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Abstract Model and Deduction System for Logic of Multiple Agent in Quantum Physics

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1 Backgrounds

Quantum logic (QL) is the field which deal with strange propositions about physical values of particle or about states in **quantum mechanics**.

Ex: "Velocity of the particle is 20".

In this logic, due to the uncertainty principle, etc., some properties that normally hold do not hold.

Ex: The **distributive law** does not hold.

$$A \wedge (B \vee C) \neq (A \wedge B) \vee (A \wedge C)$$

Extensions of quantum logics

Knowledge: **Epistemic quantum logic**

"Agent (experimenter) knows that velocity of the particle is 20"

(Beltrametti, E., Dalla Chiara M. L., Giuntini, R, Leporini, R., Sergioli, G.(2013), Baltag, A., Smets, S.(2010) (2017))

Actions: **Dynamic quantum logic**.

"After unitary transformation U, velocity of the particle is 20"

(Baltag, A., Smets, S.(2004-))

Motivation of this study

1. The logic for "knowledge + multiple observers + actions" is less advanced.

2. As models in previous studies introduce many concepts of quantum mechanics, these models are little complicated and deduction system is not much discussed.

As a part of solving these problems, in this study, new abstract models and deduction system based on **orthomodular logic** (OML) and **public announcement logic** (PAL).

Contents

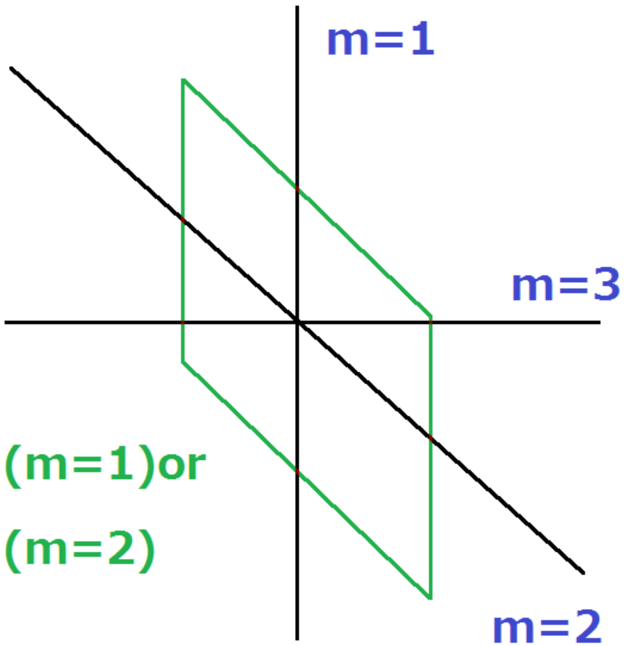
- About quantum physics and orthomodular logic
- About model and deduction system of new logic
- Additional approach to multiple agents

Quantum physics

States space : A Hilbert space H

States : 1D closed subspaces of H

Orthomodular logic



Example: in 3D Hilbert space

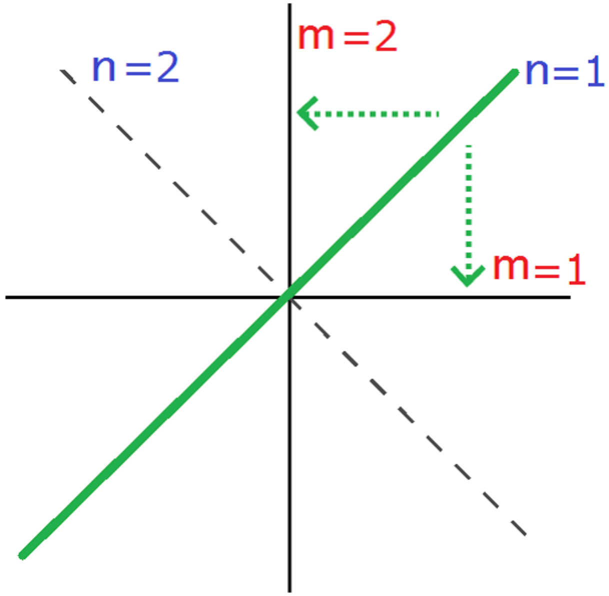
$$\sim (m=3) = (m=1) \sqcup (m=2) \qquad \neq \neg(m=3)$$

Propositions : Associated to closed subspaces of H.

Quantum disjunction $\sqcup$: Spanned space

Quantum negation : $\sim$: Orthogonal space

Observations in quantum mechanics



· After observation of truth of A, state is **projected** to the state that A is true.

· In other words, getting information from the state changes the state itself.

Orthomodular logic

$A ::= p| \sim A|A{\wedge}B$

$p, q, r, \ldots$: Propositional variables
$\sim$: quantum negation (orthogonal space)
$\wedge$: conjunction

$A \sqcup B = \ \sim (\sim A \wedge \sim B)$
quantum disjunction (spanned space)

$[A]B = \sim A \sqcup (A \wedge B)$
After get information A, B is true

SOM (strict orthmodular)-model$(W, \not\perp, V, R_A, R_B, ...)$

W: non empty set. (expressing quantum states in a Hilbert space)

$\not\perp$: binary relation on W which is **reflexive** and **symmetric**. (non-orthogonal relation between states).

V: function assigning each propositional variable p to a $\perp$-**closed** subset of W. $V(\sim A) = V(A)^{\perp}$, $V(A \wedge B) = V(A) \cap V(B)$.

R_A...: binary relations for projections

- For all $\perp$-closed subsets X, Y of X, $X \cap (X^{\perp} \sqcup (X \cap Y)) \subseteq Y$. (OM-law)

Given $X \subseteq W$,

$X^{\perp} = \{w \in W |$for all x in X, $w \perp x\}$.

We say that X is $\perp$-**closed** or **testable** if $X^{\perp\perp} = X$.

A $\perp$-closed set represents a closed subspace on the Hilbert space.

2 Multi-agent dynamic epistemic quantum logic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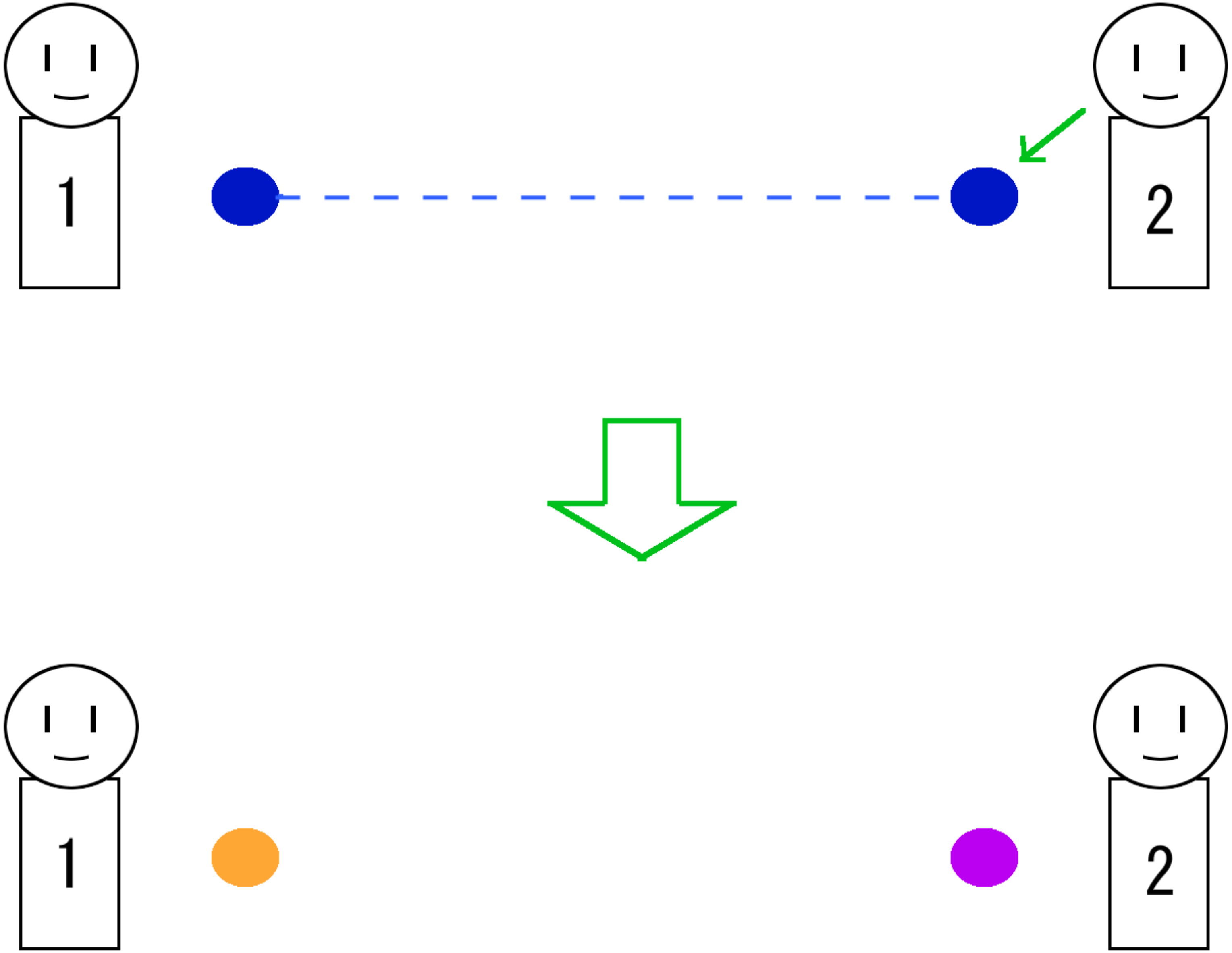
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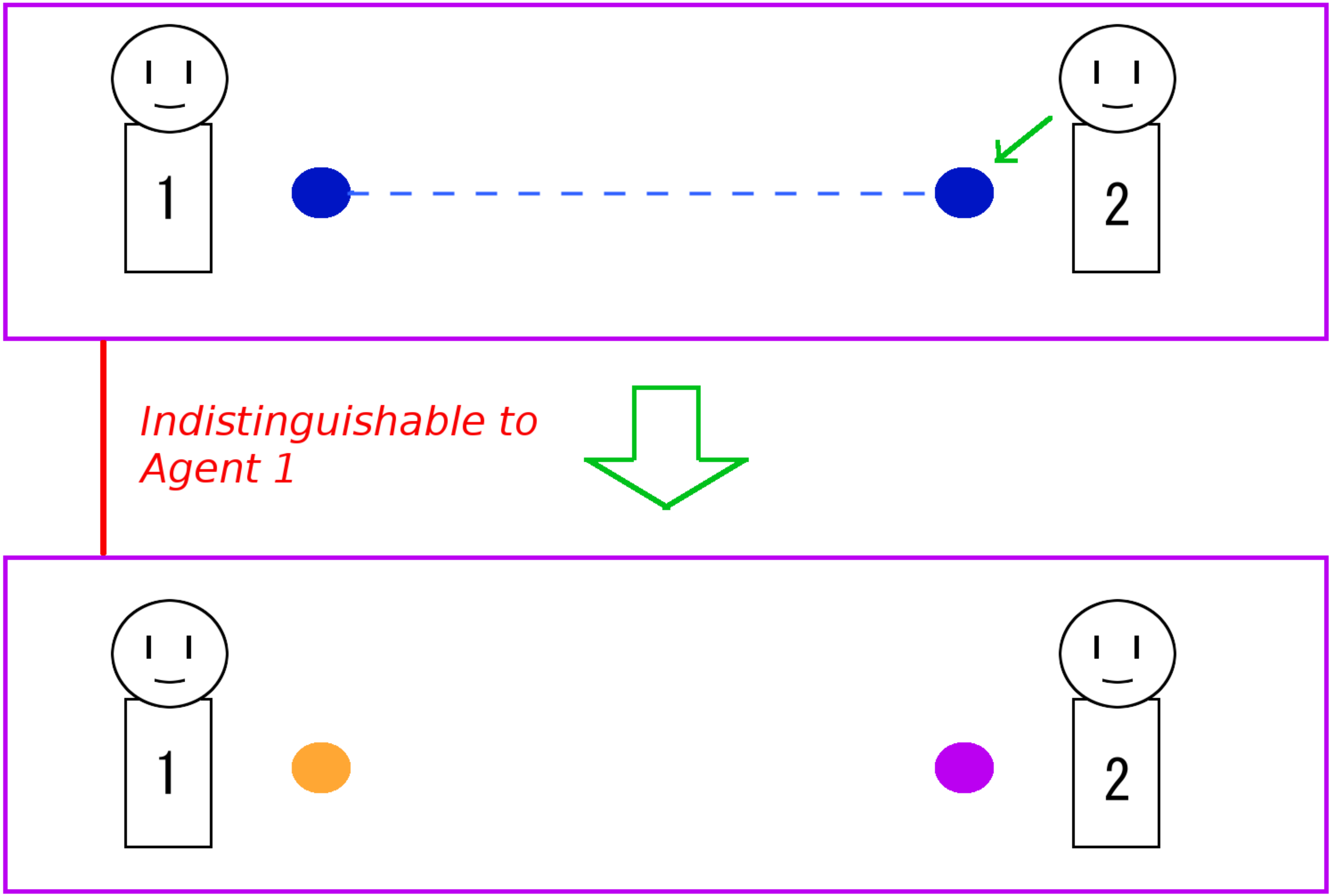
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Indistinguishable to
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Problem for expressing state of two particles by algebraic model or simple frame

State space for multiple quantum particle : $H_1 \otimes H_2 \otimes H_3 \otimes \ldots$

Suppose $H_1 \approx L_1$, $H_2 \approx L_2$ (L expresses OM-lattice)

$L_1 \otimes L_2$ does not correspond to $H_1 \otimes H_2$.

Multi-agent dynamic epistemic quantum logic (MDEQL)

Language of MDEQL:

q-formula $A ::= p_i \mid \sim A \mid A \wedge A$

g-formula $\phi ::= A \mid \neg\phi \mid \phi \wedge \phi \mid \mathsf{K}_i \phi \mid [A_i]\phi \qquad (i \in I)$

quantum-formula : express basic observational propositions.
general-formula : express propositions including knowledge and dynamism.

The propositional variables are divided into I classes, which represent the basic observational proposition of each agent.

i-pv : $p_i, q_i, r_i, \ldots$
i-formula : $A_i, B_i, C_i, \ldots$ (q-formula that includes only i-pv)

Base model for new logic

I-SOM-model $(S, \not\perp, V, R_A, R_B, ...)$

- $(S, \not\perp, V, R_A, R_B, ...)$ is an SOM-model
- V must satisfy the following spatial division conditions

  1. If $i \neq j$, then for all p_i, p_j, $V(p_i) = V((p_i \wedge p_j) \vee (p_i \wedge \sim p_j))$ (compatibility)

  2. If $i \neq j$, then for all p_i, p_j, if $V(p_i) \neq V(\top), V(p_i) \neq V(\perp), V(p_j) \neq V(\top), V(p_j) \neq V(\perp)$, then $V(p_i) \not\subseteq V(p_j)$ and $V(p_j) \not\subseteq V(p_i)$ (non-atomicity)

Model for MDEQL (MDEQ-model)

Definition *multi agent epistemic quantum model* (MDEQ-model)

is constructed from the following *base* model.

$\mathcal{M} = (S, \not\perp, V, R_A, R_B, ..., W_S, R_1, R_2, \ldots, R_n)$

- $(S, \not\perp, V, R_A, R_B, ...)$ is an I-SOM-model。
- W_S is a set of elements labeled by the elements of S.
- $R_1, R_2, \ldots, R_n$ are equivalence relations on W_S.

That is,
$W_S = \{x_s, y_t ...\}$ $(s, t, \ldots \in S)$

Additional models by obtaining information

$\mathcal{M}_{[A_i]} = (S, \not\perp, V, W_{S[A_i]}, V_{[A_i]}, R_{1[A_i]}, R_{2[A_i]}, \ldots, R_{n[A_i]})$
is defined from
$\mathcal{M} = (S, \not\perp, V, W_S, R_1, R_2, \ldots, R_n)$ as follows.

There exists a bijective partial function f from W_S to $W_{S[A_i]}$ such that

- $\mathbf{dom}(f) = \{w_s \in W_S | s \not\models \sim A_i\}$
- If $f(w_s) = x_t$, then $s(A_i)t$.

We write $w_s(A_i)x_t$ if $f(w_s) = x_t$.

Definition of $R_{j[A_i]}$

- $w_s(R_{j[A_i]})x_t$ iff $f^{-1}(w_s)(R_j)f^{-1}(x_t)$.

Global equivalence relations $R_{i[]}$

$\mathcal{M}'$ is *reachable* from $\mathcal{M}$ if there exists $A^1_\alpha, A^2_\beta, \dots A^n_\gamma$ such that $\mathcal{M}' = \mathcal{M}_{[A^1_\alpha][A^2_\beta]\dots[A^n_\gamma]}$

$r(\mathcal{M})$: the set of all reachable models from M

$R_{i[]}$ $(i \in I)$ are defined as equivalence relations on elements of $W (\in M' \in r(\mathcal{M}))$.

- In $(S, \not\angle, V, W_S, R_1, R_2, \dots, R_n) \in r(\mathcal{M})$, if $w_s(R_i)x_t$, then $w_s(R_{i[]})x_t$.
- In $(\mathcal{M}' = (S, \not\angle, V, W'_S, R'_1, R'_2, \dots, R'_n) \in r(\mathcal{M})$ and $(\mathcal{M}'' = (S, \not\angle, V, W''_S, R''_1, R''_2, \dots, R''_n) \in r(\mathcal{M})$, if $w_s \in W'_S$ and $x_t \in W''_S$ satisfy the following conditions, then $w_s(R_{i[]})x_t$.
  - $i \neq j$ and A_j exists such that $\mathcal{M}'' = \mathcal{M}'_{[A_j]}$
  - $w_s(A_j)x_t$
- $w_s(R_{i[]})x_t$ is only in the above cases.

Truth value on model

$$w_s \models A \quad \stackrel{\text{def}}{\Leftrightarrow} \quad s \in V(A) \text{(of a SOM-model } (S, \not\angle, V)\text{)},$$

$$w_s \models \neg\phi \quad \stackrel{\text{def}}{\Leftrightarrow} \quad w_s \not\models \phi,$$

$$w_s \models \phi \wedge \psi \quad \stackrel{\text{def}}{\Leftrightarrow} \quad w_s \models \phi \text{ and } w_s \models \psi,$$

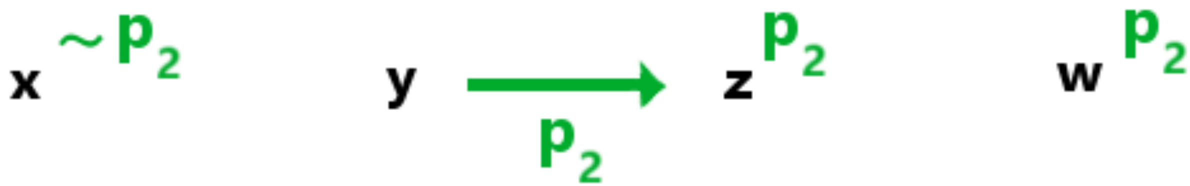
$$w_s \models \mathsf{K_i}\phi \quad \stackrel{\text{def}}{\Leftrightarrow} \quad \text{for all } x_t \in W_S \text{ such that } w_s(R_{i[]})x_t,\ x_t \models \phi,$$

$$w_s \models [A_i]\phi \quad \stackrel{\text{def}}{\Leftrightarrow} \quad \text{if } w_s(A_i)x_t, \text{ then } x_t \models \phi.$$

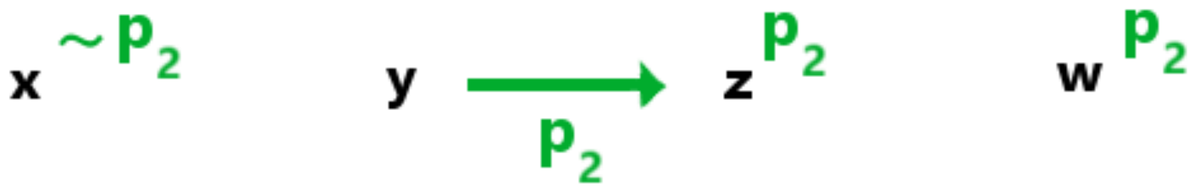
l-SOM-model W (= {x,y,z,w})

x y z w

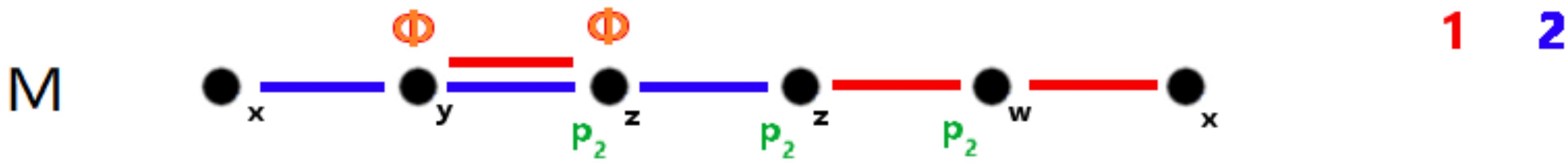
İ-SOM-model W (= {x,y,z,w})



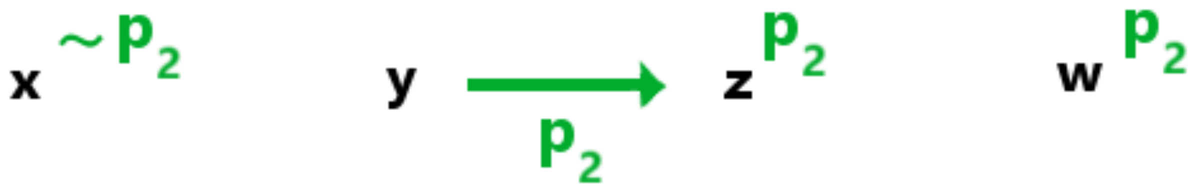
I-SOM-model   W (= {x,y,z,w})



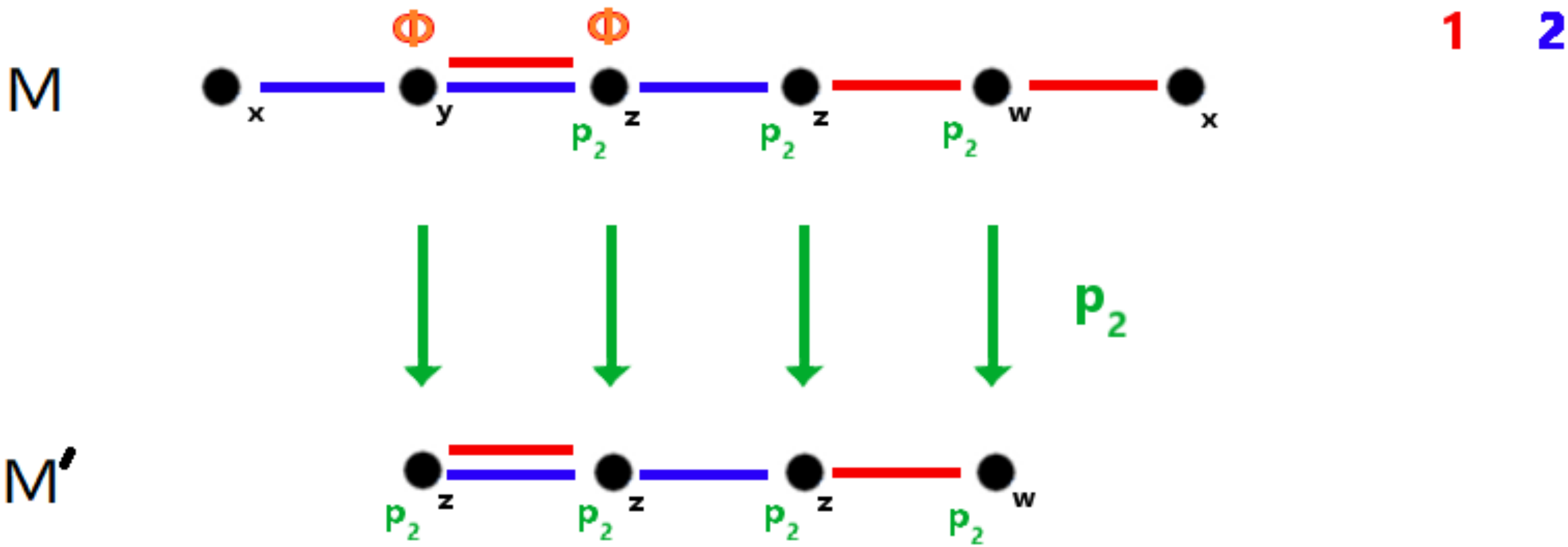
MEQ-model



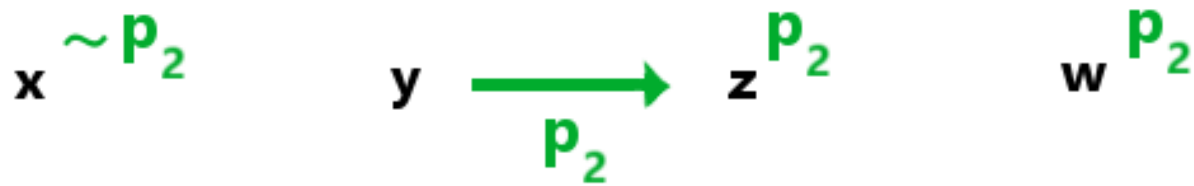
I-SOM-model W (= {x,y,z,w})



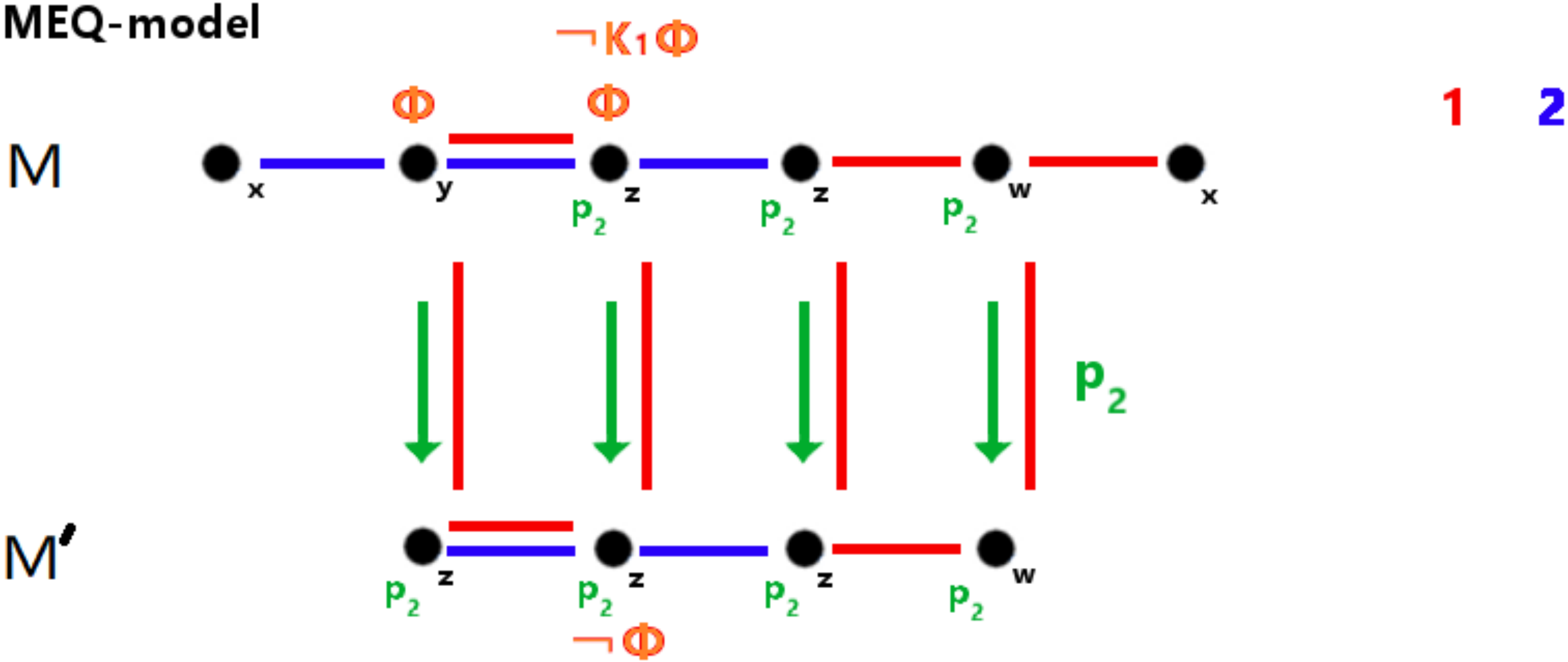
MEQ-model



İ-SOM-model   W (= {x,y,z,w})



MEQ-model



In this model the following formula is valid, which expresses that an individual's knowledge does not change with the acquisition of other agents' knowledge.

$\mathsf{K}_i A \to [B_j]\mathsf{K}_i A \quad (i \neq j)$
$\neg\mathsf{K}_i A \to [B_j]\neg\mathsf{K}_i A \quad (i \neq j)$

But, the following formula is **not valid**.

$\mathsf{K}_i A_i \to [B_i]\mathsf{K}_i A_i$

But if $A_i = (A_i \wedge B_i) \vee (A_i \wedge \sim B_i))$, then it is valid.

In PAL, the following formulas are valid.

$$[\phi]p \leftrightarrow (\phi \rightarrow p)$$
$$[\phi](\psi \wedge \chi) \leftrightarrow ([\phi]\psi \wedge [\phi]\chi)$$
$$[\phi]\neg\psi \leftrightarrow (\phi \rightarrow \neg[\phi]\psi)$$
$$[\phi]\mathsf{K}\psi \leftrightarrow (\phi \rightarrow \mathsf{K}[\phi]\psi)$$

Theorem Similar to PAL, in MDEQ-models, the following formulas are valid.

$$[A_i]B \leftrightarrow \sim A_i \sqcup (A_i \wedge B))$$
$$[A_i](\phi \wedge \psi) \leftrightarrow ([A_i]\phi \wedge [A_i]\psi)$$
$$[A_i]\neg\phi \leftrightarrow (\neg \sim A_i \rightarrow \neg[A_i]\phi)$$
$$[A_i]\mathsf{K_j}\phi \leftrightarrow (\neg \sim A_i \rightarrow \mathsf{K_j}[A_i]\phi) \;\; (i = j)$$
$$[A_i]\mathsf{K_j}\phi \leftrightarrow \mathsf{K_j}\phi \;\; (i \neq j)$$

Deduction system SDEQ

Axioms: $\phi \Rightarrow \phi$

$$A_i \Rightarrow (A_i \wedge B_j) \vee (A_i \wedge \sim B_j)$$
$$(A_i \wedge B_j) \vee (A_i \wedge \sim B_j) \Rightarrow A_i \qquad (i \neq j)$$

Rules:

$$\frac{\Gamma \Rightarrow \Delta, \phi \quad \phi, \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Delta, \Sigma} \text{ (cut)}$$

$$\frac{\Gamma \Rightarrow \Delta}{\phi, \Gamma \Rightarrow \Delta} \text{ (w L)} \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \phi} \text{ (w R)}$$

$$\frac{\Gamma \Rightarrow \Delta, A}{\sim A, \Gamma \Rightarrow \Delta} (\sim \text{L})_q \qquad \frac{A \Rightarrow \Delta}{\sim \Delta \Rightarrow \sim A} (\sim \text{R})_q$$

$$\frac{A, \Gamma \Rightarrow \Delta}{\sim\sim A, \Gamma \Rightarrow \Delta} (\sim\sim \text{L})_q \qquad \frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, \sim\sim A} (\sim\sim \text{R})_q$$

$$\frac{\sim B \Rightarrow \sim A \quad \sim A, B \Rightarrow}{\sim A \Rightarrow \sim B} (\text{OM})_q$$

$$\frac{\phi, \Gamma \Rightarrow \Delta}{\phi \wedge \psi, \Gamma \Rightarrow \Delta} \ (\wedge\mathsf{L}) \qquad \frac{\psi, \Gamma \Rightarrow \Delta}{\phi \wedge \psi, \Gamma \Rightarrow \Delta} \ (\wedge\mathsf{L}) \qquad \frac{\Gamma \Rightarrow \Delta, \phi \quad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \phi \wedge \psi} \ (\wedge\mathsf{R})$$

$$\frac{\Gamma \Rightarrow \Delta, \phi}{\neg\phi, \Gamma \Rightarrow \Delta} \ (\neg\mathsf{L}) \qquad \frac{\phi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\phi} \ (\neg\mathsf{R})$$

$$\frac{\Gamma \Rightarrow \Delta}{[A_i]\Gamma \Rightarrow [A_i]\Delta} \ ([\ ])^{**} \qquad \frac{\phi, \Gamma \Rightarrow \Delta}{\mathsf{K_i}\phi, \Gamma \Rightarrow \Delta} \ (\mathsf{K_i}1) \qquad \frac{\mathsf{K_i}\Gamma \Rightarrow \mathsf{K_i}\Delta, \phi}{\mathsf{K_i}\Gamma \Rightarrow \mathsf{K_i}\Delta, \mathsf{K_i}\phi} \ (\mathsf{K_i}2)$$

$$\frac{\Gamma \Rightarrow \Delta, A_i \quad B, \Gamma \Rightarrow \Delta}{[A_i]B, \Gamma \Rightarrow \Delta} \ ([\ ]\mathsf{L})_q \qquad \frac{< A_i > \Gamma, A_i \Rightarrow B}{\Gamma \Rightarrow [A_i]B} \ ([\ ]\mathsf{R})_q^*$$

$$\frac{[A_i]\bot, \Gamma \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, [A_i]\phi}{[A_i]\neg\phi, \Gamma \Rightarrow \Delta} \ (\neg[\ ]) \qquad \frac{[A_i]\phi, \Gamma \Rightarrow \Delta}{\mathsf{K_j}\phi, \Gamma \Rightarrow \Delta} \ (\mathsf{K}[\ ]3)$$

$$\frac{\Gamma \Rightarrow \Delta, \mathsf{K_i}[A_i]\phi}{\Gamma \Rightarrow \Delta, [A_i]\mathsf{K_i}\phi} \ (\mathsf{K}[\ ]1) \qquad \frac{[A_i]\bot, \Gamma \Rightarrow \Delta \quad \mathsf{K_i}[A_i]\phi, \Gamma \Rightarrow \Delta}{[A_i]\mathsf{K_i}\phi, \Gamma \Rightarrow \Delta} \ (\mathsf{K}[\ ]2)$$

*$< A > C = A \wedge \sim (A \wedge \sim C)$. $\quad < A > \Gamma = \{ < A > C | C \in \Gamma \}$.
**Δ must not be empty.

Theorem 2.1 The soundness and completeness theorem for **SMDEQL**
$\Gamma \Rightarrow \Delta$ is derivable in **SMDEQL** iff $\Gamma \Rightarrow \Delta$ is valid in MEQ-models.

By a standard way using canonical models.

3 Additional formula and conditions

To express detailed conditions for quantum propositions in tensor product Hilbert space $(H_1 \otimes H_2)$, we have to extend the language.

q-formula $A ::= p_i \mid \sim A \mid A \wedge A$

g-formula

$\phi ::= A \mid \neg\phi \mid \phi\wedge\phi \mid \mathsf{K}_i\phi \mid [A_i]\phi \mid \Box\phi \mid \forall p_i(A) \qquad (i \in I)$

$\Box$: Modal symbol for relation $\not\perp$.

Modal logic **B** : symmetry and reflectivity relation.

B and OML are associated by McKinsey-Tarski transfer.

$\Box\neg A \;=\, \sim A$

Conditions

For all $x, y \in W$, there exists $z \in W$ such that xRz and zRy.
$\Box\Box A \to \Box^n A \quad$ (for each $n \in \mathbb{N}$)

Each propositional variable represents a one-dimensional subspace of each Hilbert space.
$(p_i \wedge A_i) \to \Box\Box(p_i \to A_i)$

Non-implications of propositions of an individual particle.
$(\neg\Box\Box A_i \wedge \neg\Box\Box\neg A_i \wedge \neg\Box\Box B_j \wedge \neg\Box\Box\neg B_j) \to$
$\Diamond\Diamond(\neg A_i \wedge B_j) \wedge \Diamond\Diamond(A_i \wedge \neg B_j) \quad (i \neq j)$

"Particle i and j are entangled"
$\mathcal{E}_{i,j} = \forall p_i(\neg p_i) \wedge \forall q_j(\neg q_j) \wedge \forall p_i \exists q_j [p_i] q_j \wedge \forall q_j \exists p_i [q_j] p_i$

Conclusions

Abstract model for multi agent dynamic epistemic logic is constructed.

Suitable sequent calculi is constructed and it satisfies soundness and completeness.

Additional language and conditions (future works).

Problem for expressing state of two particles by algebraic model or simple frame

State space for two quantum particle : $H_1 \otimes H_2$

Suppose $H_1 \approx L_1$, $H_2 \approx L_2$ (L expresses OM-lattice)

However, $L_1 \otimes L_2$ does not correspond to $H_1 \otimes H_2$.

The solution adopted in this study

We regards single L as $H_1 \otimes H_2$,
and use **subscripted propositional variables** $p_1, q_2, r_i, \ldots)$

$L \approx H = H_1 \otimes H_2$

Thank you for listening !

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