Embeddings of metric Boolean algebras in $\mathbb{R}^N$

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Outline and Objectives

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1. Embedding isometrically the space of atoms of a metric Boolean algebra in $\mathbb{R}^N$
2. Study the topology of the probability measures for which there is an isometric embedding in $\mathbb{R}^N$.

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- Every Boolean algebra carries a finitely-additive probability measure. Not a positive one!!
- If $\mathbf{A}$ is atomic then it carries at least a positive measure (Horn-Tarski).

Metric Boolean algebras

Let $\mathbf{A}$ be a Boolean algebra with a positive finitely-additive probability measure m. For every $a, b \in A$ let:

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- If m is not positive, then $(\mathbf{A}, d_m)$ is a pseudo-metric space.

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$d_m(a, \top) = m(\neg a) = 1 - m(a)$

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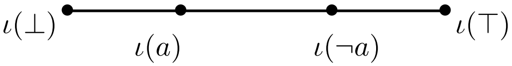

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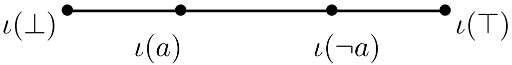

$d_m(a, \neg a) = 1$, thus $\iota(a) = \iota(\bot)$ and $\iota(\neg a) = \iota(\top)$. Contradiction!

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Embeddings of generic metric spaces into $\mathbb{R}^N$ are ruled by the following

Theorem (Morgan [6])

A metric space (X, d) embeds isometrically in $\mathbb{R}^N$ if and only if it is flat and has dimension equal to N.

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A metric space (X, d) is flat if the determinant of the $n \times n$ matrix $M(x_0, \ldots, x_n)$, whose generic ij-entry is

$$\langle x_i, x_j, x_0 \rangle = \frac{1}{2}(d(x_0, x_i)^2 + d(x_0, x_j)^2 - d(x_i, x_j)^2)$$

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Dimension

The dimension of a space (X, d) is the greatest N (if exists) such that there exists a N-simplex with positive determinant.

Simplifying Morgan's determinant

Let $\mathsf{At}(\mathbf{A}) = \{a_0, a_1, \ldots, a_k\}$; set $x_i = m(a_i)$ (for $i \in \{0, 1, \ldots, k\}$).

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Lemma

Let $\mathbf{A}$ *be a finite metric atomic Boolean algebra with* $k+1$ *atoms and* $M(x_0, \ldots, x_n)$, $2 \leq n \leq k$, *the matrix in Morgan's theorem. Then* $\det(M(x_0, \ldots, x_n)) =$

$$2^{n-1}\left[\left(\sum_{\alpha=0}^{n} x_0 \cdot \cdots \cdot \hat{x}_\alpha \cdot \cdots \cdot x_n\right)^2 - (n-1)\left(\sum_{\alpha=0}^{n} x_0^2 \cdot \cdots \cdot \hat{x}_\alpha^2 \cdot \cdots \cdot x_n^2\right)\right],$$

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Crucial: $a, b \in \mathsf{At}(\mathbf{A})$, $d_m(a, b) = m(a) + m(b)$ (since $a \leq \neg b$ and $b \leq \neg a$).

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Moreover:

- $x_i \in \mathbb{R}_+$ (not necessarily in $(0, 1)$).

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Let $\mathbf{A}$ be a finite metric Boolean algebra with $k+1$ atoms, m a finitely-additive probability measure such that $m(a_i) = \frac{1}{k+1}$, for every $a_i \in \mathrm{At}(\mathbf{A})$. Then $(\mathrm{At}(\mathbf{A}), d_m)$ embeds isometrically in $\mathbb{R}^k$ (with the Euclidean metric).

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- Not for all m, $(\mathrm{At}(\mathbf{A}), d_m) \hookrightarrow \mathbb{R}^N$!!

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- a sequence of n independent experiments (Bernoulli process), asking a "yes-no" question, each with a two-valued outcome.

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Setting $p = q = 1/2$ and using the previous Lemma, one gets $M(x_0, \ldots, x_3), M(x_0, \ldots, x_4) > 0, M(x_0, \ldots, x_5) < 0$.

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Idea

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$$\mathcal{M}_{flat}(\mathbf{A}) = \{m \colon A \to [0,1] |\ (\mathrm{At}(\mathbf{A}), d_m) \hookrightarrow \mathbb{R}^N, \text{ for some } N \in \mathbb{N}\}$$

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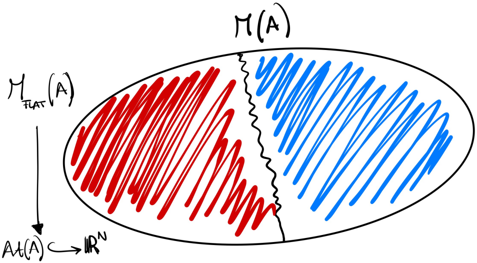

The main problem

Due to Morgan's theorem:

$$\mathcal{M}_{flat}(\mathbf{A}) = \bigcap_{n=3}^{k} C_n \cap \Pi_k,$$

$C_n = \{\vec{x} \in \mathbb{R}_+^{k+1} \mid \det(M(x_0, \ldots, x_n)) \geqslant 0\}$, with $3 \leq n \leq k$ and Π_k the simplex of probability measures.

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Problem

Study the topology of $\mathcal{M}_{flat}(\mathbf{A})$ and of its complement (with the topology induced by $(0,1)^{k+1} \subset \mathbb{R}_+^{k+1}$).

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Theorem

Let $k \geqslant 3$. Then:

1. $\mathcal{M}_{flat}(\mathrm{At}(\mathbf{A}))$ *is contractible.*
2. $\mathcal{M}(\mathrm{At}(\mathbf{A})) \setminus \mathcal{M}_{flat}(\mathrm{At}(\mathbf{A}))$ *is simply-connected (not contractible).*

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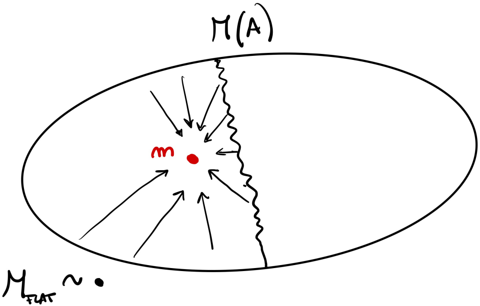

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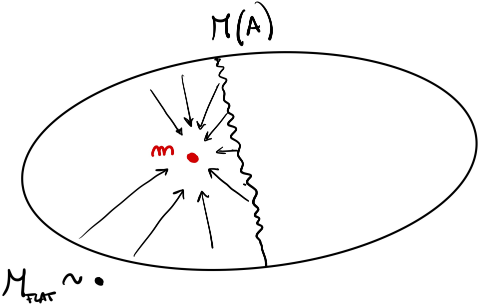

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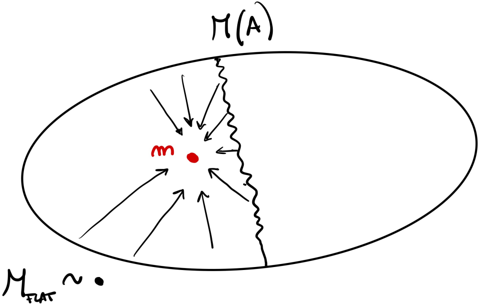

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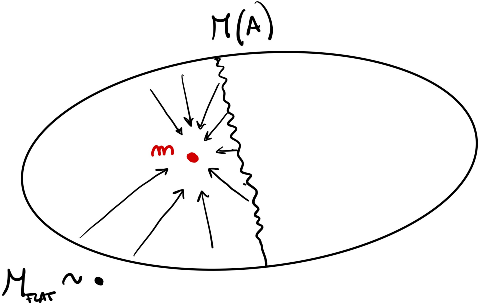

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2. Applications in probability theory.
3. Embeddings of *metric MV-algebras* (MV algebra + faithful state).

References

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Thanks!

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Proof of the main theorem

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$$s : \mathbb{R}_+^{k+1} \to \Pi_k \subset (0,1)^{k+1},$$

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$\bigcap_{n=3}^{k} C_n$ is contractible (homeomorphic to a convex), so is $s(\bigcap_{n=3}^{k} C_n)$.

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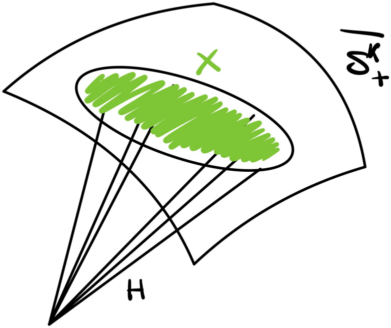

Idea of the proof (continued)

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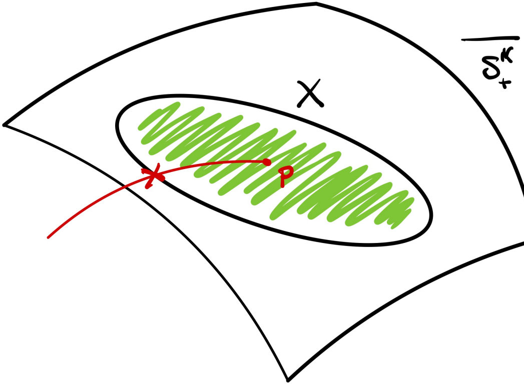
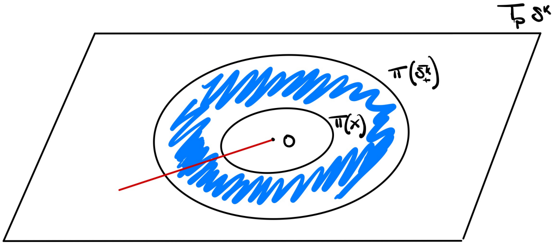
- Prove that every geodesic (from p) intersects ∂X exactly in one point.



Idea of the proof (continued)

Use stereographic projection π from $S^k \setminus \{-p\}$ to the tangent space T_pS^k.



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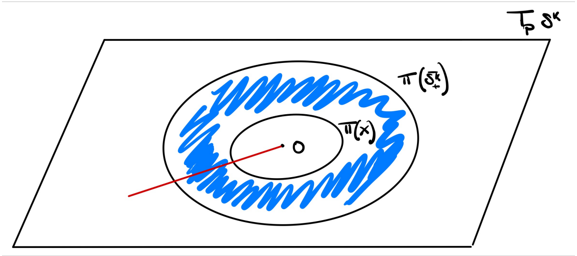

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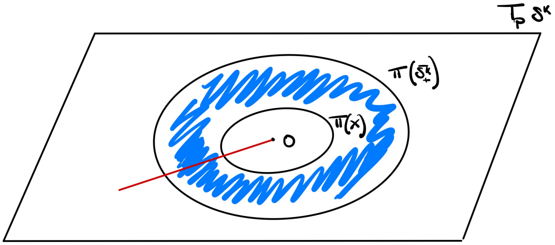

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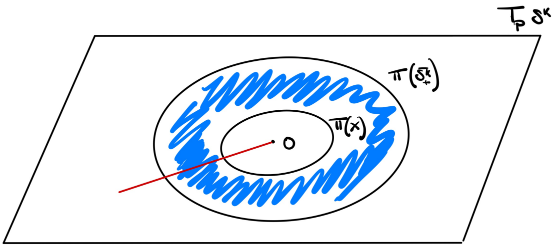

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- $\mathbb{R}^{k+1}_+ \setminus H \cong S^k_+ \setminus X \times (0,+\infty) \cong S^{k-1} \times (0,1) \times (0,+\infty)$

homotopically equivalent to S^{k-1}: simply connected ($k \geqslant 3$), not contractible.