Local Modal Product Logic is decidable

Amanda Vidal

Artificial Inteligence Research Institute IIIA - CSIC

LATD 2022, Paestum, 6 September

This project has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Sklodowska-Curie grant agreement No. 101027914.

Introduction

- Many normal (classical) modal logics: finite model property + finite axiomatizability $\Rightarrow$ decidability

Introduction

- Many normal (classical) modal logics: finite model property + finite axiomatizability $\Rightarrow$ decidability
- many-valued cases: ? no (immediate) FMP or (known) R.E axiomatization...

Introduction

- Many normal (classical) modal logics: finite model property + finite axiomatizability $\Rightarrow$ decidability
- many-valued cases: ? no (immediate) FMP or (known) R.E axiomatization...
  - Validity in the expansion of Gödel logic with modal operators does not enjoy FMP with the usual Kripke semantics, but it is decidable [Caicedo et.al., 2017]

Introduction

- Many normal (classical) modal logics: finite model property + finite axiomatizability $\Rightarrow$ decidability
- many-valued cases: ? no (immediate) FMP or (known) R.E axiomatization...
    - Validity in the expansion of Gödel logic with modal operators does not enjoy FMP with the usual Kripke semantics, but it is decidable [Caicedo et.al., 2017]
    - Validity and > 0-sat in FDL (multi-modal variation) over Product logic are decidable [Cerami et. al, 2021]

Introduction

- Many normal (classical) modal logics: finite model property + finite axiomatizability $\Rightarrow$ decidability
- many-valued cases: ? no (immediate) FMP or (known) R.E axiomatization...
  - Validity in the expansion of Gödel logic with modal operators does not enjoy FMP with the usual Kripke semantics, but it is decidable [Caicedo et.al., 2017]
  - Validity and >0-sat in FDL (multi-modal variation) over Product logic are decidable [Cerami et. al, 2021]
  - Global entailment in modal [0,1]-valued Łukasiewicz and Product logics is not even R.E [V. 2022]

Introduction

- Many normal (classical) modal logics: finite model property + finite axiomatizability ⇒ decidability
- many-valued cases: ? no (immediate) FMP or (known) R.E axiomatization...
    - Validity in the expansion of Gödel logic with modal operators does not enjoy FMP with the usual Kripke semantics, but it is decidable [Caicedo et.al., 2017]
    - Validity and > 0-sat in FDL (multi-modal variation) over Product logic are decidable [Cerami et. al, 2021]
    - Global entailment in modal [0,1]-valued Łukasiewicz and Product logics is not even R.E [V. 2022]
    - No known (explicit) R.E. axiomatization of local entailment in modal [0,1]-valued Łukasiewicz, but it is decidable.

Introduction

- Many normal (classical) modal logics: finite model property + finite axiomatizability $\Rightarrow$ decidability
- many-valued cases: ? no (immediate) FMP or (known) R.E axiomatization...
  - Validity in the expansion of Gödel logic with modal operators does not enjoy FMP with the usual Kripke semantics, but it is decidable [Caicedo et.al., 2017]
  - Validity and >0-sat in FDL (multi-modal variation) over Product logic are decidable [Cerami et. al, 2021]
  - Global entailment in modal [0,1]-valued Łukasiewicz and Product logics is not even R.E [V. 2022]
  - No known (explicit) R.E. axiomatization of local entailment in modal [0,1]-valued Łukasiewicz, but it is decidable.
  - local entailment in modal [0,1]-valued Product?

Introduction

- Many normal (classical) modal logics: finite model property + finite axiomatizability ⇒ decidability
- many-valued cases: ? no (immediate) FMP or (known) R.E axiomatization...
  - Validity in the expansion of Gödel logic with modal operators does not enjoy FMP with the usual Kripke semantics, but it is decidable [Caicedo et.al., 2017]
  - Validity and > 0-sat in FDL (multi-modal variation) over Product logic are decidable [Cerami et. al, 2021]
  - Global entailment in modal [0,1]-valued Łukasiewicz and Product logics is not even R.E [V. 2022]
  - No known (explicit) R.E. axiomatization of local entailment in modal [0,1]-valued Łukasiewicz, but it is decidable.
  - local entailment in modal [0,1]-valued Product?

Theorem

Local modal [0,1]-valued Product logic is decidable

Modal product logics

Language: $\&, \rightarrow, 0$ plus two unary (modal) symbols ($\Box, \Diamond$)

Modal product logics

Language: $\&, \rightarrow, 0$ plus two unary (modal) symbols $(\Box, \Diamond)$

Definition
A (standard crisp) product Kripke model $\mathfrak{M}$ is a tripla $\langle W, R, e\rangle$ where:

- $R \subseteq W \times W$ (Rus stands for $\langle u, s\rangle \in R$)
- $e : W \times Var \rightarrow [0, 1]$ uniquelly extended by:

$$e(u, \varphi\&\psi) := e(u, \varphi) \cdot e(u, \psi)$$

$$e(u, \varphi \rightarrow \psi) := \begin{cases} 1 & \text{if } e(u, \varphi) \leq e(u, psi) \\ e(u, \psi)/e(u, \varphi) & \text{otherwise} \end{cases}$$

$$e(u, \Box\varphi) := inf\{e(s, \varphi) \colon Rus\}$$

$$e(u, \Diamond\varphi) := sup\{e(s, \varphi) \colon Rus\}$$

Local deduction: $\Gamma \Vdash_{K\Pi} \varphi$ iff
$\forall u \in W$ $[e(u, [\Gamma]) \subseteq \{1\}$ implies $e(u, \varphi) = 1]$ for all product Kripke models $\mathfrak{M}$.

Relation to FO

The previous logic can be translated into a fragment of the corresponding FO logic.

$$\langle x, v\rangle^\sharp := P_x(v) \qquad \langle \varphi \star \psi, v\rangle^\sharp := \langle \varphi, v\rangle^\sharp \star \langle \psi, v\rangle^\sharp$$

$$\langle \Box\varphi, v\rangle^\sharp := \forall w\ R(v, w) \to \langle \varphi, w\rangle^\sharp \qquad \langle \Diamond\varphi, v\rangle^\sharp := \exists w\ R(v, w) \odot \langle \varphi, w\rangle^\sharp$$

Relation to FO

The previous logic can be translated into a fragment of the corresponding FO logic.

$$\langle x, v\rangle^\sharp := P_x(v) \qquad \langle \varphi \star \psi, v\rangle^\sharp := \langle \varphi, v\rangle^\sharp \star \langle \psi, v\rangle^\sharp$$

$$\langle \Box\varphi, v\rangle^\sharp := \forall w \; R(v,w) \to \langle \varphi, w\rangle^\sharp \qquad \langle \Diamond\varphi, v\rangle^\sharp := \exists w \; R(v,w) \odot \langle \varphi, w\rangle^\sharp$$

Observation

$\Gamma \Vdash_{K\Pi} \varphi \iff \forall v, w \; R(v,w) \vee \neg R(v,w), \forall v \; \langle \Gamma, v\rangle^\sharp \models_{\forall\Pi} \forall v \; \langle \varphi, v\rangle^\sharp$ where $\forall\Pi$ is the F.O. logic over $[0,1]_\Pi$.

Relation to FO

The previous logic can be translated into a fragment of the corresponding FO logic.

$$\langle x, v\rangle^\sharp := P_x(v) \qquad \langle \varphi \star \psi, v\rangle^\sharp := \langle \varphi, v\rangle^\sharp \star \langle \psi, v\rangle^\sharp$$

$$\langle \Box\varphi, v\rangle^\sharp := \forall w \; R(v,w) \to \langle \varphi, w\rangle^\sharp \qquad \langle \Diamond\varphi, v\rangle^\sharp := \exists w \; R(v,w) \odot \langle \varphi, w\rangle^\sharp$$

Observation

$\Gamma \Vdash_{K\Pi} \varphi \iff \forall v, w \; R(v,w) \vee \neg R(v,w), \forall v \; \langle \Gamma, v\rangle^\sharp \models_{\forall\Pi} \forall v \; \langle \varphi, v\rangle^\sharp$ where $\forall\Pi$ is the F.O. logic over $[0,1]_\Pi$.

Definition

A **Quasi-witnessed** (FO) model $\mathfrak{M}$ over an algebra $\mathbf{A}$ is s.t:

$$|\exists x \varphi(x)|_{\mathfrak{M}} = |\varphi(x)|_{\mathfrak{M}, x\mapsto p} \text{ for some } p \in W$$

$$|\forall x \varphi(x)|_{\mathfrak{M}} = \begin{cases} 0 & \text{or} \\ |\varphi(x)|_{\mathfrak{M}, x\mapsto p} \text{ for some } p \in W & \end{cases}$$

Relation to FO

The previous logic can be translated into a fragment of the corresponding FO logic.

$$\langle x, v\rangle^\sharp := P_x(v) \qquad \langle \varphi \star \psi, v\rangle^\sharp := \langle \varphi, v\rangle^\sharp \star \langle \psi, v\rangle^\sharp$$

$$\langle \Box\varphi, v\rangle^\sharp := \forall w\ R(v,w) \to \langle \varphi, w\rangle^\sharp \qquad \langle \Diamond\varphi, v\rangle^\sharp := \exists w\ R(v,w) \odot \langle \varphi, w\rangle^\sharp$$

Observation

$\Gamma \Vdash_{K\Pi} \varphi \iff \forall v,w\ R(v,w) \vee \neg R(v,w), \forall v\ \langle \Gamma, v\rangle^\sharp \models_{\forall\Pi} \forall v\ \langle \varphi, v\rangle^\sharp$
where $\forall\Pi$ is the F.O. logic over $[0,1]_\Pi$.

Definition

A **Quasi-witnessed** (Kripke) model $\mathfrak{M}$ over an algebra $\mathbf{A}$ is s.t:

$$e(v, \Diamond\varphi) = e(w,\varphi) \text{ for some } w \in W \text{ with } Rvw$$

$$e(v, \Box\varphi) = \begin{cases} 0 & \text{or} \\ e(w,\varphi) \text{ for some } w \in W \text{ with } Rvw \end{cases}$$

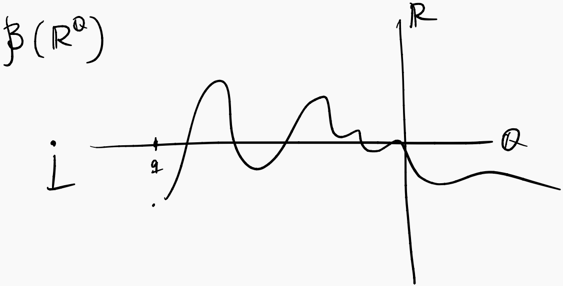
About $\forall\Pi$ and $K\Pi$

- (Laskowski-Malekpour, '07) proved $\forall\Pi$ is complete w.r.t quasi-witnessed models over $\mathfrak{B}(\mathbb{R}^{\mathbb{Q}})$, for $\mathbb{R}^{\mathbb{Q}}$ being **the Lexicographic sum group**: the ordered abelian group of functions $f\colon \mathbb{Q} \to \mathbb{R}$ whose support is well ordered (i.e., $\{q \in \mathbb{Q} \colon f(q) \neq 0\}$ is a well ordered subset of $\mathbb{Q}$). $+$ is defined component-wise and the ordering is lexicographic.
- The analogous is inherited in $K\Pi$, getting completeness w.r.t. quasi-witnessed trees over $\mathfrak{B}(\mathbb{R}^{\mathbb{Q}})$.

About $\forall\Pi$ and $K\Pi$

- (Laskowski-Malekpour, '07) proved $\forall\Pi$ is complete w.r.t quasi-witnessed models over $\mathfrak{B}(\mathbb{R}^{\mathbb{Q}})$, for $\mathbb{R}^{\mathbb{Q}}$ being **the Lexicographic sum group**: the ordered abelian group of functions $f\colon \mathbb{Q} \to \mathbb{R}$ whose support is well ordered (i.e., $\{q \in \mathbb{Q} \colon f(q) \neq 0\}$ is a well ordered subset of $\mathbb{Q}$). $+$ is defined component-wise and the ordering is lexicographic.
- The analogous is inherited in $K\Pi$, getting completeness w.r.t. quasi-witnessed trees over $\mathfrak{B}(\mathbb{R}^{\mathbb{Q}})$.

$T \models a[q] = 0 \ \forall q$



About $\forall\Pi$ and $K\Pi$

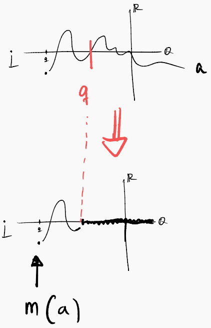
- (Laskowski-Malekpour, '07) proved $\forall\Pi$ is complete w.r.t quasi-witnessed models over $\mathfrak{B}(\mathbb{R}^{\mathbb{Q}})$, for $\mathbb{R}^{\mathbb{Q}}$ being **the Lexicographic sum group**: the ordered abelian group of functions $f\colon \mathbb{Q} \to \mathbb{R}$ whose support is well ordered (i.e., $\{q \in \mathbb{Q}\colon f(q) \neq 0\}$ is a well ordered subset of $\mathbb{Q}$). $+$ is defined component-wise and the ordering is lexicographic.
- The analogous is inherited in $K\Pi$, getting completeness w.r.t. quasi-witnessed trees over $\mathfrak{B}(\mathbb{R}^{\mathbb{Q}})$.

For an element $a \in \mathfrak{B}(\mathbb{R}^{\mathbb{Q}})$ we let:

- For $q \in \mathbb{Q}$, $a_{\hookleftarrow q}$ is $\perp$ if $a = \perp$ and, otherwise

$$a_{\hookleftarrow q}(p) = \begin{cases} a(p) & \text{if } p \leq q \\ 0 & \text{otherwise} \end{cases}$$

- $m(a) = min\{q \in \mathbb{Q}\colon a[q] < 0\}$, for $a > \perp$.



Some notation

Let Υ be a finite set of (modal) formulas with maximum modal depth $n \geq 1$. For $0 \leq i \leq n$ let:

$$\Upsilon_0 := \mathit{PropSFm}(\Upsilon) \qquad \Upsilon_{i+1} := \bigcup_{\heartsuit\psi \in \Upsilon_i} \mathit{PropSFm}(\psi)$$

Ex: $\Upsilon = \{\Box(x \to \Diamond y)\} \Rightarrow \Upsilon_0 = \Upsilon, \quad \Upsilon_1 = \{x, \Diamond y\}, \quad \Upsilon_2 = \{y\}$

Some notation

Let Υ be a finite set of (modal) formulas with maximum modal depth $n \geq 1$. For $0 \leq i \leq n$ let:

$$\Upsilon_0 := \mathit{PropSFm}(\Upsilon) \qquad \Upsilon_{i+1} := \bigcup_{\heartsuit\psi \in \Upsilon_i} \mathit{PropSFm}(\psi)$$

Consider sequences $\sigma = \langle 0, \varphi_0, \ldots, \varphi_k \rangle$ for $\varphi_i \in \Upsilon_i$ **beginning with a modality** for encoding the "witness" worlds in a model.

Some notation

Let Υ be a finite set of (modal) formulas with maximum modal depth $n \geq 1$. For $0 \leq i \leq n$ let:

$$\Upsilon_0 \coloneqq \mathit{PropSFm}(\Upsilon) \qquad \Upsilon_{i+1} \coloneqq \bigcup_{\heartsuit\psi \in \Upsilon_i} \mathit{PropSFm}(\psi)$$

Consider sequences $\sigma = \langle 0, \varphi_0, \ldots, \varphi_k \rangle$ for $\varphi_i \in \Upsilon_i$ **beginning with a modality** for encoding the "witness" worlds in a model.

For modeling the information about the unwitnessed formulas, consider also the sequences of the form $\langle \varphi_1, \ldots, \varphi'_k \rangle$ (the primed elements will be $\Box$ formulas).

Σ are all these sequences (and Σ_i the corresponding i-long sequences).

Some notation

Let Υ be a finite set of (modal) formulas with maximum modal depth $n \geq 1$. For $0 \leq i \leq n$ let:

$$\Upsilon_0 := PropSFm(\Upsilon) \qquad \Upsilon_{i+1} := \bigcup_{\heartsuit\psi \in \Upsilon_i} PropSFm(\psi)$$

Consider sequences $\sigma = \langle 0, \varphi_0, \ldots, \varphi_k \rangle$ for $\varphi_i \in \Upsilon_i$ **beginning with a modality** for encoding the "witness" worlds in a model.

For modeling the information about the unwitnessed formulas, consider also the sequences of the form $\langle \varphi_1, \ldots, \varphi_k' \rangle$ (the primed elements will be $\Box$ formulas).

Σ are all these sequences (and Σ_i the corresponding i-long sequences).

$\underline{\sigma} \equiv$ the sequence where we remove from σ the prime from all the primed formulas,
$\sigma_- \equiv$ the sequence where the prime is removed from the first appearing primed formula.

Unwitnessed formulas in $\mathfrak{B}(\mathbb{R}^{\mathbb{Q}})$-models?

Lemma

Let $\mathfrak{M}$ be a (quasi-witnessed) $\mathfrak{B}(\mathbb{R}^{\mathbb{Q}})$-Kripke model and $\Upsilon \subseteq_\omega Fm$. For any $v \in W$ and $\Box\varphi \in Fm$ such that $\Box\varphi \in UW_{\mathfrak{M}}(v, Fm)$, then there is some world $v^{\Upsilon}_{\Box\varphi} \in W$ with $Rvv^{\Upsilon}_{\Box\varphi}$ for which

$$m(e(v^{\Upsilon}_{\Box\varphi}, \varphi)) < m(e(v^{\Upsilon}_{\Box\varphi}, \chi)) \qquad \text{for any } \Box\chi \in \Upsilon \text{ s.t. } e(v, \Box\chi) > \bot.$$

Unwitnessed formulas in $\mathfrak{B}(\mathbb{R}^{\mathbb{Q}})$-models?

Lemma

Let $\mathfrak{M}$ be a (quasi-witnessed) $\mathfrak{B}(\mathbb{R}^{\mathbb{Q}})$-Kripke model and $\Upsilon \subseteq_\omega Fm$. For any $v \in W$ and $\Box\varphi \in Fm$ such that $\Box\varphi \in UW_{\mathfrak{M}}(v, Fm)$, then there is some world $v^{\Upsilon}_{\Box\varphi} \in W$ with $Rvv^{\Upsilon}_{\Box\varphi}$ for which

$$m(e(v^{\Upsilon}_{\Box\varphi}, \varphi)) < m(e(v^{\Upsilon}_{\Box\varphi}, \chi)) \qquad \text{for any } \Box\chi \in \Upsilon \text{ s.t. } e(v, \Box\chi) > \bot.$$

if $\inf e(v, \varphi) = \bot$, $\forall q \in \mathbb{Q}\ \exists v$ s.t. $m(e(v,q)) < q$.

Unwitnessed formulas in $\mathfrak{B}(\mathbb{R}^{\mathbb{Q}})$-models?

Lemma

Let $\mathfrak{M}$ be a (quasi-witnessed) $\mathfrak{B}(\mathbb{R}^{\mathbb{Q}})$-Kripke model and $\Upsilon \subseteq_\omega Fm$. For any $v \in W$ and $\Box\varphi \in Fm$ such that $\Box\varphi \in UW_{\mathfrak{M}}(v, Fm)$, then there is some world $v^{\Upsilon}_{\Box\varphi} \in W$ with $Rvv^{\Upsilon}_{\Box\varphi}$ for which

$$m(e(v^{\Upsilon}_{\Box\varphi}, \varphi)) < m(e(v^{\Upsilon}_{\Box\varphi}, \chi)) \qquad \text{for any } \Box\chi \in \Upsilon \text{ s.t. } e(v, \Box\chi) > \bot.$$

We can build $\mathfrak{M}^+$ extending $\mathfrak{M}$, with (certain) worlds labeled by elements in Σ s.t:

1. if a formula $\heartsuit\psi$ was witnessed in $\underline{\sigma}$, then $e^+(\sigma, \heartsuit\psi) = e^+(\sigma^\frown\heartsuit\psi, \psi)$,
2. if a formula $\Box\psi$ was unwitnessed in $\underline{\sigma}$, then all necessary $\sigma^\frown\Box\psi^\frown\sigma_1$ AND $\sigma^\frown\Box\psi'^\frown\sigma_1$ belong to $\mathfrak{M}^+$, and...

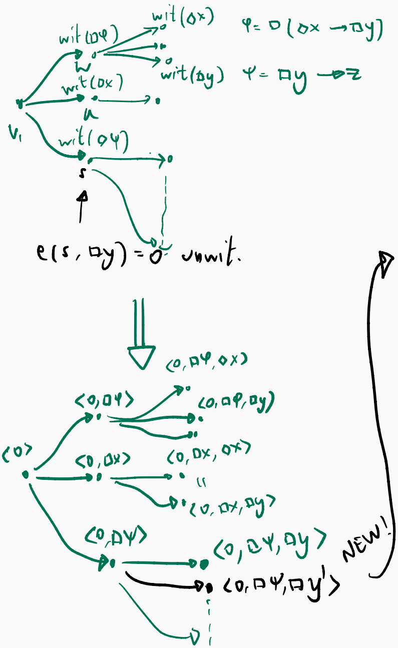
Unwitnessed formulas in $\mathfrak{B}(\mathbb{R}^{\mathbb{Q}})$-models?

3. Proposition

For each $\sigma \in W$ with $\underline{\sigma} \neq \sigma$ and for each $\chi \in \Upsilon_{|\sigma|-1}$ there is an element $\alpha_{\sigma,\chi} \in \mathfrak{B}(\mathbb{R}^{\mathbb{Q}})$ such that:

1. $e^{+}(\sigma, \chi) = e^{+}(\sigma_{-}, \chi) + \alpha_{\sigma,\chi}$;
2. $\alpha_{\sigma,\chi} = \bot$ if and only if $e^{+}(\underline{\sigma}, \chi) = \bot$;
3. For $\psi \in \Upsilon_{|\sigma|-1}$, if $e^{+}(\underline{\sigma}, \chi) \leq e^{+}(\underline{\sigma}, \psi)$, then $\alpha_{\sigma,\chi} \leq \alpha_{\sigma,\psi}$;
4. If $\sigma = \sigma_1^\frown \Box\varphi'$ then
    - 4.1 $\bot < \alpha_{\sigma,\varphi} < \top$ and,
    - 4.2 for any $\Box\chi \in \Upsilon_{|\sigma_1|-1}$ with $e^{+}(\sigma_1, \Box\chi) > \bot$, $\alpha_{\sigma,\chi} = \top$.

A picture is worth a thousand words



for each var. p:

$e(\langle 0, \Box\psi, \Box y' \rangle, p) :=$

$e(\langle 0, \Box\psi, \Box y \rangle, p) +$

$e(\langle 0, \Box\psi, \Box y \rangle, p) \leftarrow m\,(e(\langle 0, \Box\psi, \Box y \rangle, y))$

Syntactic translation of formulas

We will use the sequences Σ to generate a propositional language with variables $\mathcal{V}_\sigma$, $\heartsuit\varphi_\sigma$ and, for $\sigma \in \Sigma_i$ with some primed element, and $\chi \in \Upsilon_i$, new variables $\alpha_{\chi,\sigma}$.

Syntactic translation of formulas

We will use the sequences Σ to generate a propositional language with variables $\mathcal{V}_\sigma$, $\heartsuit\varphi_\sigma$ and, for $\sigma \in \Sigma_i$ with some primed element, and $\chi \in \Upsilon_i$, new variables $\alpha_{\chi,\sigma}$.

For each $\sigma \in \Sigma_i$ fix some set $uWit_\sigma \subseteq \Upsilon_i^\Box$.

Syntactic translation of formulas

We will use the sequences Σ to generate a propositional language with variables $\mathcal{V}_\sigma$, $\heartsuit\varphi_\sigma$ and, for $\sigma \in \Sigma_i$ with some primed element, and $\chi \in \Upsilon_i$, new variables $\alpha_{\chi,\sigma}$.

For each $\sigma \in \Sigma_i$ fix some set $uWit_\sigma \subseteq \Upsilon_i^\Box$.

Definition

- $2V(\varphi_\sigma) \coloneqq \varphi_\sigma \leftrightarrow \varphi_{\sigma_-} \odot \alpha_{\sigma,\varphi}$,
- $Imp(\varphi_\sigma, \psi_\sigma) \coloneqq \Delta(\varphi \to \psi)_\sigma \to (\alpha_{\varphi,\sigma} \to \alpha_{\psi,\sigma})$,
- $Neg(\varphi_\sigma) \coloneqq \neg\alpha_{\varphi,\sigma} \to \neg\varphi_\sigma$,
- $WV(\Upsilon) \coloneqq \bigwedge\{\neg\neg(\Box\varphi)_\sigma \to \alpha_{\varphi,\sigma\overline{\Box\chi}} \colon \alpha_{\varphi,\sigma\overline{\Box\chi}} \in \mathcal{V}, \Box\varphi \in \Upsilon_i\}$,
- $uWV(\Upsilon) \coloneqq \bigvee\{\alpha_{\chi,\sigma\overline{\Box\chi}} \colon \alpha_{\chi,\sigma\overline{\Box\chi}} \in \mathcal{V}, \Box\chi \in uWit_\sigma\}$,
- $W_\Diamond((\Diamond\psi)_\sigma) \coloneqq ((\Diamond\psi)_\sigma \leftrightarrow (\psi)_{\sigma\Diamond\psi}) \wedge (\bigvee_{\sigma\chi\in\Sigma}(\psi)_{\sigma\chi} \to (\Diamond\psi)_\sigma)$,
- $W_\Box((\Box\psi)_\sigma) \coloneqq ((\Box\psi)_\sigma \leftrightarrow (\psi)_{\sigma\Box\psi}) \wedge ((\Box\psi)_\sigma \to \bigwedge_{\sigma\chi\in\Sigma}(\psi)_{\sigma\chi})$,
- $uW((\Box\psi)_\sigma) \coloneqq \neg(\Box\psi)_\sigma$

Moving to propositional logic

Selecting only the sequences in Σ arising from the chosen $uWit_\sigma$ sets, and the previous definitions over the formulas of the corresponding level (for $|\sigma| = i$, formulas in Υ_i), we let

$$M(\Upsilon) := 2V(\Upsilon) \cup Imp(\Upsilon) \cup Neg(\Upsilon) \cup WV(\Upsilon) \cup W_\Diamond(\Upsilon) \cup W_\Box(\Upsilon) \cup uW(\Upsilon).$$

Moving to propositional logic

Selecting only the sequences in Σ arising from the chosen $uWit_\sigma$ sets, and the previous definitions over the formulas of the corresponding level (for $|\sigma| = i$, formulas in Υ_i), we let

$$M(\Upsilon) := 2V(\Upsilon) \cup Imp(\Upsilon) \cup Neg(\Upsilon) \cup WV(\Upsilon) \cup W_\diamond(\Upsilon) \cup W_\Box(\Upsilon) \cup uW(\Upsilon).$$

Theorem

Let $\Upsilon = \Gamma \cup \{\varphi\}$ be such that $\Gamma \not\Vdash_{K\Pi} \varphi$. Then, for each sequence $\sigma \in \Sigma_i$ there exists a set $uWit_\sigma \subseteq \Upsilon_i^\Box$ such that

$$\Gamma_{\langle 0 \rangle}, M(\Upsilon) \not\vdash_{\Pi_\Delta} \varphi_{\langle 0 \rangle} \vee uWV(\Upsilon)$$

Information in the propositional entailment

Proposition

Let Γ be a closed set of propositional formulas, and $h_1, h_2 \in Hom(Fm, [0,1]_\Pi)$ such that

1. For each formula $\varphi \in \Gamma$, there is some α_φ such that $h_2(\varphi) = h_1(\varphi) \cdot \alpha_\varphi$,
2. For each pair of formulas $\varphi, \psi \in \Gamma$ such that $h_1(\varphi) \leq h_1((\psi)$ it holds that $\alpha_\varphi \leq \alpha_\psi$,
3. $\alpha_\varphi = 0$ implies that $h_1(\varphi) = 0$.

Consider the family of homomorphisms h_k for $k \in \mathbb{N}$ where $h_k(x) = h(x) \cdot \alpha_x^k$ for each variable x in Γ.
Then, for each $\varphi \in \Gamma$, it holds that $h_k(\varphi) = h(\varphi) \cdot \alpha_\varphi^k$.

Information in the propositional entailment

Proposition

Let Γ be a closed set of propositional formulas, and $h_1, h_2 \in Hom(Fm, [0,1]_\Pi)$ such that

1. For each formula $\varphi \in \Gamma$, there is some α_φ such that $h_2(\varphi) = h_1(\varphi) \cdot \alpha_\varphi$,
2. For each pair of formulas $\varphi, \psi \in \Gamma$ such that $h_1(\varphi) \leq h_1((\psi)$ it holds that $\alpha_\varphi \leq \alpha_\psi$,
3. $\alpha_\varphi = 0$ implies that $h_1(\varphi) = 0$.

Consider the family of homomorphisms h_k for $k \in \mathbb{N}$ where $h_k(x) = h(x) \cdot \alpha_x^k$ for each variable x in Γ.
Then, for each $\varphi \in \Gamma$, it holds that $h_k(\varphi) = h(\varphi) \cdot \alpha_\varphi^k$.

(C1) $\alpha_{\varphi \odot \psi} = \alpha_\varphi \cdot \alpha_\psi$ and (C2) $\alpha_{\varphi \to \psi} = \alpha_\varphi \to_{[0,1]_\Pi} \alpha_\psi$.

Back to an standard Kripke model

Lemma

Let $\Upsilon = \Gamma \cup \{\varphi\} \subset Fm$, and assume that for each sequence $\sigma \in \Sigma_i$ there exists a set $uWit_\sigma \subseteq \Upsilon^{\Box}_{k+1}$ such that

$$\Gamma_{\langle 0 \rangle}, M(\Upsilon) \nvdash_{\Pi_\Delta} \varphi_{\langle 0 \rangle} \vee uWV(\Upsilon)$$

Then, $\Gamma \nVdash^{l}_{K\Pi} \varphi$.

Back to an standard Kripke model

Lemma

Let $\Upsilon = \Gamma \cup \{\varphi\} \subset Fm$, and assume that for each sequence $\sigma \in \Sigma_i$ there exists a set $uWit_\sigma \subseteq \Upsilon^{\Box}_{k+1}$ such that

$$\Gamma_{\langle 0 \rangle}, M(\Upsilon) \nvdash_{\Pi_\Delta} \varphi_{\langle 0 \rangle} \vee uWV(\Upsilon)$$

Then, $\Gamma \not\Vdash^{l}_{K\Pi} \varphi$.

$\vdash_{\Pi_\Delta}$ is decidable:

Theorem

$\Vdash^{l}_{K\Pi}$ is decidable.

Grazie mille!

References

Caicedo, X., Metcalfe, G., Rodríguez, R., and Rogger, J. (2017). Decidability of order-based modal logics. *Journal of Computer and System Sciences*, 88:53-74.

Cerami, M. and Esteva, F. (2021) On decidability of concept satisfiability in Description Logic with product semantics. *Fuzzy Sets and Systems*, (In press)

Hansoul, G. and Teheux, B. (2013). Extending Łukasiewicz logics with a modality: Algebraic approach to relational semantics. *Studia Logica*, 101(3):505–545.

M. C. Laskowski and S. Malekpour (2007) Provability in predicate product logic. *Archive for Mathematical Logic*, 46(5): 365–378.

Vidal, A. (2022) Undecidability and non-axiomatizability of modal many-valued logics. *The Journal of Symbolic Logic, (In press)*