Degrees of the finite model property

Nick Bezhanishvili
Institute for Logic, Language and Computation
University of Amsterdam
https://staff.fnwi.uva.nl/n.bezhanishvili

joint work with G. Bezhanishvili and T. Moraschini

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Problem (Fine, 1974). What is the degree of incompleteness in extensions of the basic modal logic K?

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A characterization of degrees of incompleteness in extensions of $\mathsf{K4}$, $\mathsf{S4}$ and IPC remains an outstanding open problem.

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As with the degree of incompleteness, all but one of such L' lack the fmp.

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Thus, in the lattice of all normal modal logics the dichotomy holds also for the degrees of fmp.

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Using the Blok-Esakia isomorphism this result generalizes to extensions of $\mathsf{S4.Grz}$ and with further work to extensions of $\mathsf{K4}$ and $\mathsf{S4}$.

Splittings and Jankov formulas

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An si-logic is join-splitting if it is a join in $\operatorname{Ext}\mathsf{IPC}$ of a set of splitting si-logics.

Splittings and Jankov formulas

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Jankov's Lemma. Let $\mathfrak{A}$ and $\mathfrak{B}$ be Heyting algebras with $\mathfrak{A}$ finite and SI. Then $\mathfrak{B} \not\models \mathcal{J}(\mathfrak{A})$ iff $\mathfrak{A}$ is a subalgebra of a homomorphic image of $\mathfrak{B}$.

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Jankov's Theorem. An si-logic L is a splitting logic iff there is a finite SI Heyting algebra $\mathfrak{A}$ such that $\mathsf{L} = \mathsf{IPC} + \mathcal{J}(\mathfrak{A})$.

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Corollary. L is a join-splitting logic iff L is axiomatizable by Jankov formulas.

The fmp span

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The degree of fmp of L is the cardinality of $\mathrm{fmp}(\mathsf{L})$.

The fmp span

Definition. For an si-logic L, define

1. $\mathsf{L}^+ = \mathsf{Log}(\mathsf{Fin}(\mathsf{L}))$;
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Theorem. For an si-logic L we have:

1. $\mathrm{fmp}(\mathsf{L}) = [\mathsf{L}^-, \mathsf{L}^+]$.
2. L^+ is the only member of $\mathrm{fmp}(\mathsf{L})$ that has the fmp.
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Corollary. If an si-logic L has the fmp and is axiomatizable by Jankov formulas, it has the degree of fmp 1.

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$(\mathfrak{A} + \mathfrak{B})_*$ is isomorphic to $\mathfrak{A}_* \oplus \mathfrak{B}_*$.

The Kuznetsov-Gerčiu logic

The Kuznetsov-Gerčiu logic KG is the si-logic of all Heyting algebras of the form $\mathfrak{A}_1 + \cdots + \mathfrak{A}_n$ where $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$ are one-generated.

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This implies that if $L \not\subseteq \mathsf{KG}$, then $\mathsf{Fin(L)} \neq \mathsf{Fin(KG)}$.

Thus it is enough to study the degree of fmp in extensions of KG.

The Rieger-Nishimura lattice and ladder

The free one generated Heyting algebra is the Rieger-Nishimura lattice.

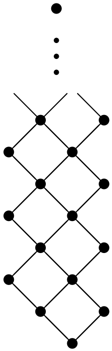

Figure: The Rieger-Nishimura lattice.

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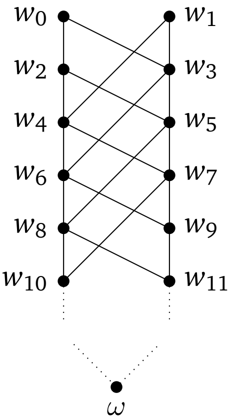

Figure: The Rieger-Nishimura ladder $\mathfrak{L}$.

Proof sketch of the Antidochotomy theorem for $\kappa < \aleph_0$

Consider the space

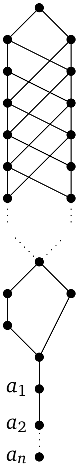

Figure: The poset underlying $\mathfrak{G}_n$.

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\mathsf{L}_0 &= \mathrm{Log}(\mathcal{R}_n) \\
\mathsf{L}_1 &= \mathrm{Log}(\mathcal{R}_n \cup \{\mathfrak{G}_1\}) \\
\mathsf{L}_2 &= \mathrm{Log}(\mathcal{R}_n \cup \{\mathfrak{G}_2\}) \\
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\end{aligned}$$

Main theorem 1.

$$\mathrm{fmp}(\mathsf{L}_0) = \{\mathsf{L}_0, \ldots, \mathsf{L}_n\}.$$

The case $\kappa = \aleph_0$

To construct an extension L of KG with $\deg(\mathsf{L}) = \aleph_0$ consider

$$\mathcal{R} = \bigcup_{n < \aleph_0} \mathcal{R}_n.$$

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$$\mathcal{R} = \bigcup_{n<\aleph_0} \mathcal{R}_n.$$

For every integer n, consider the extensions of KG

$$\mathsf{L}_n^* = \mathrm{Log}(\mathcal{R} \cup \{\mathfrak{G}_n\}) \text{ and } \mathsf{L}_\infty^* = \mathrm{Log}(\mathcal{R} \cup \{\mathfrak{G}_n : n < \aleph_0\}).$$

Main Theorem 2.

$$\mathrm{fmp}(\mathsf{L}_0^*) = \{\mathsf{L}_\infty^*\} \cup \{\mathsf{L}_n^* : n < \aleph_0\}.$$

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Main Theorem 2.

$$\mathrm{fmp}(\mathsf{L}_0^*) = \{\mathsf{L}_\infty^*\} \cup \{\mathsf{L}_n^* : n < \aleph_0\}.$$

Consequently, the logic L_0^* is an extension of KG with the degree of fmp $\aleph_0$.

Future work

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- How to characterize degress of fmp for other logical systems and varieties of algebras, e.g., for fixed-point logics (PDL, modal μ-calculus) or for many valued logics, substructural logics, etc.
- The question about the degrees of incompleteness for IPC, K4, S4 remains open.

Thank you!