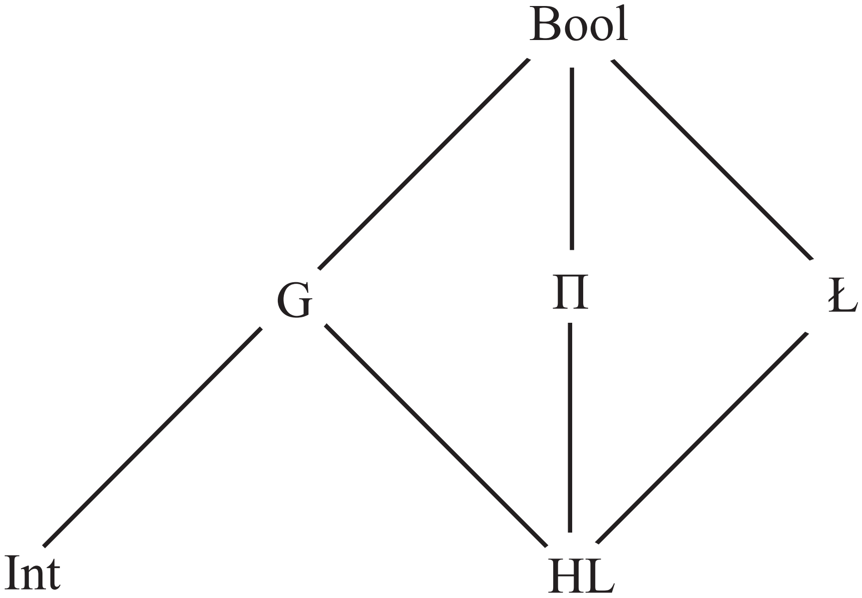
First stage: Emerging of the area (since 1965)

- 1965: Zadeh's fuzzy sets, 1968: 'fuzzy logic' (Goguen)
- 1970s: systems of fuzzy 'logic' lacking a good metatheory
- 1970s–1980s: first 'real' logics (Pavelka, Takeuti–Titani, ...),
  discussion of many-valued logics in the fuzzy context

Culminated in Hájek's monograph (1998): G, Ł, Π, HL





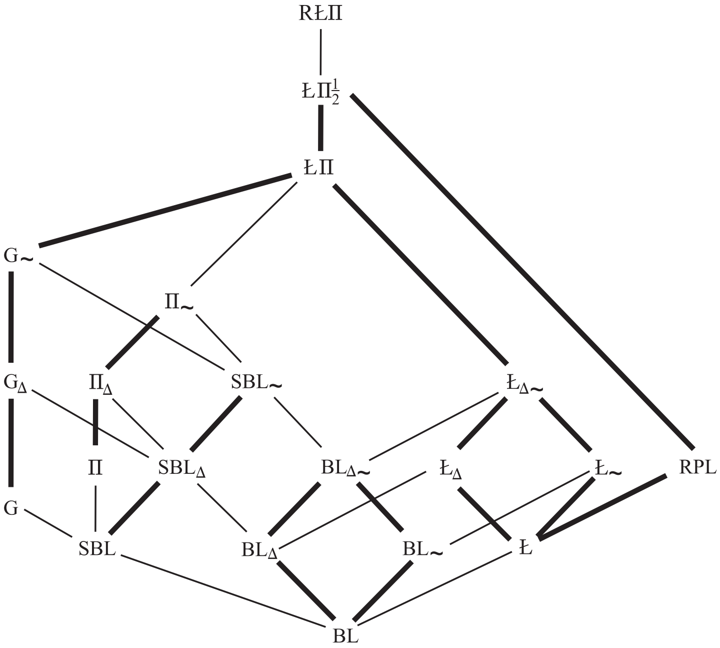

Bool
G
Π
Ł
Int
HL

Three stages of development of MFL

Second stage: development of particular logics and introduction of many new ones (since the 1990s)

- New logics: MTL, SHL, UL, $\Pi_{\sim}$, ŁΠ, …
- Algebraic semantics, proof theory, complexity
  Kripke-style and game-theoretic semantics, …
- First-order, higher-order, and modal fuzzy logics

Systematic treatment of particular fuzzy logics


RŁΠ
ŁΠ½
ŁΠ
G∼
Π∼
G∆
Π∆
SBL∼
Ł∆∼
Π
SBL∆
BL∆∼
Ł∆
Ł∼
RPL
G
SBL
BL∆
BL∼
Ł
BL

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In the 3rd EUSFLAT (Zittau, Germany, September 2003) Petr Hájek started his lecture *Fleas and fuzzy logic: a survey* with a joke.

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A group of scientists decide to investigate the ability of a flea can jump in relationship to how many legs it has.

They put the flea on a desk and said ’jump!’ The flea jumped and they noted:
“the flea with 6 legs can jump.”

They remove a leg, repeated the command, the flea jumped and they noted:
“the flea with 5 legs can jump.”

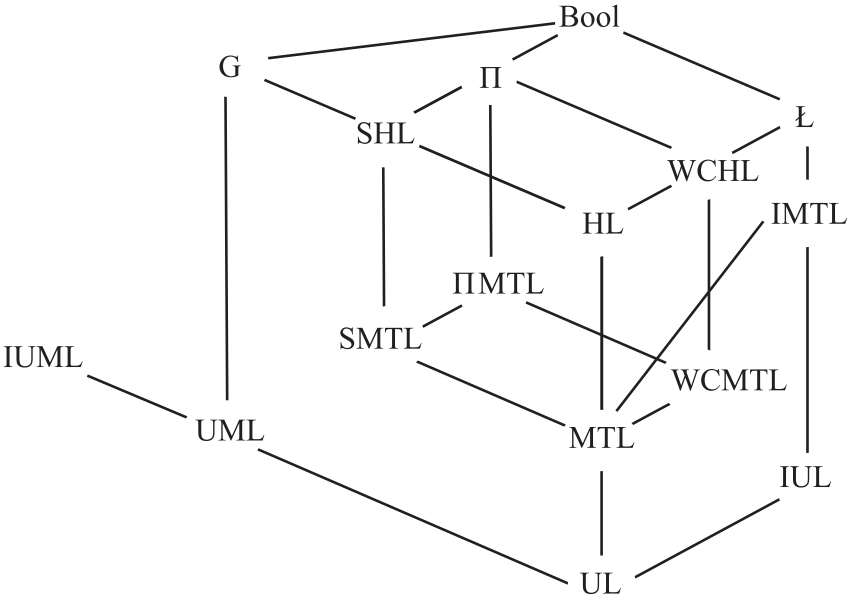
$\vdots$

Finally, they removed the last legs repeated the command but the flea didn’t move.

So they concluded:

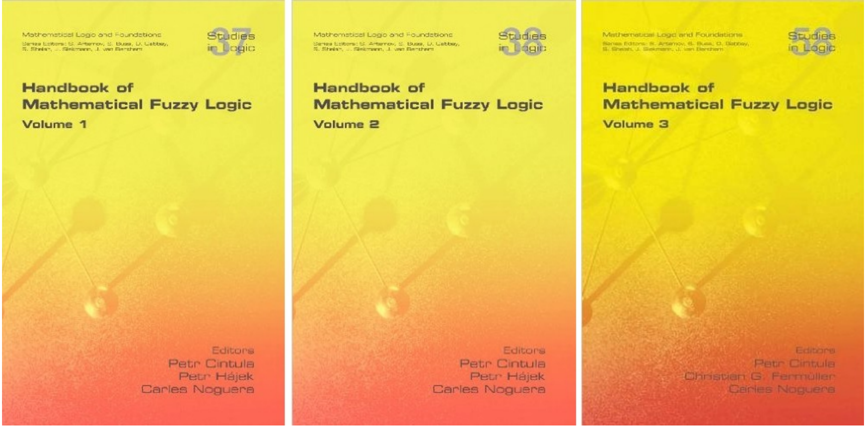
“Upon removing all its legs the flea loses sense of hearing.”

$$
\begin{tikzcd}[column sep=small]
 & \mathrm{HL} \arrow[d, dash] & \\
 & \mathrm{MTL} \arrow[dl, dash] \arrow[dr, dash] & \\
\mathrm{psMTL}^{\mathrm{r}} \arrow[dr, dash] & & \mathrm{UL} \arrow[dl, dash] \\
 & \mathrm{psUL} \arrow[d, dash] & \\
 & \mathrm{SL}^{\ell} &
\end{tikzcd}
$$


Bool
G
Π
SHL
Ł
WCHL
HL
IMTL
ΠMTL
SMTL
IUML
WCMTL
UML
MTL
IUL
UL

Three stages of development of MFL

The second stage is still ongoing; the state of the art is summarized in:



P. Cintula, C. Fermüller, P. Hájek, C. Noguera (editors). Vol. 37, 38, and 58 of *Studies in Logic: Math. Logic and Foundations*. College Publications, 2011, 2015.

Three stages of development of MFL

Third stage: universal methods

- There was a great deal of repetition in papers: slightly different logics studied by repeating the same definitions and essentially obtaining the same results by means of analogous proofs
- MFL needed general methods to prove metamathematical properties
- Classification of existing fuzzy logics
- Systematic treatment of classes of fuzzy logics
- Determining the position of fuzzy logics in the logical landscape

Basic fuzzy logic?

Hájek called the logic HL the Basic fuzzy Logic BL

HL was *basic* in the following two senses:

1. *it could not be made weaker without losing essential properties*
2. *it provided a base for the study of all fuzzy logics.*

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Because:

- HL is complete w.r.t. the semantics given by *all* continuous t-norms
- All fuzzy logics known by then were expansions of HL. The methods to introduce, algebraize, and study HL could be modified for all its expansions.

fuzzy logics = expansions of HL

(Abstract) algebraic logic

- Algebraic logic: study of particular logical systems by giving them a semantics based on some algebraic structures
- Abstract algebraic logic (AAL): aims at understanding the various ways in which an arbitrary logical system can be endowed with an algebraic semantics.

There were great works in these areas (Blok, Pigozzi, Czelakowsi, Font, Jansana, etc.), but still too detached from the specific needs of MFL.

Weakly implicative logics

Minimal reasonable behavior of an implication.

Definition

A logic L in a countable language $\mathcal{L}$ is weakly implicative if there is a binary connective $\Rightarrow$ (primitive or definable) such that:

(id)	$\vdash_{\mathrm{L}} \varphi \Rightarrow \varphi$
(MP)	$\varphi, \varphi \Rightarrow \psi \vdash_{\mathrm{L}} \psi$
(T)	$\varphi \Rightarrow \psi, \psi \Rightarrow \chi \vdash_{\mathrm{L}} \varphi \Rightarrow \chi$
(sCng)	$\varphi \Rightarrow \psi, \psi \Rightarrow \varphi \vdash_{\mathrm{L}} c(\chi_1, \ldots, \chi_i, \varphi, \ldots, \chi_n) \Rightarrow c(\chi_1, \ldots, \chi_i, \psi, \ldots, \chi_n)$ for each $\langle c, n\rangle \in \mathcal{L}$ and each $0 \leq i < n$.

Such a connective is called a weak implication.

Semilinear (a.k.a. "fuzzy") logics

Matrices of a WIL L (denoted as $\mathbf{Mod}(\mathrm{L})$) can be pre-ordered using its weak implication $\Rightarrow$:

$$a \leq_{\langle \mathbf{A}, F \rangle} b \qquad \text{iff} \qquad a \Rightarrow^{\mathbf{A}} b \in F$$

and each WIL is complete w.r.t. its ordered matrices (denoted as $\mathbf{Mod}^*(\mathrm{L})$)

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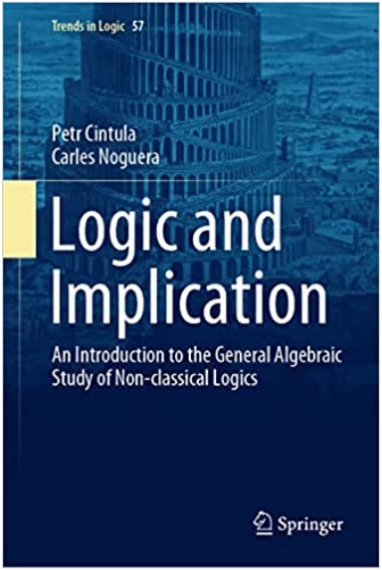
$$\mathbf{Mod}^*(\mathrm{L})_{\mathrm{RFSI}} = \mathbf{Mod}^\ell(\mathrm{L})$$

And now a book for the third stage



Contents

Sample of results – 1

Theorem

Let L *be a weakly implicative logic and* $\mathbb{K} \subseteq \mathbf{Mod}^*(\mathrm{L})$*. Then*
L *has the* $\mathbb{K}\mathrm{C}$ *if and only if* $\mathbf{Mod}^*(\mathrm{L}) \subseteq \mathbf{HSP}(\mathbb{K})$.

Sample of results – 2

Theorem

Let L *be a weakly implicative logic and* $\mathbb{K} \subseteq \mathbf{Mod}^*(\mathrm{L})$. *TFAE:*

1. L *has the* $\mathrm{S}\mathbb{K}\mathrm{C}$.
2. $\mathbf{Mod}^*(\mathrm{L}) = \mathbf{ISP}_\omega(\mathbb{K})$.
3. $\mathbf{Mod}^*(\mathrm{L})^\omega \subseteq \mathbf{ISP}(\mathbb{K})$.

If furthermore L *is finitary, then we can add:*

4. $\mathbf{Mod}^*(\mathrm{L})^\omega_{\mathrm{RSI}} \subseteq \mathbf{IS}(\mathbb{K})$.

The implication $1 \to 4$ *is always true.*

Sample of results – 3

Theorem

Let L *be a weakly implicative logic and* $\mathbb{K} \subseteq \mathbf{Mod}^*(\mathrm{L})$. *TFAE:*

1. L *has the* $\mathrm{FS}\mathbb{K}\mathrm{C}$.
2. $\mathbf{Mod}^*(\mathrm{L}) \subseteq \mathbf{ISPP}_{\mathrm{U}}(\mathbb{K})$.

Furthermore, if L *is finitary, then we can add:*

3. $\mathbf{Mod}^*(\mathrm{L}) = \mathbf{ISPP}_{\mathrm{U}}(\mathbb{K})$.
4. $\mathbf{Mod}^*(\mathrm{L})^{\omega}_{\mathrm{RSI}} \subseteq \mathbf{ISP}_{\mathrm{U}}(\mathbb{K})$.

If the language of L *is finite, we can add:*

6. $\mathbf{Mod}^*(\mathrm{L})^{\omega}_{\mathrm{RSI}}$ *is partially embeddable into* $\mathbb{K}$.

Sample of results – 4

Theorem

Let L be a weakly implicative logic with a "reasonable" generalized disjunction ∇ (has $\nabla\mathrm{PEP}$). Then,

$$\models_{\{\mathbf{B}\in\mathbf{Mod}^*(\mathrm{L})\mid \mathbf{B}\text{ is linearly ordered}\}} = \mathrm{L} + (\varphi \Rightarrow \psi)\nabla(\psi \Rightarrow \varphi).$$

Theorem

Let L be a logic with the $\nabla\mathrm{PEP}$ and let L_1 and L_2 be axiomatic extensions of L by sets of axioms $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively. Then, $\mathrm{L}_1 \cap \mathrm{L}_2$ is an axiomatic extension of L and

$$\mathrm{L}_1 \cap \mathrm{L}_2 = \mathrm{L} + \bigcup\{\varphi\nabla\psi \mid \varphi \in \mathcal{A}_1, \psi \in \mathcal{A}_2\}.$$

Therefore, the axiomatic extensions of L form a sublattice of its extensions.

Linear extension property and semilinearity property

Definition

We say that a weakly implicative logic L has the

- linear extension property, LEP for short, if linear theories form a basis of $\mathcal{T}h(\mathrm{L})$, i.e. for every theory T and every formula $\varphi \in Fm_{\mathcal{L}} \setminus T$, there is a linear theory $T' \supseteq T$ such that $\varphi \notin T'$.
- semilinearity property, SLP for short, if, for each set of formulas $\Gamma \cup \{\varphi, \psi, \chi\}$,

$$\frac{\Gamma, \varphi \Rightarrow \psi \vdash_{\mathrm{L}} \chi \qquad \Gamma, \psi \Rightarrow \varphi \vdash_{\mathrm{L}} \chi}{\Gamma \vdash_{\mathrm{L}} \chi}.$$

Sample of results – 5

Theorem

Let L *be a weakly implicative logic. Then, the following are equivalent:*

1. L *is semilinear.*
2. L *has the* LEP, *i.e., linear theories form a basis of* $\mathcal{T}h(\mathrm{L})$,
3. L *has the* IPEP *and* SLP, *i.e., fin-meet-irred theories form a basis of* $\mathcal{T}h(\mathrm{L})$ *(e.g., if* L *is finitary) and*

$$\frac{\Gamma, \varphi \Rightarrow \psi \vdash_{\mathrm{L}} \chi \qquad \Gamma, \psi \Rightarrow \varphi \vdash_{\mathrm{L}} \chi}{\Gamma \vdash_{\mathrm{L}} \chi}.$$

4. L *is* RFSI*-complete (i.e..* $\mathrm{L} = \models_{\mathbf{Mod}^*(\mathrm{L})_{\mathrm{RFSI}}}$, *e.g., if* L *is finitary) and any of the following (in this context equivalent) conditions holds:*
   1. $\mathrm{Fi}(X, a \Rightarrow^{\mathbf{A}} b) \cap \mathrm{Fi}(X, b \Rightarrow^{\mathbf{A}} a) = \mathrm{Fi}(X)$, *for each* $\mathcal{L}$*-alg.* $\mathbf{A}$ *and* $X \cup \{a, b\} \subseteq A$,
   2. *Linear filters coincide with fin-meet-irred filters in each* $\mathcal{L}$*-algebra.*
   3. $\mathbf{Mod}^*(\mathrm{L})_{\mathrm{RFSI}} = \mathbf{Mod}^\ell(\mathrm{L})$.