Positive (Modal) Logic Beyond Distributivity

Anna Dmitrieva (University of East Anglia)

joint work with Nick Bezhanishvili, Jim de Groot and Tommaso Moraschini

September 6, 2022


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Introduction

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This talk is based on our paper [BDdGM22] with Nick Bezhanishvili, Jim de Groot and Tommaso Moraschini. The paper itself is partially based on my Master thesis [Dmi21], supervised by Nick Bezhanishvili and Tommaso Moraschini.

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- Sahlqvist canonicity and correspondence results are some of the important applications of the duality theory [SV89].
- In [CJ99], Celani and Jansana developed a Priestley-like duality for modal distributive lattices, leading to Sahlqvist theory for positive distributive modal logic.

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- However, each of these uses either a ternary relation, or two-sorted frames, making them quite different from known dualities, such as Stone and Priestley dualities.
- Instead, we concentrate on the the duality for meet-semilattices developed by Hofmann, Mislove and Stralka [HMS74] and its generalization to lattices by Jipsen and Moshier [MJ14].

Priestley duality

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Definition

An L-space is an HMS space such that for every pair of clopen filters a, b, the filter $a \curlyvee b := a \cup b \cup \uparrow\{x \wedge y \mid x \in a, y \in b\}$ is clopen as well.

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A consequence pair is simply an expression of the form $\phi \trianglelefteq \psi$, where ϕ and ψ are formulae in L.

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- But we can also forget about the topology and look at the valuations of the underlining L-frame.
- This gives rise to the Sahlqvist results.

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3. If Γ is a set of consequence pairs, then $\mathcal{L}(\Gamma)$ is sound and complete with respect to the class of L-frames validating all consequence pairs in Γ.

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- An alternative approach to Sahlqvist correspondence and canonicity for non-distributive logics has been undertaken by Conradie and Palmigiano in [CP19]. But this approach is purely algebra based and does not use duality theory.

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This is an analogue of the result by Baker and Hales [BH74, Theorem B] that ideal completions preserve lattice equations. Our Sahlqvist results also give a proof via duality for this theorem and extend it to the modal case.

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$$\begin{tikzcd}
y \arrow[r, "R"] \arrow[d, dash] & w \arrow[d, dash, dashed] \\
x \arrow[r, dashed, "R"'] & z
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(2)

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 & & y \arrow[rr, dashed, "R"] & & w \arrow[dddl, dash, dashed] \\
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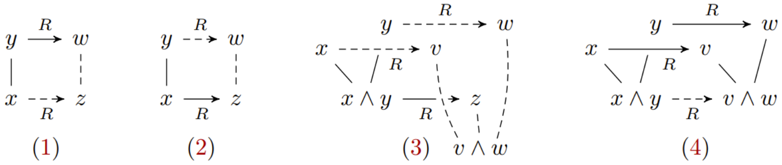
4. If xRv and yRw then $(x \wedge y)R(v \wedge w)$;



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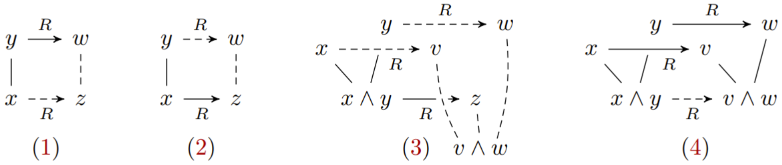
4. If xRv and yRw then $(x \wedge y)R(v \wedge w)$;
5. For all x there exists an y such that xRy.



Modal L-models

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Propositional connectives are interpreted as before, and

$$[\![\Box\phi]\!] = \{x \in X \mid \forall y \in X, xRy \text{ implies } \mathfrak{M}, y \Vdash \phi\}$$
$$[\![\Diamond\phi]\!] = \{x \in X \mid \exists y \in X \text{ such that } xRy \text{ and } \mathfrak{M}, y \Vdash \phi\}$$

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$\Box(p \wedge q) \trianglelefteq \Box p \wedge \Box q \quad \Diamond p \trianglelefteq \Diamond(p \vee q) \qquad$ (monotonicity)

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A modal lattice $(A, \Box, \Diamond)$ is a lattice A with two maps $\Box, \Diamond : A \to A$ satisfying the inequalities above.

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We denote the corresponding logic by $\mathcal{L}_{\Box\Diamond}$.

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It can be shown that $(X, \wedge, R)$ is a modal L-frame.

Modal duality

$$\begin{tikzcd}[column sep=huge]
\text{Modal Lattices} \arrow[r, bend left, "\text{Non-empty proper filters with } pRq \text{ iff } \Box^{-1}(p)\subseteq q\subseteq\Diamond^{-1}(p)"] & \text{Modal L-Spaces} \arrow[l, bend left, "\text{Clopen filters with [R] and } \langle R\rangle"]
\end{tikzcd}$$

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Note that $\Diamond(a \vee b) \trianglelefteq \Diamond a \vee \Diamond b$ (the normality of $\Diamond$) is a Sahlqvist consequence pair.

Sahlqvist results

Theorem

Let $\phi \trianglelefteq \psi$ be a Sahlqvist consequence pair.

1. $\phi \trianglelefteq \psi$ locally corresponds to a first-order formula with one free variable.
2. For every modal L-space $\mathbb{X}$, if $\mathbb{X} \Vdash \phi \trianglelefteq \psi$ then $\kappa\mathbb{X} \Vdash \phi \trianglelefteq \psi$.
3. If Γ is a set of Sahlqvist consequence pairs, then $\mathcal{L}_{\Box\Diamond}(\Gamma)$ is sound and complete with respect to the class of L-frames validating all consequence pairs in Γ.

Thank you for your attention!

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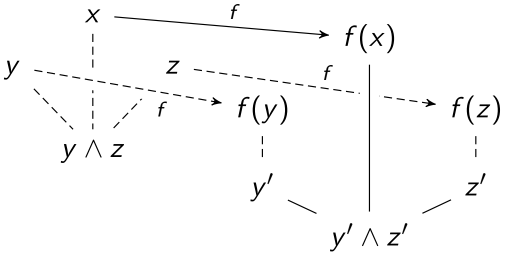
Morphism slides

- An HMS morphism is a continuous meet-semilattice morphism.
- An L-morphism from $(X, \wedge)$ to $(X', \wedge')$ is a meet-preserving function $f : (X, \wedge) \to (X', \wedge')$ that satisfies for all $x \in X$ and $y', z' \in X'$: if $y' \wedge z' \leq f(x)$ then there exist $y, z \in X$ such that $y' \leq' f(y)$ and $z' \leq' f(z)$ and $y \wedge z \leq x$.



- An L-space morphism is a continuous L-morphism.

Morphism slides

- A bounded L-morphism from $(X, \wedge, R)$ to $(X', \wedge', R')$ is a function $f : X \to X'$ such that $f : (X, \wedge) \to (X', \wedge')$ is an L-morphism and for all $x, y \in X$ and $z' \in X'$:
    1. If xRy then $f(x)R'f(y)$;
    2. If $f(x)R'z'$ then there exists a $z \in X$ such that xRz and $f(z) \leq z'$;
    3. If $f(x)R'z'$ then there exists a $w \in X$ such that xRz and $z' \leq' f(w)$.
- A modal lattice homomorphism is a lattice homomorphism that preserves modal operators.
- A modal L-space morphism is a continuous bounded L-morphism.