Quantificational issues in Prawitzian validity

Antonio Piccolomini d'Aragona
antonio.piccolomini-d-aragona@univ-amu.fr

Institute of Philosophy, Czech Academy of Sciences
Aix-Marseille University (Centre Gilles Gaston Granger)
University of Siena (DISPOC)

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Deductions in Gentzen's natural deduction may contain **peaks**, which can be removed through **reductions**. Deductions without peaks are called **normal**.

$$\dfrac{\dfrac{\begin{matrix}[A] \\ \mathscr{D}_1 \\ B\end{matrix}}{A \to B} \qquad \begin{matrix}\mathscr{D}_2 \\ A\end{matrix}}{B} \quad \to \ \texttt{Red} \qquad \begin{matrix}\mathscr{D}_2 \\ [A] \\ \mathscr{D}_1 \\ B\end{matrix}$$

Prawitz's normalisation for Σ

For every $\mathscr{D} \in \Sigma$ for A from Γ, there is normal $\mathscr{D}^*$ for A from $\Gamma^* \subseteq \Gamma$.

Generalising normalisation

Gentzen's semantic claim: introductions determine the meaning of the logical constants, eliminations are unique functions of the introductions. In certain important systems (like intuitionistic logic), we can prove the following.

Fundamental corollary

If $\mathscr{D}$ is closed and normal, then it ends by an introduction.

Dummett's fundamental assumption: if something is provable at all, there is a **proof** of it ending by an introduction. We moved from deductions in systems to a semantically unrestricted notion of proof.

Deductions with formal rules ⇒ **argument structures** with **arbitrary inferences**.

Reductions for eliminating peaks ⇒ **Justifications** for showing that generalised eliminations **preserve validity**.

Atomic bases

A semantics should say how non-logical meanings are fixed. This is done in Prawitz's semantics through atomic bases.

An atomic base $\mathfrak{B}$ is a pair $\langle \mathscr{L}, \mathscr{S} \rangle$, where $\mathscr{L}$ is a background (logical) language and $\mathscr{S}$ is an atomic system, i.e. a set of **production rules**

$$\frac{A_1, \ldots, A_n}{B}$$

where $A_1, \ldots, A_n, B \in \mathtt{ATOM}_{\mathscr{L}}$ and $A_i \neq \bot$ $(i \leq n)$.

Additional restrictions may be put on bases. These may influence completeness. However, we shall abstract from these details, since they are not needed for the incompleteness proof we shall refer to.

Validity of arguments over a base

Validity of arguments over a base

$\langle \mathscr{D}, \mathscr{J} \rangle$ is valid over $\mathfrak{B}$ iff:

- $\mathscr{D}$ is closed $\Rightarrow$ it reduces through $\mathscr{J}$ to a closed canonical $\mathscr{D}^*$ whose immediate sub-arguments, paired with $\mathscr{J}$, are valid over $\mathfrak{B}$;
- $\mathscr{D}$ is open $\Rightarrow$ after replacing the unbound assumptions of $\mathscr{D}$ with closed arguments structures which, paired with some extension $\mathscr{J}^+$ of $\mathscr{J}$, are valid over $\mathfrak{B}$, we obtain a closed argument structure which, paired with $\mathscr{J}^+$, is valid over $\mathfrak{B}$.

Logical validity of arguments

Consequence over a base

$\Gamma \vDash_{\mathfrak{B}} A$ iff for some $\mathscr{D}$ from Γ to A, some $\mathscr{J}$, $\langle \mathscr{D}, \mathscr{J} \rangle$ is valid over $\mathfrak{B}$.

System-rooted validity

$\Gamma \vDash_{SR} A$ iff for every $\mathfrak{B}$, $\Gamma \vDash_{\mathfrak{B}} A$.

Schematic validity

$\Gamma \vDash_{S} A$ iff for some $\mathscr{D}$ from Γ to A, some $\mathscr{J}$, for every $\mathfrak{B}$, $\langle \mathscr{D}, \mathscr{J} \rangle$ is valid over $\mathfrak{B}$.

SR-validity: for every base there is a valid argument.
S-validity: there is an argument valid on every base.

Inferential validity

Validity of an inference (rule) over a base

$\langle R, \{\phi\}\rangle$ is $\mathfrak{B}$-valid iff, for every $\mathscr{D}$ which, paired $\mathfrak{J}$, is valid over $\mathfrak{B}$, it holds that $\langle \mathscr{D}/R, \mathscr{J} \cup \{\phi\}\rangle$ is valid over $\mathfrak{B}$.

System-rooted inferential validity

R is SR-valid iff for every $\mathfrak{B}$, there is ϕ such that $\langle R, \{\phi\}\rangle$ is $\mathfrak{B}$-valid.

Schematic inferential validity

R is S-valid iff, for some ϕ, for every $\mathfrak{B}$, $\langle R, \{\phi\}\rangle$ is $\mathfrak{B}$-valid.

SR-validity: for every base, there is a justification for R.
S-validity: there is a justification for R which works on all bases.

Prawitz's conjecture

Prawitz's conjecture

Let R be an inference (rule)

$$\frac{\begin{array}{c}\Gamma_1, [\Delta_1] \\ \vdots \\ A_1\end{array} \quad \ldots \quad \begin{array}{c}\Gamma_n, [\Delta_n] \\ \vdots \\ A_n\end{array}}{B}$$

If R is logically valid, then R is derivable in $\mathtt{IL}$ (respecting bindings).

Is this for SR-validity or for S-validity?

P&S refutation 1: disregarding justifications

Once we have introduced the relations $\vDash_{\mathfrak{B}}$ and $\vDash_{SR/S}$, we can actually disregard justifications.

Consequence over a base without justifications

Given $\mathfrak{B} = \langle \mathscr{L}, \mathscr{S} \rangle$, we prove by induction on A e.g.

- $\vDash_{\mathfrak{B}} A \Leftrightarrow \vdash_{\mathscr{S}} A$ for $A \in \mathtt{ATOM}_{\mathscr{L}}$
- $\vDash_{\mathfrak{B}} A \to B \Leftrightarrow A \vDash_{\mathfrak{B}} B$
- $\vDash_{\mathfrak{B}} A \vee B \Leftrightarrow \vDash_{\mathfrak{B}} A$ or $\vDash_{\mathfrak{B}} B$

P&S refutation 2: basic principles

The refutation of Prawitz's conjecture due to Piecha and Schroeder-Heister is based on two semantic principles.

Semantic admissibility principle (SAP)

$\Gamma \vDash_{\mathfrak{B}} A \Leftrightarrow (\vDash_{\mathfrak{B}} \Gamma \Rightarrow \vDash_{\mathfrak{B}} A)$

Semantic disjunction property (SDP)

$\vee$ does not occur in $\Gamma \Rightarrow (\Gamma \vDash_{\mathfrak{B}} A \vee B \Rightarrow \Gamma \vDash_{\mathfrak{B}} A$ or $\Gamma \vDash_{\mathfrak{B}} B)$

We also have the following result.

Correctness of intuitionistic logic

$\Gamma \vdash_{\mathrm{IL}} A \Rightarrow \Gamma \vDash_{SR/S} A$.

P&S refutation 3: proof-sketch

1. Disjunctions can be eliminated from negated formulas.
2. Harrop's rule H is not derivable in $\mathtt{IL}$.

(Piecha & Schroeder-Heister 2018)

(SDP) holds for every $\mathfrak{B} \Rightarrow H$ is SR-valid.

3. Proved by using (SAP), correctness and 1 above.

(Piecha & Schroder-Heister 2018)

(SDP) holds for every $\mathfrak{B}$.

4. Proved using classical logic in the meta-language.

(Piecha & Schroeder-Heister 2018)

$\mathtt{IL}$ is incomplete with respect to SR-validity.

Failure of the refutation for the ∃∀ order

Because of the use of (SAP), Piecha and Schroeder-Heister argue Prawitz's semantics may be a semantics for intuitionistically *admissibile* rules.

(SAP) fails if we replace $\vDash_{\mathfrak{B}}$ with $\vDash_S$.

Let $\mathfrak{B} = \emptyset$. Then $\nvDash_S p$, then $\vDash_S p \Rightarrow \vDash_S q$, but of course $p \nvDash_S q$.

The proof above does not work for S-validity.

More in general, it seems that $\Gamma \vDash_{SR} A$ does not imply $\Gamma \vDash_S A$. This is seen if we *do not* disregard justifications.

Suppose that for every $\mathfrak{B}$ there is $\mathscr{D}_{\mathfrak{B}}$ from Γ to A valid over $\mathfrak{B}$. It may well be that for any such $\mathscr{D}_{\mathfrak{B}}$ there is $\mathfrak{B}^*$ such that $\mathscr{D}_{\mathfrak{B}}$ is not valid over $\mathfrak{B}^*$. Suppose that for every $\mathfrak{B}$ there is a $\phi_{\mathfrak{B}}$ which justifies R over $\mathfrak{B}$. It may well be that, for any such $\phi_{\mathfrak{B}}$ there is $\mathfrak{B}^*$ such that $\phi_{\mathfrak{B}}$ does not justify R over $\mathfrak{B}^*$.

Choice-validity

R is C-valid iff it can be justified by a choice function.

$SR \Rightarrow C$

R is SR-valid $\Rightarrow$ R is C-valid.

Suppose that for every $\mathfrak{B}$, there is $\phi_{\mathfrak{B}}$ such that $\langle R, \{\phi_{\mathfrak{B}}\}\rangle$ is valid over $\mathfrak{B}$. Then we define the following choice function C for justifying R:

$$\dfrac{\overset{\mathscr{D}_{\mathfrak{B}}}{\Gamma}}{A}\, R \quad \overset{\texttt{C}}{\Longrightarrow} \quad \phi_{\mathfrak{B}}(\mathscr{D}_{\mathfrak{B}}/R)$$

Since C is "one and the same" on all atomic bases, one may be tempted to say also that C-validity implies S-validity. Then, IL is incomplete also with respect to S-validity.

Criticism of choice-validity

Atomic bases cannot be recursively generated. So $\mathtt{C}$ is not allowed [but...we should read meta-logical constants constructively, so $\mathtt{C}$ is after all allowed (similar to proof of AC in intuitionistic type theory)].

Compare $\mathtt{C}$ with $\rightarrow \mathtt{Red}$. The latter is defined *only* on arguments for the premises of *modus ponens*:

$$\mathtt{App}(\lambda x^A(f(x^A)), c^A) = f(c^A)$$

Instead for the choice function we have

$$\phi(c^\Gamma, \mathtt{h}(c^\Gamma)) = \mathtt{C}(\mathtt{h}(c^\Gamma))(c^\Gamma)$$

where $\mathtt{h}$ associates c^Γ to its atomic base. This can be said to schematic, but we have an additional parameter, which may not be what we wanted when requiring schematic validity.

The justifications-class

The problem thus is "what does *schematic* mean?". Prawitz gives only three conditions for ϕ to be a justification function:

1. ϕ must be effectively computable;
2. when applied to an argument for A from Γ, ϕ must yield an argument for A from $\Gamma^* \subseteq \Gamma$;
3. ϕ must be linear over substitution, i.e.

$$\phi(\mathscr{D})[\star/\square] = \phi(\mathscr{D}[\star/\square])$$

But we have just seen that we may have at least another parameter, i.e. ϕ must not be defined on atomic bases (or functions from arguments to atomic bases).

It is difficult to say when such a list is complete. Perhaps, we should just try to *define* a notion of schematicity.

Provable validity 1: main idea

$$\dfrac{\dfrac{\begin{matrix}\mathscr{D}_1 & \mathscr{D}_2\\ A_1 & A_2\end{matrix}}{A_1 \wedge A_2}}{A_i} \quad \wedge\,\texttt{Red} \quad \begin{matrix}\mathscr{D}_i\\ A_i\end{matrix}$$

Suppose $\mathscr{D}_i$ is valid over $\mathfrak{B}$. Then, the reduction simply extracts the correct argument for proving that elimination of conjunction preserves validity. To this we should add that $\mathscr{D}_i$ may not be in canonical form but, if it is valid, then it must reduce to this form.

We have used only: logic, general principles about proofs (the fact that non-canonical valid reduces to canonical valid or that canonical for formulas of a certain kind must have a certain form), and how $\wedge$ Red works. No references to bases.

Provable validity 2: systems of grounding

Let us develop a language $\mathfrak{L}$ for speaking about proofs, containing:

- x^A for proofs for A
- primitive functionals for introductions, e.g. $\wedge I : A, B \triangleright A \wedge B$
- non-primitive functionals, e.g. $\wedge E : A_1 \wedge A_2 \triangleright A_i$
- logical constants, say $=_\pi, \wedge_\pi, \rightarrow_\pi, \forall_\pi$

Let us develop a *minimal grounding system* Σ over $\mathfrak{L}$, containing:

- type introductions
- type elimination (Dummett's rule)
- (extensional) equality rules
- defining equations (reductions)
- logical rules (intuitionistic? intermediate? classical?)

WARNING: difference between an object-constant k and an $\mathfrak{L}$ constant k_π.

Provable validity 2.1: toy example 1

$$\frac{T : A \qquad U : B}{\wedge I(U, T) : A \wedge B} \qquad \frac{T : A \wedge B \qquad \begin{matrix}[T =_\pi \wedge I(x, y)] \quad [x : A] \quad [y : B] \\ \vdots \\ \mathfrak{A}\end{matrix}}{\mathfrak{A}}$$

$$\frac{T_1 =_\pi U_1 \qquad T_2 =_\pi U_2}{\wedge I(T_1, T_2) =_\pi \wedge I(U_1, U_2)} \qquad \frac{\wedge I(T_1, T_2) =_\pi \wedge I(U_1, U_2)}{T_i =_\pi U_i}$$

$$\frac{T =_\pi U}{\wedge E(T) =_\pi \wedge E(U)} \qquad \frac{}{\wedge E(\wedge I(T_1, T_2)) =_\pi T_i}$$

$$\frac{T : A \qquad T =_\pi U}{U : A}$$

Provable validity 2.2: toy example 2

$$
\dfrac{
  \dfrac{
    \dfrac{
      \overset{1}{[x : A \wedge B]}
      \qquad
      \dfrac{
        \dfrac{
          \dfrac{\overset{2}{[x =_\pi \wedge I(y,z)]}}{\wedge E(x) =_\pi \wedge E(\wedge I(y,z))}
          \qquad
          \dfrac{}{\wedge E(\wedge I(y,z)) =_\pi y}
        }{\wedge E(x) =_\pi y}
        \qquad
        \overset{3}{[y : A]}
      }{\wedge E(x) : A}
    }{\wedge E(x) : A}\; 2,3
  }{x : A \wedge B \to_\pi \wedge E(x) : A}\; 1
}{\forall_\pi x(x : A \wedge B \to_\pi \wedge E(x) : A)}
$$

So, the equation provides a good-definition of $\wedge E$. This proof can be given in any system of grounding where we add individual constants for accounting for proofs in atomic bases.

Provable validity 3: sketch of definition

Provable validity

Let R be an inference (rule)

$$\frac{A_1 \qquad \dots \qquad A_n}{B}$$

Then R is *provably valid* (PV) iff there is Σ with functional ϕ^R [and defining equation ε for ϕ^R] such that

$$\vdash_\Sigma \forall_\pi x_1, ..., x_n, (\bigwedge_{i \leq n} x_i : A_i \rightarrow_\pi \phi^R(x_1, ..., x_n) : B)$$

Correctness of intuitionistic logic

$\Gamma \vdash_{\mathrm{IL}} A \Rightarrow \Gamma \vDash_{PV} A$.

Provable validity 4: measure for inference (rule)

R can be said to have:

- pre-rank 0 if there is Σ such that R is PV wrt Σ and the only defining equation of Σ is one associated to R;
- pre-rank n if there is Σ such that R is PV wrt Σ, $n-1$ is the max of the pre-ranks "determined" by Σ and, provided ε is the defining equation associated to R in Σ, for no $\Sigma - \varepsilon^*$ with $\varepsilon^* \neq \varepsilon$, R is PV wrt $\Sigma - \varepsilon^*$.

R can be called *strict* if the max of its pre-ranks is 0. Are Gentzen's eliminations the only strict rules?

R can be called PV-connected to all the rules with maximal pre-rank in the systems where it is PV. Is it now true that any PV rule has a PV-path ending with Gentzen's eliminations?

THANK YOU

antonio.piccolomini-d-aragona@univ-amu.fr

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