# On Equational Completeness Theorems

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$$\Gamma \vdash_{\mathsf{CPC}} \varphi \iff$$
 for every  $\mathbf{A} \in \mathsf{BA}$  and every hom  $f : \mathbf{Fm} \to \mathbf{A}$ ,  
if  $f(\gamma) = 1^{\mathbf{A}}$  for all  $\gamma \in \Gamma$ , then  $f(\varphi) = 1^{\mathbf{A}}$ .

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• Given a class of similar algebras K and a set of equations  $\Theta \cup \{\varphi \approx \psi\}$ , we write  $\Theta \vDash_{\mathsf{K}} \varphi \approx \psi$  when

$$\begin{split} \Gamma \vdash_{\mathsf{CPC}} \varphi & \Longleftrightarrow \text{ for every } \mathbf{A} \in \mathsf{BA} \text{ and every hom } f \colon \mathbf{Fm} \to \mathbf{A}, \\ & \text{ if } f(\gamma) = 1^{\mathbf{A}} \text{ for all } \gamma \in \mathbf{\Gamma}, \text{ then } f(\varphi) = 1^{\mathbf{A}}. \end{split}$$

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equational consequence relative to K.

In this terminology, the equational completeness theorem of CPC can be written, more concisely, as

$$\Gamma \vdash_{\mathsf{CPC}} \varphi \Longleftrightarrow \{ \gamma \approx 1 : \gamma \in \Gamma \} \vDash_{\mathsf{BA}} \varphi \approx 1.$$

$$\begin{split} \boldsymbol{\tau}(\varphi) &\coloneqq \{ \boldsymbol{\epsilon}(\varphi) \approx \delta(\varphi) : \boldsymbol{\epsilon} \approx \delta \in \boldsymbol{\tau} \} \\ \boldsymbol{\tau}[\Gamma] &\coloneqq \bigcup_{\gamma \in \Gamma} \boldsymbol{\tau}(\gamma). \end{split}$$

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• Taking 
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 $\psi \longmapsto \tau(\psi)$ , i.e.,  $\{\psi \approx 1\}$ .

A (propositional) logic ⊢ is a consequence relation on the set of formulas *Fm* of an arbitrary algebraic language

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**Example. CPC** admits an equational completeness theorem w.r.t. Boolean algebras.

Pandora's box

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Notably, the situation does not improve if we restrict to the case where τ(x) = {x ≈ 1}. Actually, there is no escape from nonstandard equational completeness theorems.

Sometimes **nonstandard** equational completeness theorems are the sole possible ones.

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$\wedge$	0-	0+	1	$\vee$	0-	0+	1
0-	0+	0+	0+	0-	0+	0+	1
0+		0-	0+	0+		0-	1
1			1	1			1

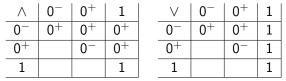
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► Then CPC<sub>∧∨</sub> admits an equational completeness theorem w.r.t. K := {A} witnessed by the set of equations

$$\boldsymbol{\tau}(\boldsymbol{x}) = \{ \boldsymbol{x} \approx \boldsymbol{x} \wedge \boldsymbol{x} \}.$$

# A sufficient condition

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► More concretely, in CPC we have

$$\begin{split} \varphi \equiv_{\mathsf{CPC}} \psi & \Longleftrightarrow \oslash \vdash_{\mathsf{CPC}} (\varphi \to \psi) \land (\psi \to \varphi) \\ & \Longleftrightarrow \varphi \dashv \vdash_{\mathsf{CPC}} \psi. \end{split}$$

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**Example.** The  $\langle \diamondsuit, 0, 1 \rangle$ -fragment of any modal logic is graph-based, while the  $\langle \diamondsuit, \Box, 0, 1 \rangle$ -one is not.

Let  $\vdash$  a logic that is not graph-based.

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then  $\vdash$  admits an equational completeness theorem.

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# The case of logics with theorems

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- 2.  $\vdash$  is not graph-based and there are distinct  $\varphi$  and  $\psi$  s.t.

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# A logic $\vdash$ is **protoalgebraic** if there is a set of formulas $\Delta(x, y)$ s.t.

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#### Observation

The local consequence of the modal system K (resp. K4, S4) does not admit an equational completeness theorem w.r.t. the variety of modal algebras (resp. of K4-algebras, resp. of interior algebras).

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 $\Gamma \vdash_{\mathbf{K}}^{\mathscr{L}} \varphi \iff \text{for every Kripke frame } \langle W, R \rangle,$ valuation  $v \colon \text{Var} \to W$ , and world  $w \in W$ , if  $w, v \Vdash \Gamma$ , then  $w, v \Vdash \varphi$ .

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# **Computational aspects**

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# Summary.

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Thank you very much for your attention!