Modal Nelson lattices and their associated twist structures

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Context

- Possibilistic logic is one of the best-known logical systems proposed for handling uncertainty in Approximated Reasoning.
- We are interested in studying Possibilistic Logic by means modal Nilpotent Minimum (NM) logic using an algebraic approach.
- Modal NML can be considered as an involutive version of modal Gödel logic.
- In this work, we explode the fact that the class of NMAs is a subvariety of Nelson lattices.
- It is very well known that every Nelson lattice (N3) can be generated from a Heyting algebra using a twist-construction. The same construction works for NMAs from a Gödel algebra.
- We will expand this construction for N3 with modal operators.
- We will attempt to obtain the most general possible characterization.

Preliminaries: Nelson lattices

Nelson lattices

A bounded integral commutative residuated lattice is a Nelson lattice $\mathbf{A} = \langle A, *, \rightarrow, \wedge, \vee, \bot, \top \rangle$ of type $(2, 2, 2, 2, 0, 0)$ such that

- $\langle A, *, \top \rangle$ is a commutative monoid.
- $\langle A, \wedge, \vee, \bot, \top \rangle$ is a bounded lattice.
- The following residuated property holds:

$$a * b \leq c \quad \text{iff} \quad a \leq b \rightarrow c.$$

- The negation $\neg a = a \rightarrow \bot$ is involutive, i.e. $a = \neg\neg a$.
- The following property holds:

$$((a^2 \rightarrow b) \wedge ((\neg b)^2 \rightarrow \neg a)) \rightarrow (a \rightarrow b) = \top.$$

Preliminaries: Twist construction

Let $\mathbf{H} = \langle H, \wedge, \vee, \to, \bot, \top \rangle$ be a Heyting algebra.

Definition

A filter F of $\mathbf{H}$ is said to be Boolean provided the quotient $\mathbf{H}/F$ is a Boolean algebra.

- It is well known and easy to check that a filter F of the Heyting algebra $\mathbf{H}$ is Boolean if and only if $D(\mathbf{H}) = \{a \in H : \neg a = \bot\} \subseteq F$. (dense elements of H)
- Boolean filters of $\mathbf{H}$, ordered by inclusion, form a lattice, having the improper filter H as the greatest element and $D(\mathbf{H})$ as the smallest element.

Preliminaries: Twist construction

Theorem (Sendlewski + Busaniche&Cignoli)

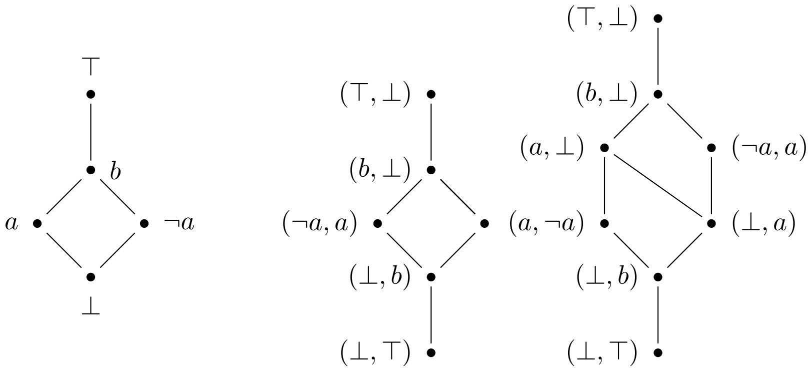
Given a Heyting algebra $\mathbf{H}$ and a Boolean filter F of $\mathbf{H}$ let

$$R(\mathbf{H}, F) := \{(x, y) \in H \times H : x \wedge y = \bot \text{ and } x \vee y \in F\}.$$

Then

1. $\mathbf{R}(\mathbf{H}, F) = (R(\mathbf{H}, F), \wedge, \vee, *, \Rightarrow, \bot, \top)$ *is a Nelson lattice, where*
   - $(x, y) \vee (s, t) = (x \vee s, y \wedge t)$,
   - $(x, y) \wedge (s, t) = (x \wedge s, y \vee t)$,
   - $(x, y) * (s, t) = (x \wedge s, (x \to t) \wedge (s \to y))$,
   - $(x, y) \Rightarrow (s, t) = ((x \to s) \wedge (t \to y), x \wedge t)$,
   - $\top = (\top, \bot)$, $\bot = (\bot, \top)$.
2. $\neg(x, y) = (y, x)$,
3. *Given a Nelson lattice $\mathbf{A}$, there is a (unique up to isomorphisms) Heyting algebra $\mathbf{H}_{\mathbf{A}}$ and a unique Boolean filter $F_{\mathbf{A}}$ of $\mathbf{H}_{\mathbf{A}}$ such that $\mathbf{A}$ is isomorphic to $\mathbf{R}(\mathbf{H}_{\mathbf{A}}, F_{\mathbf{A}})$.*

Examples



$\mathbf{H}$ $\mathbf{R}(\mathbf{H}, F')$ $\mathbf{R}(\mathbf{H}, F'')$

$F' = \{b, \top\}$ $F'' = \{a, b, \top\}$

From Nelson lattices to Heyting algebras

Important: Nelson lattice $\mathbf{A}$ satisfies 3-potency, i.e, $\forall a \in A : a^3 = a^2$.

On each Nelson lattice $\mathbf{A}$, we can define a congruence $\equiv$ on $\mathbf{A}$ by

$$x \equiv y \text{ if and only if } x^2 = y^2.$$

Let $H = \{a^2 : a \in \mathbf{A}\}$ and operations $a \star^* b = (a \star b)^2$ for every binary operation $\star \in \mathbf{A}$.Then

$$\mathbf{H}^* = (H, \vee^*, \wedge^*, \rightarrow^*, 0, 1)$$

is a Heyting algebra and $F = \{(a \vee \neg a)^2 : a \in A\}$ is a Boolean filter.

From Nelson lattices to Heyting algebras

Theorem

Let $\mathbf{N}$ *be a Nelson lattice. Then* $\mathbf{N}$ *is isomorphic to*

$$R(\mathbf{H}^*, F) := \{(x, y) \in H \times H : x \wedge y = \bot \text{ and } x \vee y \in F\}$$

where $F = \{(a \vee \neg a)^2 : a \in N\}$.

$$i \colon N \to R(\mathbf{H}^*, F)$$

$$i(a) = (a^2, (\neg a)^2)$$

Modal N3-lattices

A modal N3-lattices is an algebra $\langle \mathbf{A}, \blacksquare, \blacklozenge \rangle$ such that the reduct $\mathbf{A}$ is an N3-lattice and, for all $a, b \in A$,

(1) $\blacklozenge a = \neg \blacksquare \neg a$,

(2) if $a^2 = b^2$ then $(\blacksquare a)^2 = (\blacksquare b)^2$ and $(\blacklozenge a)^2 = (\blacklozenge b)^2$,

(3) If $(a \wedge b)^2 = \perp$ then $(\blacksquare a \wedge \blacklozenge b)^2 = \perp$.

$\mathbf{A}$ is said to be regular if it further satisfies

(4) $\blacksquare(a \wedge b) = \blacksquare a \wedge \blacksquare b$.

Moreover, by using (1) and (4), we can conclude:

(4′) $\blacklozenge(a \vee b) = \blacklozenge a \vee \blacklozenge b$.

$\mathbf{A}$ is normal if it is regular and

(5) $\blacksquare \top = \top$.

Comparison with existing work

U. Rivieccio. Paraconsistent modal logics. Electronic Notes in Theoretical Computer Science,278:173–186, 2011.

Rivieccio studied Modal N4-lattices and since Nelson algebras conform a subclass of N4-lattices, we can compare the results in the N3 context because Nelson algebras and Nelson residuated lattices are term equivalent.

$$\text{Nelson algebras} = \text{N4-lattices} + x \wedge \neg x \preceq y$$

Comparison with existing work

Definition (Rivieccio)

A monotone modal N4-lattice is an algebra $\mathbf{B} = \langle B, \wedge, \vee, \Rightarrow, \neg, \blacksquare \rangle$ such that the reduct $\langle B, \wedge, \vee, \Rightarrow, \neg \rangle$ is an N4-lattice and, for all $a, b \in B$,

- if $a \preceq b$, then $\blacksquare a \preceq \blacksquare b$,
- if $\neg a \preceq \neg b$, then $\neg \blacksquare a \preceq \neg \blacksquare b$.

Comparison with existing work

In the N3 context, we have

Monotone modal N4-lattice

$+$

$(x \wedge \neg x) \preceq y$

$\Longrightarrow$

Monotone N3-lattice

$$a^2 \leq b \rightarrow (\blacksquare a)^2 \leq \blacksquare b$$
$$(\neg a)^2 \leq \neg b \rightarrow (\neg \blacksquare a)^2 \leq \neg \blacksquare b$$

which is subclass of

Modal N3-lattices

Modal Heyting algebras

A modal Heyting algebra $M\mathbf{A}$ is an algebra $\langle \mathbf{A}, \Box, \Diamond \rangle$ such that the reduct $\mathbf{A}$ is an Heyting algebra and

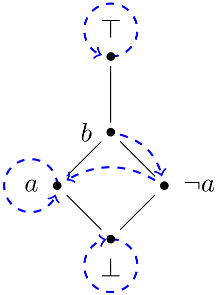
$$\text{If } a \wedge b = \bot \text{ then } \Box a \wedge \Diamond b = \bot.$$

$\mathbb{MH}$ denotes the quasi-variety of modal Heyting algebras.

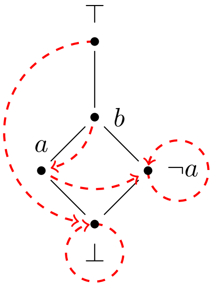
For example, an extension of this quasi-variety is the variety of *normal* modal Heyting algebras which is obtained by further considering

1. $\neg \Diamond a = \Box \neg a$,
2. $\Box(a \to b) \to (\Box a \to \Box b) = \top$ and
3. $\Box \top = \top$.

Modal Heyting example



$\Box\top = \top$; $\Diamond\top = \bot$;
$\Box\bot = \bot$; $\Diamond\bot = \bot$;
$\Box a = a$; $\Diamond a = \neg a$;
$\Box\neg a = a$; $\Diamond\neg a = \neg a$;
$\Box b = \neg a$; $\Diamond b = a$;



First result

Theorem

Let $\mathbf{H}$ be a modal Heyting algebra and let F be a Boolean filter such that

$$\text{if } a \wedge b = \bot \text{ and } a \vee b \in F \text{ then } \Box a \vee \Diamond b \in F.$$

Then $\mathbf{R}(\mathbf{H}, F) = (R(\mathbf{H}, F), \wedge, \vee, *, \Rightarrow, \bot, \top, \blacksquare, \blacklozenge)$ is a Modal Nelson lattice, where the operators $\blacksquare, \blacklozenge$ are defined as follows:

$$\blacksquare(x, y) = (\Box x, \Diamond y), \qquad \blacklozenge(x, y) = (\Diamond x, \Box y).$$

$$i \colon N \to R(H^*, F)$$

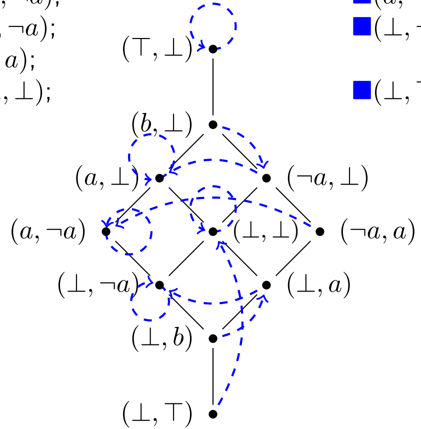
$$\begin{aligned} i(\blacksquare a) &= ((\blacksquare a)^2, (\neg \blacksquare a)^2) \\ &= ((\blacksquare a)^2, (\blacklozenge \neg a)^2) \\ &= (\Box^* a^2, \Diamond^* (\neg a)^2) \end{aligned}$$

Example

- $(\top,\bot) = (\top,\bot)$;
- $(\neg a,\bot) = (a,\bot)$;
- $(\neg a, a) = (a,\neg a)$;
- $(\bot, a) = (\bot,\neg a)$;
- $(\bot, b) = (\bot, a)$;
- $(\bot,\bot) = (\bot,\bot)$;

- $(b,\bot) = (\neg a,\bot)$;
- $(a,\bot) = (a,\bot)$;
- $(a,\neg a) = (a,\neg a)$;
- $(\bot,\neg a) = (\bot,\neg a)$;

- $(\bot,\top) = (\bot,\bot)$;



Next results

Lemma

Let $\mathbf{N}$ *be a modal N3 lattice. Then* $\mathbf{H}^* = (H, \vee^*, \wedge^*, \to^*, \neg^*, 0, 1, \Box^*, \Diamond^*)$ *with* $H = \{a^2 : a \in N\}$, $F = \{(a \vee \neg a)^2 : a \in N\}$ *and modal operators*

$$\Box^* a^2 = (\blacksquare a)^2, \qquad \Diamond^* a^2 = (\blacklozenge a)^2,$$

is a modal Heyting algebra. In addition, if $a^2 \vee^* b^2 \in F$ *and* $a^2 \wedge^* b^2 = 0$ *then* $\Box^* a^2 \vee^* \Diamond^* b^2 \in F$.

Theorem

Let $\mathbf{N}$ *be a modal N3 lattice. Then* $\mathbf{N}$ *is isomorphic to*

$$R(\mathbf{H}^*, F) := \{(x, y) \in H \times H : x \wedge y = \bot \text{ and } x \vee y \in F\}$$

where $F = \{(a \vee \neg a)^2 : a \in N\}$.

Final comments

Modal Nilpotent Minimum algebras

They are modal Nelson lattices which further satisfy

$$\text{(Prelinearity)} \quad (x \to y) \vee (y \to x) = \top$$

$$(a * b \to \bot) \vee (a \wedge b \to a * b) = \top$$

Modal Gödel algebras

They are modal Heyting algebras which further satisfy

$$\text{(Prelinearity)} \quad (x \to y) \vee (y \to x) = \top$$

All mentioned connections between modal N3 lattices and modal Heyting algebras can be established between Modal Nilpotent Minimum algebras and Modal Gödel algebras.

Conclusions and future works

- Our results generalize the existing conditions regarding modal operators on twist-structures in the $N3$-context.
- We want to provide a topological duality for these structures by means of Esakia spaces endowed with (non-monotonic) neighborhood functions.
- We would like to explore the notions of N3-neighborhood frame and N3.Kripke frame as alternative semantics, and their connections with the algebraic semantics introduced before.
- We plan to study more deeply the connection between to Modal NM-algebras and modal Gödel algebras when more axioms are added.

Thank you for your attention

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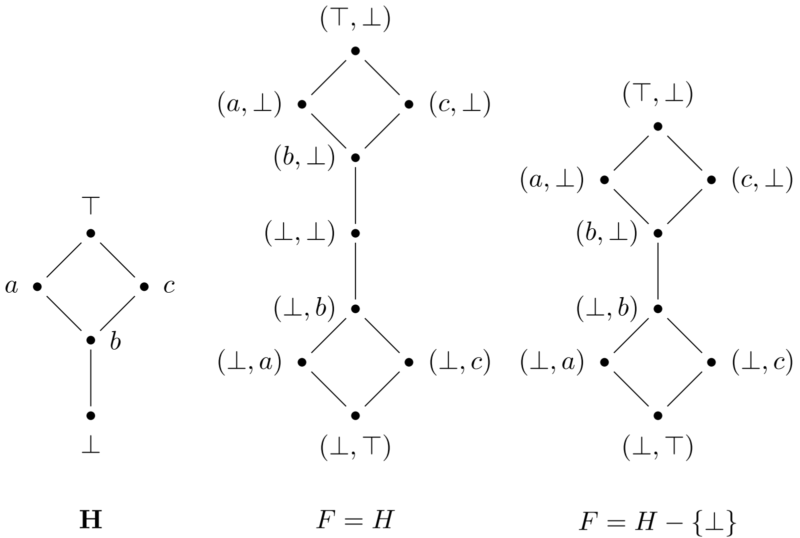
Regular Nelson lattices

- A Nelson lattice is Regular if and only if the Heyting algebra $\mathbf{H}^*$ satisfies the Stone identity $\neg x \vee \neg\neg x = 1$.
- $\mathcal{NR}$ is a subvariety of the variety of Nelson residuated lattices generated by the connected rotations of generalized Heyting algebras.
- Let $A \in \mathcal{NR}$ be directly indecomposable. Then either $A \cong DR(A_{\mathbf{H}})$ or $A \cong CR(A_{\mathbf{H}})$. (disconnected or connected rotations of generalized H.A., respectively).

Bonus track: Regular Nelson lattices d.i.



Regular Nelson lattices d.i.

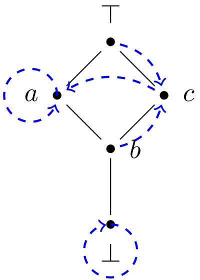
With negation fixed point:
If there exist $x, y \in H$ such that $\Box x > \bot$ and $\Diamond y > \bot$ then the operators are defined:

$$\blacksquare(x,y) = \begin{cases} \text{if } y = \bot & \text{then } (\Box x, \bot) \\ \text{if } x = \bot & \text{then } (\bot, \Diamond y) \end{cases}$$

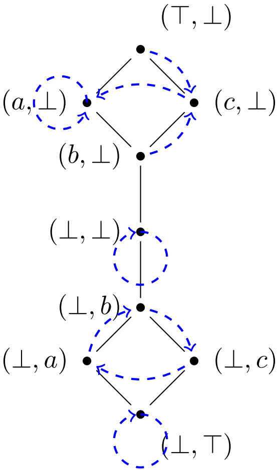
$$\blacklozenge(x,y) = \begin{cases} \text{if } y = \bot & \text{then } (\Diamond x, \bot) \\ \text{if } x = \bot & \text{then } (\bot, \Box y) \end{cases}$$

$$\blacklozenge(\bot,\bot) = \blacksquare(\bot,\bot) = (\bot,\bot)$$

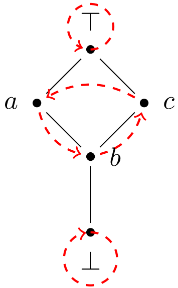
Regular Nelson lattices d.i.



$\mathbf{H}$

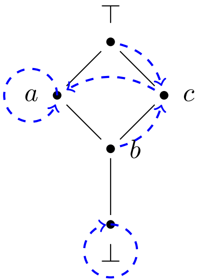


$F = H$

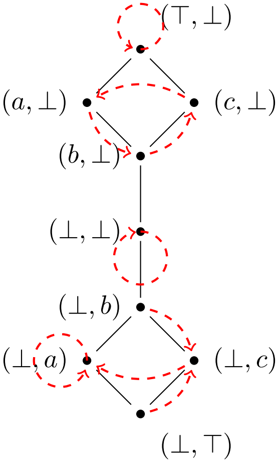


$\mathbf{H}$

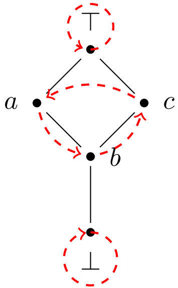
Regular Nelson lattices d.i.



$\mathbf{H}$



$F = H$



$\mathbf{H}$

Regular Nelson lattices d.i.

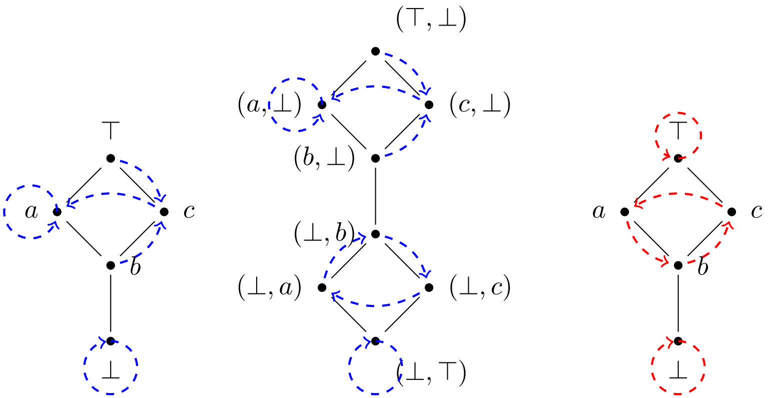
Without negation fixed point:
If there exist $x, y \in H$ such that $\Box x > \bot$ and $\Diamond x > \bot$. The operators are defined:

$$\blacksquare(x,y) = \begin{cases} \text{if } y = \bot & \text{then } (\Box x, \bot) \\ \text{if } x = \bot & \text{then } (\bot, \Diamond y) \end{cases}$$

$$\blacklozenge(x,y) = \begin{cases} \text{if } y = \bot & \text{then } (\Diamond x, \bot) \\ \text{if } x = \bot & \text{then } (\bot, \Box y) \end{cases}$$

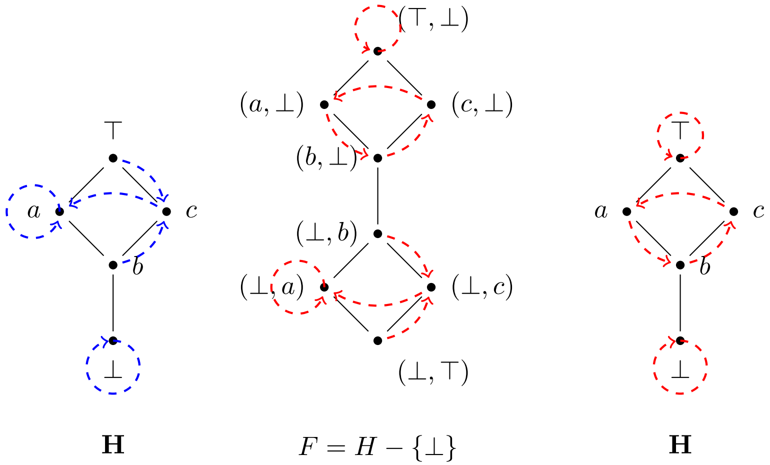
If $x \in H$ such that $x > \bot$ then $\Box x > \bot$ and $\Diamond x > \bot$.

Regular Nelson lattices d.i.



H $\qquad F = H - \{\perp\}$ $\qquad$ **H**

Regular Nelson lattices d.i.



Regular Nelson lattices d.i.

With negation fixed point:
If $\Box[H] = \{\bot\}$, then the operators are defined:

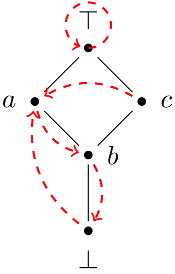
$$\blacksquare(x, y) = (\bot, \Diamond y)$$

$$\blacklozenge(x, y) = (\Diamond x, \bot)$$

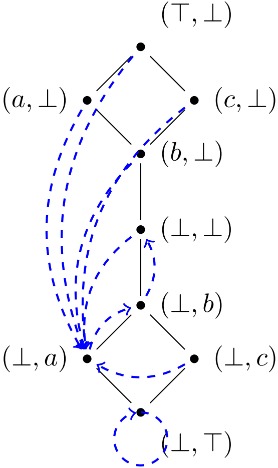
In particular

$$\blacklozenge(\bot, \bot) = (\Diamond\bot, \bot) \quad \text{and} \quad \blacksquare(\bot, \bot) = (\bot, \Diamond\bot)$$

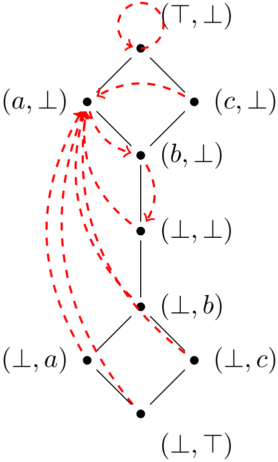
Regular Nelson lattices d.i.



$\mathbf{H}$



$F = H$



$F = H$

Regular Nelson lattices d.i.

Without negation fixed point:
If $\Box[H] = \{\bot\}$, then the operators are defined:

$$\begin{aligned} \blacksquare(x,y) &= (\bot, \Diamond y) \\ \blacklozenge(x,y) &= (\Diamond x, \bot) \end{aligned}$$

$\Diamond x > \bot$ for all $x \in H$.

Regular Nelson lattices d.i.