Towards a non-integral variant of Łukasiewicz logic

Marta Bílková[1] Petr Cintula[1] Carles Noguera[2]

[1]Institute of Computer Science
Czech Academy of Sciences

[2]Department of Information Engineering and Mathematics
University of Siena

Let us fix the languages

commutative residuated lattice: $\mathbf{A} = \langle A, \wedge, \vee, \&, \rightarrow, \mathtt{t} \rangle$

Let us fix the languages

commutative residuated lattice: $\mathbf{A} = \langle A, \wedge, \vee, \&, \to, \mathtt{t} \rangle$

$\mathrm{FL_e}$-algebra: $\mathbf{B} = \langle B, \wedge, \vee, \&, \to, \mathtt{f}, \mathtt{t} \rangle$

Let us fix the languages

commutative residuated lattice: $\mathbf{A} = \langle A, \wedge, \vee, \&, \rightarrow, \mathtt{t} \rangle$

$\mathrm{FL_e}$-algebra: $\mathbf{B} = \langle B, \wedge, \vee, \&, \rightarrow, \mathtt{f}, \mathtt{t} \rangle$

$\mathrm{FL_e^+}$-algebra: $\mathbf{C} = \langle B, \wedge, \vee, \&, \rightarrow, \mathtt{f}, \mathtt{t}, \bot, \top \rangle$

Let us fix the languages

commutative residuated lattice: $\mathbf{A} = \langle A, \wedge, \vee, \&, \to, \mathtt{t} \rangle$

$\mathrm{FL_e}$-algebra: $\mathbf{B} = \langle B, \wedge, \vee, \&, \to, \mathtt{f}, \mathtt{t} \rangle$

$\mathrm{FL_e^+}$-algebra: $\mathbf{C} = \langle B, \wedge, \vee, \&, \to, \mathtt{f}, \mathtt{t}, \bot, \top \rangle$

By $\mathbf{B}^-$ we denote $\mathtt{f}$-free reduct of an $\mathrm{FL_e}$-algebra $\mathbf{B}$

By $\mathbf{A}_a$ we denote the $\mathrm{FL_e}$-algebra st. $\mathbf{A}_a^- = \mathbf{A}$ and $a = \mathtt{f}^{\mathbf{A}_a}$

Łukasiewicz unbound

Consider algebra $\mathbf{Lu} = \langle \mathbb{R}, \wedge, \vee, \&, \to, 0, 1 \rangle$, where

$$x \mathbin{\&} y = x + y - 1 \qquad\qquad x \to y = 1 - x + y$$

$\mathbf{Lu}$ is involutive $\mathrm{FL_e}$-chain with the negation defined

$$\neg x = 1 - x$$

Łukasiewicz unbound vs. Lukasiewicz

Consider algebra $\text{Ł} = \langle [0, 1], \wedge, \vee, \&, \rightarrow, 0, 1 \rangle$, where

$$x \,\&\, y = \max(0,\ x + y - 1) \qquad\qquad x \rightarrow y = \min(1,\ 1 - x + y)$$

Ł is involutive integral $\mathrm{FL}_{\mathrm{e}}^{+}$-chain with the negation and bounds defined

$$\neg x = 1 - x \qquad\qquad \bot = 0 \qquad\qquad \top = 1$$

Łukasiewicz unbound

Consider algebra $\mathbf{Lu} = \langle \mathbb{R}, \wedge, \vee, \&, \rightarrow, 0, 1 \rangle$, where

$$x \mathbin{\&} y = x + y - 1 \qquad\qquad x \rightarrow y = 1 - x + y$$

$\mathbf{Lu}$ is involutive $\mathrm{FL_e}$-chain with the negation defined

$$\neg x = 1 - x$$

Łukasiewicz unbound vs. Abel

Consider algebra $\mathbf{R} = \langle \mathbb{R}, \wedge, \vee, \&, \rightarrow, 0, 0 \rangle$, where

$$x \,\&\, y = x + y \qquad\qquad x \rightarrow y = -x + y$$

$\mathbf{R}$ is involutive $\mathrm{FL_e}$-chain with the negation defined

$$\neg x = -x$$

Problem with f

$\mathbf{R}^-$ and $\mathbf{Lu}^-$ are isomorphic

Problem with f

$\mathbf{R}^{-}$ and $\mathbf{Lu}^{-}$ are isomorphic

BUT

$\mathbf{R}$ and $\mathbf{Lu}$ are NOT isomorphic

Note that $\mathbf{Lu}$ is isomorphic to $\mathbf{R}^{-}_{f}$ for any $f < 1$:

$$h(x) = 1 - \frac{x}{f}$$

WHY?

A sidestep

PC, B. Grimau, C. Noguera, N. Smith: “These Degrees go to Eleven: Fuzzy Logics and Graded Predicates” Under review or to appear (depending on who you ask)

A sidestep

PC, B. Grimau, C. Noguera, N. Smith: "These Degrees go to Eleven: Fuzzy Logics and Graded Predicates" Under review or to appear (depending on who you ask)

We are defending there agains one of the usual attacks to "fuzzy" treatment of vagueness:

- Suppose we have associated comparative forms of graded predicates: 'taller', 'heavier', and 'more acute'.
- **Natural proposal:** a is Fer than b iff a's degree of Fness is greater than b's degree of Fness
- Consider the first member in a sorites series for 'tall'— Adam, who is definitely tall.
- Consider someone, Bob, who is even taller than Adam.
- But Adam's degree of tallness is already 1 on the standard fuzzy approach, and so there is nowhere to go to make Bob's degree of tallness greater than Adam's: there are no degrees above 1!

Inspiration from *This Is Spiñal Tap* (1984)

- Nigel Tufnel: …the numbers all go to eleven. Look, right across the board, eleven, eleven, eleven and
- Marty DiBergi: Oh, I see. And most amps go up to ten?

NT: Exactly.

MD: Does that mean it's louder? Is it any louder?

NT: Well, it's one louder, isn't it? It's not ten. You see, most, most blokes, you know, be playing at ten, you're on ten here, all the way up, all the way up, all the way up, you're on ten on your guitar, where can you go from there? Where?

MD: I don't know.

NT: Nowhere. Exactly. What we do is, if we need that extra push over the cliff, you know what we do?

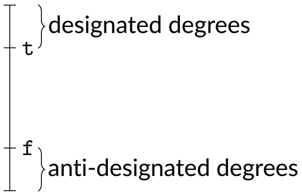
MD: Put it up to eleven.

NT: Eleven. Exactly. One louder.

MD: Why don't you just make ten louder and make ten be the top number and make that a little louder?

NT: [long pause] These go to eleven.

Why not Abelian logic?



Why not Abelian logic?

- **Highest-standard**:

  Your glass is more full than my glass. $\vdash$ *My glass is not full.*
  Your glass is more full than my glass. $\nvdash$ *Your glass is full.*

- **Lowest-standard**:

  The iron rod is more bent than the steel rod. $\nvdash$ *The steel rod is not bent.*
  The iron rod is more bent than the steel rod. $\vdash$ *The iron rod is bent.*

Vague	**Highest-standard**	**Lowest-standard**	**Intermediate-st.**	**Bigraded**
True False	• True False	True • False	True False	• True • False

Why not Abelian logic?

- **Intermediate-standard**:

  Bleach is more basic than ammonia. $\nvdash$ *Ammonia is not basic.*
  Vinegar is more basic than hydrochloric acid. $\nvdash$ *Vinegar is basic.*
  Alex is not underage. $\vdash$ *Alex is overage.*

- **Bigraded**:

  Number 2 is not odd. $\vdash$ *Number 2 is even.*
  They cannot appear in degree constructions.

Vague	**Highest-standard**	**Lowest-standard**	**Intermediate-st.**	**Bigraded**
True False	• True False	True • False	True False	• True • False

Back to math . . .

Axiomatizing Łukasiewicz unbound logic

First question: which LU logic?

Axiomatizing Łukasiewicz unbound logic

First question: which LU logic?

Observation:

$$\models_{\langle \mathbf{Lu},\ \{x \mid x \geq 1\}\rangle} \text{ is not finitary}$$

Axiomatizing Łukasiewicz unbound logic

First question: which LU logic?

Observation:

$\vDash_{\langle \mathbf{Lu},\ \{x \mid x \geq 1\}\rangle}$ is not finitary

So we start with:

$$\mathcal{FC}\left(\vDash_{\langle \mathbf{Lu},\ \{x \mid x \geq 1\}\rangle}\right)$$

Recall: axiomatization of Abelian logic

First question: which Abelian logic?

$$\mathcal{FC}(\models_{\langle \mathbf{R},\ \{x \mid x \geq 0\}\rangle})$$

Recall: axiomatization of Abelian logic

First question: which Abelian logic?

$$\mathcal{FC}(\models_{\langle \mathbf{R},\ \{x \mid x \geq 0\}\rangle})$$

$$\mathrm{Ab} \quad = \quad \mathrm{FL_e} \quad + \quad \begin{array}{c} ((\varphi \to \psi) \to \psi) \to \varphi \\ \mathtt{f} \to \mathtt{t} \\ \mathtt{t} \to \mathtt{f} \end{array}$$

Recall: axiomatization of Abelian logic

First question: which Abelian logic?

$$\mathcal{FC}(\models_{\langle \mathbf{R},\ \{x \mid x \geq 0\}\rangle})$$

$$\mathrm{Ab} \quad = \quad \mathrm{FL_e} \quad + \quad \begin{array}{c} ((\varphi \to \psi) \to \psi) \to \varphi \\ \mathtt{f} \to \mathtt{t} \\ \mathtt{t} \to \mathtt{f} \end{array}$$

The completeness theorem: For each finite Γ we have:

$$\Gamma \vdash_{\mathrm{Ab}} \varphi \qquad \text{iff} \qquad \Gamma \models_{\langle \mathbf{R},\ \{x \mid x \geq 0\}\rangle} \varphi$$

Axiomatizing Łukasiewicz unbound logic

$$\mathrm{Ab} \quad = \quad \mathrm{FL_e} \quad + \quad \begin{array}{c} ((\varphi \to \psi) \to \psi) \to \varphi \\ \mathtt{f} \to \mathtt{t} \\ \mathtt{t} \to \mathtt{f} \end{array}$$

$$\mathrm{Lu} \quad = \quad \mathrm{FL_e} \quad + \quad \begin{array}{c} ((\varphi \to \psi) \to \psi) \to \varphi \\ \varphi \vee \mathtt{f} \blacktriangleright \varphi \end{array}$$

The completeness theorem: For each finite Γ we have:

$$\Gamma \vdash_{\mathrm{LU}} \varphi \qquad \text{iff} \qquad \Gamma \models_{\langle \mathbf{Lu},\, \{x \mid x \geq 1\} \rangle} \varphi$$

Some facts about the logic $\mathrm{pAb} = \mathrm{FL_e} + ((\varphi \to \psi) \to \psi) \to \varphi$

Note: $\mathrm{Ab} = \mathrm{pAb}$ + $\mathtt{t} \leftrightarrow \mathtt{f}$ and $\mathrm{Lu} = \mathrm{pAb}$ + $\varphi \vee \mathtt{f} \blacktriangleright \varphi$

pAb proves

$$\varphi \,\&\, \psi \leftrightarrow \neg(\varphi \to \neg\psi) \qquad\qquad \varphi \vee \psi \leftrightarrow ((\varphi \to \psi) \wedge \mathtt{t} \to \psi)$$

Some facts about the logic $\mathrm{pAb} = \mathrm{FL_e} + ((\varphi \to \psi) \to \psi) \to \varphi$

Note: $\mathrm{Ab} = \mathrm{pAb} + \mathtt{t} \leftrightarrow \mathtt{f}$ and $\mathrm{Lu} = \mathrm{pAb} + \varphi \vee \mathtt{f} \blacktriangleright \varphi$

pAb proves

$$\varphi \,\&\, \psi \leftrightarrow \neg(\varphi \to \neg\psi) \qquad \varphi \vee \psi \leftrightarrow ((\varphi \to \psi) \wedge \mathtt{t} \to \psi)$$

For Daniele:

$$\mathrm{pAb} = \mathrm{FL_e} \quad + \quad [(\varphi \to \psi) \wedge \mathtt{t} \to \psi] \to [(\psi \to \varphi) \wedge \mathtt{t} \to \varphi]$$

Some facts about the logic $\mathrm{pAb} = \mathrm{FL_e} + ((\varphi \to \psi) \to \psi) \to \varphi$

Note: $\mathrm{Ab} = \mathrm{pAb}$ $+$ $\mathtt{t} \leftrightarrow \mathtt{f}$ and $\mathrm{Lu} = \mathrm{pAb}$ $+$ $\varphi \vee \mathtt{f} \blacktriangleright \varphi$

pAb proves

$$\varphi \mathbin{\&} \psi \leftrightarrow \neg(\varphi \to \neg\psi) \qquad\qquad \varphi \vee \psi \leftrightarrow ((\varphi \to \psi) \wedge \mathtt{t} \to \psi)$$

$\vee$ is a disjunction in pAb (i.e., enjoys proof by case property; cf. not in $\mathrm{FL_e}$!)

pAb is semilinear logic

Some facts about the logic $\mathrm{pAb} = \mathrm{FL_e} + ((\varphi \to \psi) \to \psi) \to \varphi$

Note: $\mathrm{Ab} = \mathrm{pAb} + \mathtt{t} \leftrightarrow \mathtt{f}$ and $\mathrm{Lu} = \mathrm{pAb} + \varphi \vee \mathtt{f} \blacktriangleright \varphi$

pAb proves

$$\varphi \mathbin{\&} \psi \leftrightarrow \neg(\varphi \to \neg\psi) \qquad \varphi \vee \psi \leftrightarrow ((\varphi \to \psi) \wedge \mathtt{t} \to \psi)$$

$\vee$ is a disjunction in pAb (i.e., enjoys proof by case property; cf. not in $\mathrm{FL_e}$!)

pAb is semilinear logic

$\vee$ is a disjunction in LU; due new rule is its own $\vee$-form

LU is semilinear logic and

$$\vdash_{\mathrm{LU}} \mathtt{f} \to \mathtt{t} \qquad \nvdash_{\mathrm{LU}} \mathtt{t} \to \mathtt{f}$$

How do we prove our “main” result?

We show: $\mathbf{Alg}^*(\mathrm{pAb}) = \{\mathbf{A}_a \mid \mathbf{A}$ is Abelian ℓ-group and $a \in A\}$

How do we prove our “main” result?

We show: $\mathbf{Alg}^*(\mathrm{pAb}) = \{\mathbf{A}_a \mid \mathbf{A}$ is Abelian ℓ-group and $a \in A\}$

We recall: For each Abelian ℓ-group $\mathbf{A}$ we have $\mathbf{A} \in \mathbf{ISPP_u}(\mathbf{R}^-)$

How do we prove our “main” result?

We show: $\mathbf{Alg}^*(\mathrm{pAb}) = \{\mathbf{A}_a \mid \mathbf{A}$ is Abelian ℓ-group and $a \in A\}$

We recall: For each Abelian ℓ-group $\mathbf{A}$ we have $\mathbf{A} \in \mathbf{ISPP_u}(\mathbf{R}^-)$

Therefore we can show:

$$\mathbf{Alg}^*(\mathrm{pAb}) = \mathbf{ISPP_u}(\{\mathbf{R}_a^- \mid a \in R\})$$

How do we prove our “main” result?

We show: $\mathbf{Alg}^*(\mathrm{pAb}) = \{\mathbf{A}_a \mid \mathbf{A}$ is Abelian ℓ-group and $a \in A\}$

We recall: For each Abelian ℓ-group $\mathbf{A}$ we have $\mathbf{A} \in \mathbf{ISPP_u}(\mathbf{R}^-)$

Therefore we can show:

$$\mathbf{Alg}^*(\mathrm{pAb}) = \mathbf{ISPP_u}(\{\mathbf{R}^-_a \mid a \in R\}) = \mathbf{ISPP_u}(\mathbf{Lu}, \mathbf{R}, \mathbf{R}^-_1)$$

How do we prove our “main” result?

We show: $\mathbf{Alg}^*(\mathrm{pAb}) = \{\mathbf{A}_a \mid \mathbf{A}$ is Abelian ℓ-group and $a \in A\}$

We recall: For each Abelian ℓ-group $\mathbf{A}$ we have $\mathbf{A} \in \mathbf{ISPP_u}(\mathbf{R}^-)$

Therefore we can show:

$$\mathbf{Alg}^*(\mathrm{pAb}) = \mathbf{ISPP_u}(\{\mathbf{R}^-_a \mid a \in R\}) = \mathbf{ISPP_u}(\mathbf{Lu}, \mathbf{R}, \mathbf{R}^-_1)$$

And thus it is easy to entail that

$$\mathbf{Alg}^*(\mathrm{Lu}) = \mathbf{ISPP_u}(\mathbf{Lu})$$

How do we prove our "main" result?

We show: $\mathbf{Alg}^*(\mathrm{pAb}) = \{\mathbf{A}_a \mid \mathbf{A}$ is Abelian ℓ-group and $a \in A\}$

We recall: For each Abelian ℓ-group $\mathbf{A}$ we have $\mathbf{A} \in \mathbf{ISPP_u}(\mathbf{R}^-)$

Therefore we can show:

$$\mathbf{Alg}^*(\mathrm{pAb}) = \mathbf{ISPP_u}(\{\mathbf{R}_a^- \mid a \in R\}) = \mathbf{ISPP_u}(\mathbf{Lu}, \mathbf{R}, \mathbf{R}_1^-)$$

And thus it is easy to entail that

$$\mathbf{Alg}^*(\mathrm{Lu}) = \mathbf{ISPP_u}(\mathbf{Lu})$$

the rest is in the book ...

How do we prove our “main” result?

We show: $\mathbf{Alg}^*(\mathrm{pAb}) = \{\mathbf{A}_a \mid \mathbf{A}$ is Abelian ℓ-group and $a \in A\}$

We recall: For each Abelian ℓ-group $\mathbf{A}$ we have $\mathbf{A} \in \mathbf{ISPP_u}(\mathbf{R}^-)$

Therefore we can show:

$$\mathbf{Alg}^*(\mathrm{pAb}) = \mathbf{ISPP_u}(\{\mathbf{R}^-_a \mid a \in R\}) = \mathbf{ISPP_u}(\mathbf{Lu}, \mathbf{R}, \mathbf{R}^-_1)$$

And thus it is easy to entail that

$$\mathbf{Alg}^*(\mathrm{Lu}) = \mathbf{ISPP_u}(\mathbf{Lu})$$

the rest is in the book ...

including infinitary axiomatization by a variant of Hay rule ...

Bounding Łukasiewicz unbound again (but a bit more loosely)

$\mathbf{C} = \langle \overline{\mathbb{R}}, \wedge, \vee, \&, \rightarrow, 0, 1, -\infty, +\infty \rangle$

$x \mathbin{\&}^{\mathbf{C}} y$	$y = -\infty$	$y \in \mathbb{R}$	$y = +\infty$
$x = -\infty$	$-\infty$	$-\infty$	$-\infty$
$x \in \mathbb{R}$	$-\infty$	$x + y - 1$	$+\infty$
$x = +\infty$	$-\infty$	$+\infty$	$+\infty$

$x \rightarrow^{\mathbf{C}} y$	$y = -\infty$	$y \in \mathbb{R}$	$y = +\infty$
$x = -\infty$	$+\infty$	$+\infty$	$+\infty$
$x \in \mathbb{R}$	$-\infty$	$1 - x + y$	$+\infty$
$x = +\infty$	$-\infty$	$-\infty$	$+\infty$

$\mathbf{C}$ is a IUL-*chain* with the negation

$$\neg^{\mathbf{C}} x = x \rightarrow^{\mathbf{C}} \mathtt{f}^{\mathbf{C}} = \begin{cases} 1 - x & \text{for } x \in \mathbb{R} \\ -\infty & \text{for } x = +\infty \\ +\infty & \text{for } x = -\infty \end{cases}$$

Squeezing $\mathbf{C}$ into the real unit interval

Using suitable isomorphism of $\overline{\mathbb{R}}$ and $[0, 1]$ we can transform $\mathbf{C}$ into the UL-algebra $\mathbf{CR}$ based on the cross ratio uninorm and its residuum:

$$a \circ_{CR} b = \begin{cases} \dfrac{ab}{ab + (1-a)(1-b)}, & \text{if } \{a, b\} \neq \{0, 1\}, \\ 0, & \text{otherwise,} \end{cases}$$

Squeezing $\mathbf{C}$ into the real unit interval

Using suitable isomorphism of $\overline{\mathbb{R}}$ and $[0,1]$ we can transform $\mathbf{C}$ into the UL-algebra $\mathbf{CR}$ based on the cross ratio uninorm and its residuum:

$$a \circ_{CR} b = \begin{cases} \dfrac{ab}{ab + (1-a)(1-b)}, & \text{if } \{a,b\} \neq \{0,1\}, \\ 0, & \text{otherwise,} \end{cases}$$

Gabbay and Metcalfe in *Fuzzy logics based on [0,1)-continuous uninorms (2007)* "almost" axiomatize this logic in a rather complex way …

Squeezing $\mathbf{C}$ into the real unit interval

Using suitable isomorphism of $\overline{\mathbb{R}}$ and $[0,1]$ we can transform $\mathbf{C}$ into the UL-algebra $\mathbf{CR}$ based on the cross ratio uninorm and its residuum:

$$a \circ_{CR} b = \begin{cases} \dfrac{ab}{ab + (1-a)(1-b)}, & \text{if } \{a,b\} \neq \{0,1\}, \\ 0, & \text{otherwise,} \end{cases}$$

Gabbay and Metcalfe in *Fuzzy logics based on [0,1)-continuous uninorms (2007)* "almost" axiomatize this logic in a rather complex way ...

Our conjecture: it is UL plus "restricted Abel axiom":

$$(\psi \to \bot) \vee (\top \to \psi) \vee [((\varphi \to \psi) \to \psi) \to \varphi]$$