Logics of upsets of De Morgan lattices

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Finite basis theorems

A **finite basis theorem** gives a sufficient condition for the theory of a finite set of finite algebras in a finite signature K to be finitely axiomatizable.

Equational theory (Baker '77):

$$\mathbb{V}(\mathsf{K}) \text{ congruence distributive} \implies \text{finitely axiomatizable.}$$

Quasiequational theory (Pigozzi, '88):

$$\mathbb{Q}(\mathsf{K}) \text{ relatively congruence distributive} \implies \text{finitely axiomatizable.}$$

In the logical setting, K consists of matrices, i.e. algebras with a designated subset. Its logical theory is the set of logical rules $(\Gamma \vdash \varphi)$ valid in K.

Logical theory (Palasińska '94):

$$\mathrm{Log}(\mathsf{K}) \text{ filter distributive and protoalgebraic} \implies \text{finitely axiomatizable.}$$

Finite basis theorems

The above finite basis theorems are deep and general theorems.

In comparison, what I shall prove a shallow and specific theorem.

However, it introduces an idea which may be worth exploring: a logic with no finite Hilbert-style axiomatization (by logical rules $\gamma_1, \ldots, \gamma_n \vdash \varphi$) may still have a finite **Gentzen-style** axiomatization (by rules **and meta-rules**).

Another theme is the distinction between filter classes and logical classes.

Filter classes

Which families of upsets of distributive lattices are there that behave like the family of all lattice filters?

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Which families of upsets of distributive lattices are there
that behave like the family of all lattice filters?

Key properties of lattice filters: closed under

- homomorphic preimages (F filter $\implies h^{-1}[F]$ filter),
- arbitrary intersections (F_i filters $\implies \bigcap_{i \in I} F_i$ filter),
- directed unions (F_i dir. family of filters $\implies \bigcup_{i \in I} F_i$ filter).

Definition. A **(finitary) filter class** of upsets of distributive lattices is one closed under homomorphic preimages and intersections (and dir. unions).

Filter classes

Equivalent definition. A **filter class** (of upsets of distributive lattices) is a class of structures of the form $\langle \mathbf{A}, F \rangle$ (where $\mathbf{A}$ is a distributive lattice and F is an upset of $\mathbf{A}$) which is closed under

- ($\mathbb{H}_{\mathrm{S}}^{-1}$) surjective homomorphic preimages,
- ($\mathbb{S}$) substructures,
- ($\mathbb{P}$) products of structures.

A **finitary** filter class is moreover closed under

- ($\mathbb{P}_{\mathrm{U}}$) ultraproducts.

Fact. The filter class generated by K is $\mathbb{H}_{\mathrm{S}}^{-1}\mathbb{S}\mathbb{P}(\mathsf{K})$. Equivalently, it is the class of all intersections of homomorphic preimages of structures in K.

Filter classes

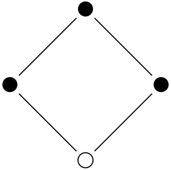
Theorem. The lattices of non-trivial finitary filter classes upsets of distributive lattices looks like this:

DL_∞	upsets	generated by $\{\mathbb{B}_n \mid n \geq 1\}$
$\vdots$		
DL_3	3-filters	generated by $\mathbb{B}_3$
DL_2	2-filters	generated by $\mathbb{B}_2$
DL_1	filters	generated by $\mathbb{B}_1$

Proof. Find the generators. Show that DL_n and DL_{n+1} form a splitting.

2-filters

Consider $\mathbb{B}_2 := \langle \mathbf{B}_2, P_2 \rangle$ where $\mathbf{B}_2 := (\mathbf{B}_1)^2$ and $P_2 := \{a \in \mathbf{B}_2 \mid a > \mathsf{f}\}$:



This is what we call a **2-filter**: an upset F such that

$$x \wedge y, y \wedge z, z \wedge x \in F \implies x \wedge y \wedge z \in F.$$

(In this case because given 3 elements in F at least 2 are comparable.)

n-filters

Consider $\mathbb{B}_n := \langle \mathbf{B}_n, P_n \rangle$ where $\mathbf{B}_n := (\mathbf{B}_1)^n$ and $P_n := \{a \in \mathbf{B}_n \mid a > \mathsf{f}\}$.

This is an n**-filter**: an upset F such that for each non-empty finite $X \subseteq F$

$$\bigwedge Y \in F \text{ for each } Y \subseteq X \text{ with } 1 \leq |Y| \leq n \implies \bigwedge X \in F.$$

Without loss of generality we may take $|X| = n + 1$ in this definition.

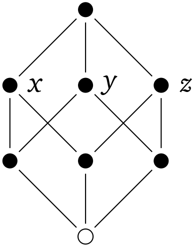
Fact. n-filters form a finitary filter class of upsets of distributive lattices.

Fact. The n-filters of a distributive lattice $\mathbf{L}$ form a distributive lattice $\mathrm{Fi}_n\,\mathbf{L}$.

Examples

Fact. The set $P_n \subseteq \mathbf{B}_n$ is an n-filter but not an m-filter for any $m < n$.

Example. $P_3 \subseteq \mathbf{B}_3$ is not a 2-filter ($x \wedge y, y \wedge z, z \wedge x \in F$ but $x \wedge y \wedge z \notin F$):

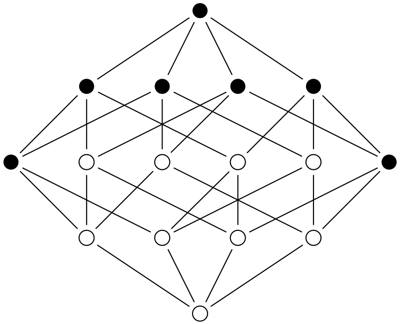


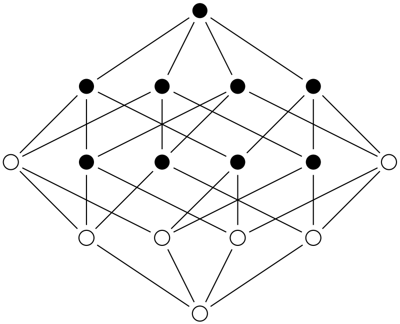
Examples

Fact. Each union of n ordinary filters is an n-filter (but not vice versa).

Corollary. Each upset of a finite lattice is an n-filter for some n.

Example. The following are 2-filters on $\mathbf{B}_4$:





Prime n-filters and k-prime filters

Definition. A meet k-prime element of a lattice $\mathbf{L}$ is an $a \in \mathbf{L}$ such that for each non-empty finite $X \subseteq \mathbf{L}$ (w.l.o.g. with $|X| = k+1$)

$$\bigwedge X \leq a \implies \bigwedge Y \leq a \text{ for some } Y \subseteq X \text{ with } 1 \leq |Y| \leq k.$$

For $k := 1$ this yields the usual definition of meet primes.

Definition. An **k-prime n-filter** on $\mathbf{L}$ is a meet k-prime element of $\mathrm{Fi}_n\, \mathbf{L}$.

Fact. An n-filter is prime if its complement is an ideal.

Fact. A filter is k-prime if its complement is an k-ideal.

Generating the filter class DL_n

How do we find the generators of DL_1?

Lemma. Each filter is an intersection of prime filters.

(**Proof** relies on having a description of the filter generated by an upset.)

Lemma. Each prime filter is a homomorphic preimage of $P_1 \subseteq \mathbf{B}_1$.

Theorem. The filter class DL_1 of all filters is generated by $\langle \mathbf{B}_1, P_1 \rangle$.

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Why do the structures $\langle \mathbf{B}_n, P_n \rangle$ arise in this context?

Because of the following lemma and the **dual product** construction.

Lemma. Each prime n-filter is a union of at most n prime filters.

Dual products of matrices

Consider a family $\langle \mathbf{A}_i, F_i \rangle$ for $i \in I$ with projection maps $\pi_i \colon \prod_{i \in I} \mathbf{A}_i \to \mathbf{A}_i$.

The **direct product** of this family is the matrix

$$\prod_{i \in I} \langle \mathbf{A}_i, F \rangle := \Big\langle \prod_{i \in I} \mathbf{A}_i, \bigcap_{i \in I} \pi_i^{-1}[F_i] \Big\rangle.$$

The **dual product** of this family is the matrix

$$\bigotimes_{i \in I} \langle \mathbf{A}_i, F_i \rangle := \Big\langle \prod_{i \in} \mathbf{A}_i, \bigcup_{i \in I} \pi_i^{-1}[F_i] \Big\rangle.$$

The dual product construction was studied by Badia & Marcos (2018).

Key example. The matrix $\langle \mathbf{B}_n, \{\mathtt{t}\} \rangle$ is the n-th direct power of $\langle \mathbf{B}_1, \{\mathtt{t}\} \rangle$.
The matrix $\mathbb{B}_n := \langle \mathbf{B}_n, P_n \rangle$ is the n-th dual power of $\mathbb{B}_1 := \langle \mathbf{B}_1, \{\mathtt{t}\} \rangle$.

Dual products of matrices

Fact. Consider a family of strict homomorphisms

$$h_i\colon \langle \mathbf{A}, F_i\rangle \to \langle \mathbf{B}_i, G_i\rangle \text{ for } i \in I.$$

Then the product map yields two strict homomorphisms, namely

$$h\colon \langle \mathbf{A}, \bigcap_{i\in I} F_i\rangle \to \prod_{i\in I} \langle \mathbf{B}_i, G_i\rangle, \qquad h\colon \langle \mathbf{A}, \bigcup_{i\in I} F_i\rangle \to \bigotimes_{i\in I} \langle \mathbf{B}_i, G_i\rangle.$$

Key example. Consider some prime n-filter F on $\mathbf{A}$. Then $F = F_1 \cup \dots \cup F_n$ for some prime filters F_i. This yields a family of strict $h_i\colon \langle \mathbf{A}, F_i\rangle \to \mathbb{B}_1$. The product of these maps is a strict $h\colon \langle \mathbf{A}, F\rangle \to \mathbb{B}_n$.

Key example. Consider some n-prime F on $\mathbf{A}$. Then $F = F_1 \cap \dots \cap F_n$ for some prime filters F_i. This yields a family of strict $h_i\colon \langle \mathbf{A}, F_i\rangle \to \mathbb{B}_1$. The product of these maps is a strict $h\colon \langle \mathbf{A}, F\rangle \to (\mathbb{B}_1)^n$.

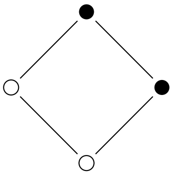
De Morgan lattices

We can now play the same game with De Morgan lattices (DMLs), i.e. distributive lattices equipped with an order-inverting involution $\neg$.

Let DM_n (DM_∞) be the class of all matrices $\langle \mathbf{A}, F \rangle$ where $\mathbf{A}$ is a DML and $F \subseteq \mathbf{A}$ is an n-filter (upset). Let BD_n (BD_∞) be the corresponding logic.

Remark. BD_1 is known as Belnap–Dunn logic, or FDE.

The role of $\mathbb{B}_1$ will be taken over by the matrix $\mathbb{DM}_1 := \langle \mathbf{DM}_1, Q_1 \rangle$:



Lemma. Each prime filter on a DML is a homomorphic preimage of Q_1.

Generating the filter class DM_n

The proof for distributive lattices carries over to DMLs.

Lemma. Each n-filter on a DML is an intersection of prime n-filters.

Lemma. Each prime n-filter is a union of at most n prime filters.

Lemma. Each prime filter is a homomorphic preimage of $Q_1 \subseteq \mathbf{DM}_1$.

Theorem. The filter class DM_n of n-filters of upsets of De Morgan lattices is generated by $\mathbb{DM}_n := (\mathbb{DM}_1)^{\otimes n}$ (the n-th dual power of $\mathbb{DM}_1$).

Remark. There are continuum many other filter classes of upsets of DMLs.

Logical classes vs. filter classes

Definition. A **logical class** is a filter class which is moreover closed under

- ($\mathbb{H}_S$) strict surjective homomorphic images.

That is, $\langle \mathbf{A}, h^{-1}[G] \rangle \in \mathsf{K}$ implies $\langle \mathbf{B}, G \rangle \in \mathsf{K}$ for $h\colon \mathbf{A} \to \mathbf{B}$ surjective.

Fact. The logical class generated by K is $\mathbb{H}_S^{-1}\mathbb{H}_S\mathbb{S}\mathbb{P}(\mathsf{K})$.

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Fact. The logical class generated by K is $\mathbb{H}_S^{-1}\mathbb{H}_S\mathbb{S}\mathbb{P}(\mathsf{K})$.

Theorem (Dellunde & Jansana '96). Logical classes are precisely the classes axiomatized by (possibly infinitary) logical rules:

$$\mathrm{True}(\gamma_1) \;\&\; \dots \;\&\; \mathrm{True}(\gamma_n) \implies \mathrm{True}(\varphi).$$

Theorem (Stronkowski '18). Filter classes are precisely the classes axiomatized by (possibly infinitary) generalized logical rules:

$$\alpha_1 \approx \beta_1 \;\&\; \dots \;\&\; \alpha_m \approx \beta_m \;\&\; \mathrm{True}(\gamma_1) \;\&\; \dots \;\&\; \mathrm{True}(\gamma_n) \implies \mathrm{True}(\varphi).$$

That is, we can use equations in the premises (but not in the conclusion).

The n-PCP

Definition. A logic has the **proof by cases property** (**PCP**) if

$$\Gamma, \alpha \vdash \varphi \;\&\; \Gamma, \beta \vdash \varphi \iff \Gamma, \alpha \vee \beta \vdash \varphi.$$

Definition. A logic has the 2-**proof by cases property** (2-**PCP**) if

$$\Gamma, \alpha \vee \beta \vdash \varphi \;\;\&\;\; \Gamma, \beta \vee \gamma \vdash \varphi \;\;\&\;\; \Gamma, \gamma \vee \alpha \vdash \varphi \implies \Gamma, \alpha \vee \beta \vee \gamma \vdash \varphi.$$

The n-**PCP** is the obvious generalization. The 1-PCP is just the PCP.

Fact. For finitary extensions of BD_∞

$$\text{PCP} \iff \text{complete w.r.t. a set of DMLs with prime upsets.}$$

Fact. For finitary extensions of BD_1

$$n\text{-PCP} \iff \text{complete w.r.t. a set of DMLs with } n\text{-prime filters.}$$

Hierarchies of extensions

Theorem. The following are equivalent:

- L is a finitary extension of BD_n with the PCP,
- L is complete w.r.t. some finite set of DMLs with prime n-filters,
- L is complete w.r.t. some set of submatrices of $\mathbb{DM}_n$.

Some such n exists for each fin. generated extension of BD_∞ with the PCP.

Theorem. The following are equivalent:

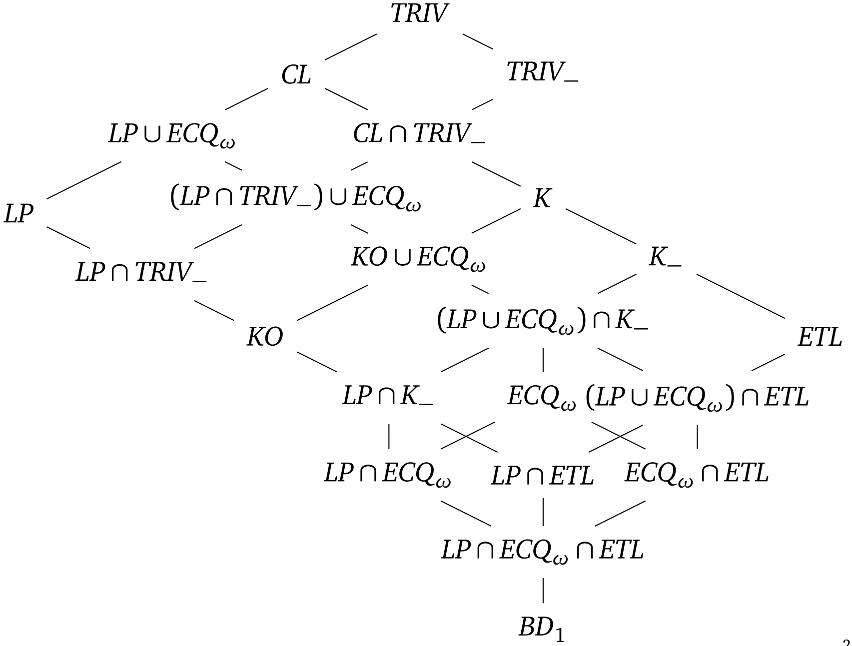
- L is a finitary extension of BD_1 with the n-PCP,
- L is complete w.r.t. some finite set of DMLs with n-prime filters,
- L is complete w.r.t. some set of submatrices of $(\mathbb{DM}_1)^n$.

Some such n exists for each fin. generated extension of BD_1.

Lattice of extensions of BD_1 with the PCP:

$$
\begin{tikzcd}[column sep=small]
 & & TRIV \arrow[dl, dash] \arrow[dr, dash] & & \\
 & CL \arrow[dl, dash] \arrow[dr, dash] & & TRIV_{-} \arrow[dl, dash] & \\
LP \arrow[dr, dash] & & CL \cap TRIV_{-} \arrow[dl, dash] \arrow[dr, dash] & & \\
 & LP \cap TRIV_{-} \arrow[dr, dash] & & K \arrow[dl, dash] & \\
 & & KO \arrow[d, dash] & & \\
 & & BD_1 & &
\end{tikzcd}
$$

Lattice of extensions of BD_1 with the 2-PCP:



Finite basis theorems

Finite basis theorem. Each logic determined by a finite set of finite DMLs **with prime upsets** is axiomatized by a finite set of logical rules.

Proof. There are finitely many finitary extensions of BD_n with the PCP, since each is determined by a family of submatrices of $\mathbb{DM}_n$. Thus, each such extension is axiomatized relative to BD_n by the PCP and a finite set of logical rules. The PCP can be eliminated by the transformation

$$\gamma_1, \dots, \gamma_n \vdash \varphi \qquad \mapsto \qquad \gamma_1 \vee x, \dots, \gamma_n \vee x \vdash \varphi \vee x.$$

Finite basis theorem. Each logic determined by a finite set of finite DMLs **with filters** is axiomatized by a finite set of logical rules and meta-rules.

Proof. Same argument, but we cannot eliminate the n-PCP.

Side remark: filter classes vs. logical classes

A standard way to prove that a logic L is complete w.r.t. a matrix $\langle \mathbf{A}, F \rangle$ is:

1. Find the algebraic counterpart K of L.
2. Find a good definition of what it means for an L-filter to be prime.
3. Show that L-filters on K-algebras are intersections of prime L-filters.
4. Each prime L-filter on a K-algebra is a homomorphic preimage of F.

This shows that L is in fact complete **as a filter class** w.r.t. $\langle \mathbf{A}, F \rangle$.

In a way, completeness theorems which **only** establish completeness as a logical class are the exception rather than the rule. But they exist!

Side remark: filter classes vs. logical classes

Example. The family of 2-filters on distributive lattices is generated by $\mathbb{B}_2$ as a filter class, but the family of 2-filters on meet semilattices is generated by (the semilattice reduct of) $\mathbb{B}_2$ only as a logical class, **not** as a filter class.

In particular, the 2-filter of non-zero elements of the diamond $\mathbf{M_5}$ is not an intersection of preimages of $\{\mathtt{t}\} \subseteq \mathbf{B}_1$, as witnessed by the generalized rule

$$x \wedge y \approx y \wedge z \approx z \wedge x \mathbin{\&} \mathrm{True}(x) \mathbin{\&} \mathrm{True}(y) \mathbin{\&} \mathrm{True}(z) \implies \mathrm{True}(x \wedge y \wedge z).$$

Example. DL_∞ is generated by $\{\mathbb{B}_n \mid n \in \omega\}$ as a finitary filter class, and also as a logical class, but **not** as a filter class. A syntactic witness for this is a generalized rule stating, roughly, that no infinite (anti)chains exist.

Open problems. Axiomatize the filter class generated by upsets of finite distributive lattices, or the logic determined by finite upsets of finite BAs.

Thank you for your attention!

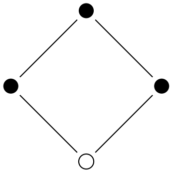
Example. The logic ECQ_∞ determined by the 8-element matrix $\mathbb{DM}_1 \times \mathbb{B}_1$ has no finite Hilbert-style axiomatization. It extends BD_1 by the rules

$$(x_1 \wedge \neg x_1) \vee \cdots \vee (x_n \wedge \neg x_n) \vdash y.$$

But it has a finite Gentzen-style axiomatization: it is the smallest extension of BD_1 with the 2-PCP which validates the rule

$$(x_1 \wedge \neg x_1) \vee (x_2 \wedge \neg x_2) \vdash y.$$

Example. Shramko's logic of Anything but Falsehood is determined by:



Theorem. The logic of anything but falsehood extends BD_2 by the excluded middle $\emptyset \vdash x \vee \neg x$ and the rule $x \vee y, \neg x \vee y \vdash (x \wedge \neg x) \vee y$.

Compare:

Theorem (Shramko). The FMLA-FMLA logic of anything but falsehood extends the FMLA-FMLA version of BD by the rule $x \vdash (x \wedge \neg y) \vee y$.

The $\mathbb{H}_S\mathbb{S}$-order on the substructures of $\mathbb{DM}_2$: