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Cut-elimination for a Hypersequent Calculus for First-order Gödel Logic over $[0,1]$ with $\triangle$

Matthias Baaz[1] Chris Fermüller[1] Norbert Preining[2]

[1]Theory and Logic Group, TU Wien
[2]Mercari Inc., Tokyo

Gödel Logic $\mathbf{G}_{[0,1]}^{\forall\triangle}$

Semantics:

$\mathfrak{I}$ extends a $[0,1]$-assignment to atomic formulas over domain D:

$$
\begin{aligned}
\mathfrak{I}(\bot) &= 0 \\
\mathfrak{I}(A \wedge B) &= \min(\mathfrak{I}(A), \mathfrak{I}(B)) \\
\mathfrak{I}(A \vee B) &= \max(\mathfrak{I}(A), \mathfrak{I}(B)) \\
\mathfrak{I}(A \to B) &= \begin{cases} 1 & \text{if } \mathfrak{I}(A) \leq \mathfrak{I}(B) \\ \mathfrak{I}(B) & \text{if } \mathfrak{I}(A) > \mathfrak{I}(B) \end{cases} \\
\mathfrak{I}(\triangle A) &= \begin{cases} 1 & \text{if } \mathfrak{I}(A) = 1 \\ 0 & \text{if } \mathfrak{I}(A) < 1 \end{cases} \\
\mathfrak{I}(\forall x\, A(x)) &= \inf\{\mathfrak{I}(A(u)) : u \in D\} \\
\mathfrak{I}(\exists x\, A(x)) &= \sup\{\mathfrak{I}(A(u)) : u \in D\}
\end{aligned}
$$

$$\neg A =^{df.} A \to \bot$$

The calculus HGIF (1)

based on hypersequents = finite multisets of squents:

$$\Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n$$

Axioms:

$$A \Rightarrow A \qquad \bot \Rightarrow$$

Internal structural rules:

$$\frac{G \mid \Gamma \Rightarrow \Delta}{G \mid A, \Gamma \Rightarrow \Delta}\ iw\Rightarrow \qquad \frac{G \mid \Gamma \Rightarrow}{G \mid \Gamma \Rightarrow A} \Rightarrow iw \qquad \frac{G \mid A, A, \Gamma \Rightarrow \Delta}{G \mid A, \Gamma \Rightarrow \Delta}\ ic\Rightarrow$$

External structural rules:

$$\frac{G}{G \mid \Gamma \Rightarrow \Delta}\ ew \qquad \frac{G \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta}{G \mid \Gamma \Rightarrow \Delta}\ ec$$

Communication rule:

$$\frac{G \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta \quad G \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta'}{G \mid \Gamma_1 \Rightarrow \Delta \mid \Gamma_2 \Rightarrow \Delta'}\ cm$$

The hypersequent calculus HGIF (2)

Logical rules (standard connectives and quantifiers):

$$\frac{G \mid \Gamma \Rightarrow A}{G \mid \neg A, \Gamma \Rightarrow} \neg\Rightarrow \qquad \frac{G \mid A, \Gamma \Rightarrow}{G \mid \Gamma \Rightarrow \neg A} \Rightarrow\neg$$

$$\frac{G \mid A, \Gamma \Rightarrow \Delta \quad G \mid B, \Gamma \Rightarrow \Delta}{G \mid A \vee B, \Gamma \Rightarrow \Delta} \vee\Rightarrow \qquad \frac{G \mid \Gamma \Rightarrow A \quad G \mid \Gamma \Rightarrow B}{G \mid \Gamma \Rightarrow A \wedge B} \Rightarrow\wedge$$

$$\frac{G \mid \Gamma \Rightarrow A}{G \mid \Gamma \Rightarrow A \vee B} \Rightarrow\vee_1 \qquad \frac{G \mid A, \Gamma \Rightarrow \Delta}{G \mid A \wedge B, \Gamma \Rightarrow \Delta} \wedge\Rightarrow_1$$

$$\frac{G \mid \Gamma \Rightarrow B}{G \mid \Gamma \Rightarrow A \vee B} \Rightarrow\vee_2 \qquad \frac{G \mid B, \Gamma \Rightarrow \Delta}{G \mid A \wedge B, \Gamma \Rightarrow \Delta} \wedge\Rightarrow_2$$

$$\frac{G \mid \Gamma_1 \Rightarrow A \quad G \mid B, \Gamma_2 \Rightarrow \Delta}{G \mid A \to B, \Gamma_1, \Gamma_2 \Rightarrow \Delta} \to\Rightarrow \qquad \frac{G \mid A, \Gamma \Rightarrow B}{G \mid \Gamma \Rightarrow A \to B} \Rightarrow\to$$

$$\frac{G \mid A(t), \Gamma \Rightarrow \Delta}{G \mid \forall x\, A(x), \Gamma \Rightarrow \Delta} \forall\Rightarrow \qquad \frac{G \mid \Gamma \Rightarrow A(a)}{G \mid \Gamma \Rightarrow \forall x\, A(x)} \Rightarrow\forall$$

$$\frac{G \mid A(a), \Gamma \Rightarrow \Delta}{G \mid \exists x\, A(x), \Gamma \Rightarrow \Delta} \exists\Rightarrow \qquad \frac{G \mid \Gamma \Rightarrow A(t)}{G \mid \Gamma \Rightarrow \exists x\, A(x)} \Rightarrow\exists$$

The hypersequent calculus HGIF (3)

Rules for $\triangle$:

$$\frac{G \mid \quad A, \Gamma \Rightarrow \Delta}{G \mid \triangle A, \Gamma \Rightarrow \Delta}\ \triangle\Rightarrow \qquad \frac{G \mid \triangle\Gamma \Rightarrow \quad A}{G \mid \triangle\Gamma \Rightarrow \triangle A}\ \Rightarrow\triangle$$

$$\frac{G \mid \triangle\Gamma, \Gamma' \Rightarrow \Delta}{G \mid \triangle\Gamma \Rightarrow \mid \Gamma' \Rightarrow \Delta}\ \triangle cl$$

Cut rule:

$$\frac{G \mid \Gamma \Rightarrow A \quad G \mid A, \Pi \Rightarrow \Lambda}{G \mid \Gamma, \Pi \Rightarrow \Lambda}\ cut$$

Example of an HGIF derivation (1)

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A}{\triangle A \Rightarrow A}\;\triangle\Rightarrow}{\triangle A \Rightarrow \;\mid\; \Rightarrow A}\;\triangle cl}{\triangle A \Rightarrow \;\mid\; \Rightarrow \triangle A}\;\Rightarrow\triangle}{\Rightarrow \neg\triangle A \;\mid\; \Rightarrow \triangle A}\;\Rightarrow\neg}{\Rightarrow \triangle A \vee \neg\triangle A \;\mid\; \Rightarrow \triangle A}\;\Rightarrow\vee_2}{\Rightarrow \triangle A \vee \neg\triangle A \;\mid\; \Rightarrow \triangle A \vee \neg\triangle A}\;\Rightarrow\vee_1}{\Rightarrow \triangle A \vee \neg\triangle A}\;ec
$$

Example of an HGIF derivation (2)

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A(a) \Rightarrow A(a)}{\triangle A(a) \Rightarrow A(a)}\ \triangle\Rightarrow}{\triangle A(a) \Rightarrow\ \mid\ \Rightarrow A(a)}\ \triangle cl}{\forall x \triangle A(x) \Rightarrow\ \mid\ \Rightarrow A(a)}\ \forall\Rightarrow}{\forall x \triangle A(x) \Rightarrow\ \mid\ \Rightarrow \forall x A(x)}\ \Rightarrow\forall}{\forall x \triangle A(x) \Rightarrow\ \mid\ \Rightarrow \triangle \forall x A(x)}\ \Rightarrow\triangle}{\forall x \triangle A(x) \Rightarrow \triangle \forall x A(x)\ \mid\ \Rightarrow \triangle \forall x A(x)}\ iw}{\forall x \triangle A(x) \Rightarrow \triangle \forall x A(x)\ \mid\ \forall x \triangle A(x) \Rightarrow \triangle \forall x A(x)}\ iw}{\forall x \triangle A(x) \Rightarrow \triangle \forall x A(x)}\ ec
$$

Soundness and Completeness of HGIF

Theorem [Baaz, Preining, Zach, 2006]
HGIF is sound and complete for $\mathbf{G}^{\forall\triangle}_{[0,1]}$.

More precisely:
For every formula A of $\mathbf{G}^{\forall\triangle}_{[0,1]}$: $\mathfrak{I}(A) = 1$ in every interpretation iff the hypersequent $\Rightarrow A$ is derivable in **HGIF**.

Cut Elimination (1)

We apply the method of Schütte-Tait:
The highest occurrence of the largest cut gets eliminated.

Critical case:
Applications of the $\Rightarrow\triangle$-rule in combination with $\triangle cl$.

Remark:
Rule $\Rightarrow\triangle$ could be replaced by

$$\frac{G \mid A, \Gamma \Rightarrow \Delta}{G \mid \Gamma \Rightarrow \Delta \mid \triangle A \Rightarrow} \; \triangle\Rightarrow^{*}$$

Rule *cm* (communication) could be replaced by

$$\frac{G \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta}{G \mid \Gamma_1 \Rightarrow \Delta \mid \Gamma_2 \Rightarrow \Delta} \; split \qquad \frac{G \mid \Gamma_1 \Rightarrow \Delta_2 \quad G \mid \Gamma_2 \Rightarrow \Delta_1}{G \mid \Gamma_1 \Rightarrow \Delta_1 \mid \Gamma_2 \Rightarrow \Delta_2} \; cm^{*}$$

Cut Elimination (2)

Proposition:

All axioms can be derived cut-free from atomic axioms.

Lemma:

Let H be derived from $H_1, \ldots, H_n$.
Consider an occurrence of A in H and trace this occurrence through the derivation. Assume that the predecessors are inferred by weakening.

1. The occurrence of A in H is positive:
   Replace A everywhere by B, where B does not contain eigenvariables.
   The resulting derivation from modified $H_1, \ldots, H_n$ is correct.
2. The occurrence of A in H is negative:
   Replace A everywhere by $B_1, \ldots, B_n$, not containing eigenvariables.
   2.1 $A \not\equiv \triangle A'$:
   The resulting derivation from modified $H_1, \ldots, H_n$ is correct.
   2.2 $A \equiv \triangle A'$ and $B_1, \ldots, B_n \equiv \triangle B'_1, \ldots, \triangle B'_n$:
   The resulting derivation from modified $H_1, \ldots, H_n$ is correct.

Cut Elimination (3)

Proposition:

The following hold without additional cuts and without increase of depth.

$\vdash_{\mathbf{HGIF}} G \mid \Gamma \Rightarrow \neg A$	implies	$\vdash_{\mathbf{HGIF}} G \mid A, \Gamma \Rightarrow$
$\vdash_{\mathbf{HGIF}} G \mid \Gamma \Rightarrow A \wedge B$	implies	$\vdash_{\mathbf{HGIF}} G \mid \Gamma \Rightarrow A$ and $\vdash_{\mathbf{HGIF}} G \mid \Gamma \Rightarrow B$
$\vdash_{\mathbf{HGIF}} G \mid A \vee B, \Gamma \Rightarrow \Delta$	implies	$\vdash_{\mathbf{HGIF}} A, G \mid \Gamma \Rightarrow \Delta$ and $\vdash_{\mathbf{HGIF}} B, G \mid \Gamma \Rightarrow \Delta$
$\vdash_{\mathbf{HGIF}} G \mid \Gamma \Rightarrow A \rightarrow B$	implies	$\vdash_{\mathbf{HGIF}} G \mid A, \Gamma \Rightarrow B$
$\vdash_{\mathbf{HGIF}} G \mid \Gamma \Rightarrow \forall x A(x)$	implies	$\vdash_{\mathbf{HGIF}} G \mid \Gamma \Rightarrow A(t)$ for any t
$\vdash_{\mathbf{HGIF}} G \mid \exists x A(x) \Gamma \Delta$	implies	$\vdash_{\mathbf{HGIF}} G \mid A(t), \Gamma \Rightarrow \Delta$ for any t

Main Theorem: **HGIF** admits cut elimination.

Proof (sketch):

1. The largest cuts are atomic: Cut the left side of the highest atomic cut with the ancestors of the cut in the axioms. This is correct by the above Lemma (case 2.1). Cuts with axioms can be deleted.
2. The highest occurrence of a largest cut is on $A \not\equiv \triangle A'$:
   Use reductions to cut immediately after the inference on the non-reduced side. Again, this is correct by the above Lemma.
3. The highest occurrence of a largest cut is on $\triangle A$:
   Determine the inferences of $\Rightarrow \Delta$ which result in the cut-formula $\triangle A$. In

   $$\frac{G \mid \Gamma \Rightarrow \triangle A \quad \triangle A, \Pi \Rightarrow \Delta}{G \mid \Gamma, \Pi \Rightarrow \Delta} \; cut$$

   on the left side cut the inferences

   $$\frac{G_i' \mid \triangle \Pi_i' \Rightarrow A}{G_i' \mid \triangle \Pi_i' \Rightarrow \triangle A} \Rightarrow \triangle$$

   with the right side to obtain $G_i' \mid G \mid \triangle \Pi_i', \Pi_i \Rightarrow \Delta$.
   From these premises use the Lemma to derive $G \mid \Gamma, \Pi \Rightarrow \Delta$.

Consequences of cut elimination (1)

Corollary:
There is a tower-of-exponentials function bounding the depth of the cut-free proof with respect to the depth of the original proof and the logical complexity of the end-hypersequent.

Corollary (Mid-sequent theorem):
For every provable hypersequent containing only prenex formulas there is a proof π with a sequent S, such that above S all inferences are propositional or structural and below S all inferences are quantifier inferences or contractions.

Corollary (Herbrand's Theorem):
$\vdash_{\mathbf{HGIF}} \Rightarrow \exists \vec{x} A(\vec{x})$, where A is quantifier free implies
$\vdash_{\mathbf{HGIF}} \Rightarrow A(\vec{t}_1) \vee \ldots \vee A(\vec{t}_n)$ for some tuples $\vec{t}_1, \ldots, \vec{t}_n$ of terms with function symbols occurring in A.

Consequences of cut elimination (2)

Corollary:

The set $\{\forall\vec{x}\exists\vec{y}A(\vec{x},\vec{y}) \mid A$ quantifier and function free$\}$ is decidable.

Corollary:

For quantifier free A, the rule

$$\frac{\Rightarrow \triangle\exists\vec{x}A(\vec{x})}{\Rightarrow \exists\vec{x}\triangle A(\vec{x})}\ \triangle\exists\text{-shift}$$

is dependent (admissible).

Remark:

$\triangle\exists\vec{x}A(x) \to \exists\vec{x} \triangle A(\vec{x})$ is not valid (consider a proper supremum at 1).

Open Problem: Complexity of eliminating $\triangle\exists$-shift.

Remark:

$\triangle\exists$-shift is also admissible for prenex formulas.

Consequences of cut elimination (3)

Proposition:

The propositional logic $\mathbf{G}^{\triangle}_{[0,1]}$ admits interpolation.

Proof Idea:

Replace A and B in $A \to B$ by their respective chain normal forms.
The rest follows by case distinction.

Proposition:

The fragment of formulas $A \to B$, where A and B are prenex, admits interpolation in $\mathbf{G}^{\forall\triangle}_{[0,1]}$.

Proof Idea:

Use the expansion method of [Baaz, Lolic, 2020].

References (selection)

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