On Equational Completeness Theorems

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- In this terminology, the equational completeness theorem of **CPC** can be written, more concisely, as

$$\Gamma \vdash_{\mathbf{CPC}} \varphi \iff \{\gamma \approx 1 : \gamma \in \Gamma\} \vDash_{\mathsf{BA}} \varphi \approx 1.$$

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Example. **CPC** admits an equational completeness theorem w.r.t. Boolean algebras.

Pandora's box

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- Notably, the situation does not improve if we restrict to the case where $\boldsymbol{\tau}(x) = \{x \approx 1\}$. Actually, there is no escape from **nonstandard** equational completeness theorems.

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with commutative operations defined by the tables

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- Then $\mathbf{CPC}_{\wedge\vee}$ admits an equational completeness theorem w.r.t. $\mathsf{K} := \{\boldsymbol{A}\}$ witnessed by the set of equations

$$\boldsymbol{\tau}(x) = \{x \approx x \wedge x\}.$$

A sufficient condition

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- More concretely, in **CPC** we have

$$\begin{aligned} \varphi \equiv_{\mathbf{CPC}} \psi &\iff \emptyset \vdash_{\mathbf{CPC}} (\varphi \to \psi) \land (\psi \to \varphi) \\ &\iff \varphi \dashv\vdash_{\mathbf{CPC}} \psi. \end{aligned}$$

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Example. The $\langle \Diamond, 0, 1 \rangle$-fragment of any modal logic is graph-based,

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Example. The $\langle \Diamond, 0, 1 \rangle$-fragment of any modal logic is graph-based, while the $\langle \Diamond, \Box, 0, 1 \rangle$-one is not.

Theorem

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The case of logics with theorems

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2. $\vdash$ is not graph-based and there are distinct φ and ψ s.t.

$$\mathrm{Var}(\varphi) \cup \mathrm{Var}(\psi) = \{x\} \text{ and } \varphi \equiv_{\vdash} \psi.$$

Definition

A logic $\vdash$ is **protoalgebraic** if there is a set of formulas $\Delta(x, y)$ s.t.

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Observation

The local consequence of the modal system **K** (resp. **K4**, **S4**) does **not** admit an equational completeness theorem w.r.t. the variety of modal algebras (resp. of K4-algebras, resp. of interior algebras).

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$$\Gamma \vdash^{\mathscr{L}}_{\mathbf{K}} \varphi \iff \text{for every Kripke frame } \langle W, R\rangle, \text{ valuation } v\colon \mathrm{Var} \to W, \text{ and world } w \in W, \text{ if } w, v \Vdash \Gamma, \text{ then } w, v \Vdash \varphi.$$

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Computational aspects

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Thank you very much for your attention!