Intuitionistic Sahlqvist correspondence for deductive systems

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Joint work with Damiano Fornasiere

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- In addition, every Heyting algebra $\boldsymbol{A}$ embeds into $\mathsf{Up}(\boldsymbol{A}_*)$ via the map defined by the rule

  $$a \longmapsto \{F \in \boldsymbol{A}_* : a \in F\}.$$

Definition

A Sahlqvist quasiequation is an expression of the form

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where $\varphi_1, \ldots, \varphi_n$ are Sahlqvist formulas.

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$$(x_1 \to x_2) \wedge y \leqslant z \,\&\, (x_2 \to x_1) \wedge y \leqslant z \Longrightarrow y \leqslant z;$$

- The bounded top width n axiom as

$$\mathop{\&}_{1 \leqslant i \leqslant n+1} \Big(\neg(\neg x_i \wedge \bigwedge_{0<j<i} x_j) \wedge y \leqslant z \Big) \Longrightarrow y \leqslant z.$$

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$$\text{if } \boldsymbol{A} \models \Phi, \text{ then } \mathsf{Up}(\boldsymbol{A}_*) \models \Phi;$$

2. Correspondence: There is an effectively computable sentence $\mathsf{tr}(\Phi)$ in the language of posets such that for every poset $\mathbb{X}$,

$$\mathbb{X} \models \mathsf{tr}(\Phi) \quad \text{iff} \quad \mathsf{Up}(\mathbb{X}) \models \Phi.$$

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Representation Theorem

Every algebraic lattice L is isomorphic to the lattice of filters of L^{ω}.

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Example. Most logics with a respectable implication connective $x \to y$. To see this, take $\Delta := \{x \to y\}$.

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- the inconsistency lemma (IL) when for every $n \in \mathbb{Z}^+$ there exists a finite set ${\sim}_n(x_1, \ldots, x_n)$ of formulas s.t.

$$\Gamma \cup \{\varphi_1, \ldots, \varphi_n\} \text{ is inconsistent } \text{ iff } \Gamma \vdash {\sim}_n(\varphi_1, \ldots, \varphi_n);$$

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- the proof by cases (PC) when there exists a finite set $x \curlyvee y$ of formulas s.t.

$$\Gamma, \varphi \vdash \gamma \text{ and } \Gamma, \psi \vdash \gamma \text{ iff } \Gamma, \varphi \curlyvee \psi \vdash \gamma.$$

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Let $\vdash$ be a logic. A subset F of an algebra $\boldsymbol{A}$ is said to be a deductive filter of $\vdash$ on $\boldsymbol{A}$ if for every set $\Gamma \cup \{\varphi\}$ of formulas,

if $\Gamma \vdash \varphi$, then for every homomorphism $f\colon \boldsymbol{Fm} \to \boldsymbol{A}$ we have that if $f[\Gamma] \subseteq F$, then $f(\varphi) \in F$.

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Theorem (Czelakowski, Dziobiak, and Raftery)

A protoalgebraic logic $\vdash$ has the IL (resp. DT, PC) iff the semilattice $\mathsf{Fi}^{\omega}_{\vdash}(\boldsymbol{A})$ is pseudocomplemented (resp. implicative semilattice, distributive lattice) for every algebra $\boldsymbol{A}$.

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More precisely, given $k \in \mathbb{Z}^+$, the k-translation of φ is the finite set

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- Similarly, for $\vee$ and $\to$.

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- The logic $\vdash$ validates the metarules

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- For every algebra $\boldsymbol{A}$, we have $\mathsf{Spec}_{\vdash}(\boldsymbol{A}) \vDash \mathsf{tr}(\Phi)$.

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Lastly, every axiomatic extension of ILL *has the* PC.

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Thank you very much for your attention!