Game semantics for constructive modal logic

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Background

$$\mathcal{A} = \{1, a, b, c, d \ldots\}$$

$$\mathcal{F} := \mathcal{A} | \mathcal{F} \wedge \mathcal{F} | \mathcal{F} \supset \mathcal{F} | \Diamond\mathcal{F} | \Box\mathcal{F}$$

The modal logic CK is the smallest set of formulas containing:

- any instance of an intuitionistic theorem ;
- any instance of the axiom $\Box(A \supset B) \supset (\Box A) \supset (\Box B)$;
- any instance of the axiom $\Box(A \supset B) \supset (\Diamond A) \supset (\Diamond B)$

and closed for :

- **modus ponens**: if A and $A \supset B$ are in CK so is B;
- **necessitation**: if A is in CK so is $\Box A$.
- **substitution**: if A is in CK so is $A[B_1/a_1, \ldots, B_n/a_n]$

$$\frac{}{A \vdash A}\ \text{AX} \qquad \frac{}{\vdash 1}\ 1 \qquad \frac{\Gamma \vdash C}{\Gamma, A \vdash C}\ \text{W} \qquad \frac{\Gamma, A, A \vdash C}{\Gamma, A \vdash C}\ \text{C}$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \supset B}\ \supset^{\text{R}} \qquad \frac{\Gamma \vdash A, \quad \Delta, B \vdash C}{\Gamma, \Delta, A \supset B \vdash C}\ \supset^{\text{L}} \qquad \frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \wedge B}\ \wedge^{\text{R}} \qquad \frac{\Gamma, A, B \vdash C}{\Gamma, A \wedge B \vdash C}\ \wedge^{\text{L}}$$

$$\frac{A_1, \ldots, A_n \vdash C}{\Box A_1, \ldots, \Box A_n \vdash \Box C}\ \text{k}^{\Box} \qquad \frac{A_1, \ldots, A_n, B \vdash C}{\Box A_1, \ldots, \Box A_n, \Diamond B \vdash \Diamond C}\ \text{k}^{\Diamond}$$

$$\frac{\Gamma \vdash A \qquad \Delta, A \vdash C}{\Gamma, \Delta \vdash C}\ \text{cut}$$

Theorem

There is a derivation $\mathscr{D}$ of the sequent $\vdash A$ iff $A \in$ CK.

Theorem

There is a procedure P that turns every derivation $\mathscr{D}$ in which the cut rule is used in a derivation $\mathscr{D}'$ of the same sequent in which the cut rule is never used.

Proof Semantics

$$\{\!\{-\}\!\}: \quad \{\text{ derivations }\} \quad \to \quad \{\text{mathematical objects }\}$$

$$\mathscr{D} \quad \to \quad \{\!\{\mathscr{D}\}\!\}$$

Denotational Semantics

$$\mathscr{D} \rightsquigarrow \mathscr{D}' \quad \Rightarrow \quad \{\!\{\mathscr{D}\}\!\} = \{\!\{\mathscr{D}'\}\!\}$$

Denotational Semantics for Constructives Modal Logics (Bellin-De Paiva-Ritter)

$$\frac{\{\lambda\text{-termes}\}}{\beta\text{-reduction}}$$

Morally

$$\frac{\{\text{Proofs}\}}{\text{Cut elimination}}$$

Game Semantics

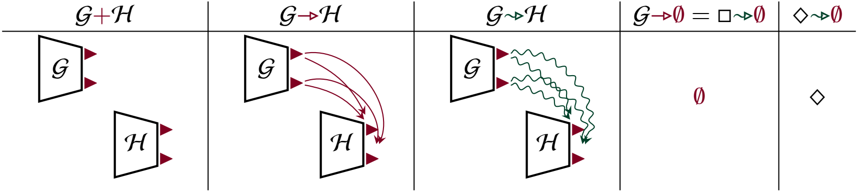
$$\{\text{Derivations}\} \quad \rightarrow \quad \{\text{Winning Strategies}\}$$

$$\mathscr{D}_F \quad \rightarrow \quad \text{Winning Strategy over} [\![F]\!]$$

- $[\![F]\!]$ is a finite graph representing F;
- A strategy is a particular set of plays over $[\![F]\!]$;
- A play is a particular sequence of nodes of $[\![F]\!]$.

Arenas

Let $\mathcal{G}$ and $\mathcal{H}$ two bi-colored DAGS and let $\emptyset$ be the empty DAG.



$\llbracket a \rrbracket = a$ $\quad$ $\llbracket 1 \rrbracket = \emptyset$ $\quad$ $\llbracket A \wedge B \rrbracket = \llbracket A \rrbracket + \llbracket B \rrbracket$ $\quad$ $\llbracket A \supset B \rrbracket = \llbracket A \rrbracket \rightarrowtriangle \llbracket B \rrbracket$

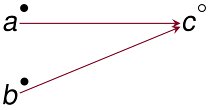
$\llbracket \Box A \rrbracket = \Box \rightsquigarrow \llbracket A \rrbracket$ $\quad$ $\llbracket \Diamond A \rrbracket = \Diamond \rightsquigarrow \llbracket A \rrbracket$

Arenas

Each vertex v of an arena has a polarity. Such a polarity, positive (∘) or negative (•), is the same as that of the occurrence of the atomic formula (or modality) of A that labels v.

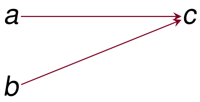
$$[\![(a \wedge b) \supset c]\!] \quad = \quad [\![a \wedge b]\!] \;\rightarrow\; [\![c]\!]$$

$$=$$



$$[\![a \supset (b \supset c)]\!] \quad = \quad [\![a]\!] \;\rightarrow\; [\![b \supset c]\!]$$

$$=$$

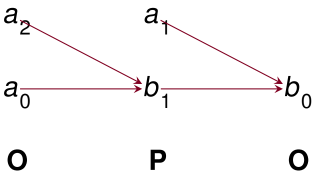


An **intuitionistic move** in $[\![F]\!]$ is a node v of $[\![F]\!]$ labeled by a propositional variable. It is a **P**-move if v is of negative polarity and an **O**-move otherwise

An **intuitionistic play** for F is a finite alternate sequence of moves of $[\![F]\!]$ such that:

- **O**-starts : the first node of the sequence is an arena-root.
- any move w of the play, but the first, is justified by a preceding move made by the other player : $w \rightarrow v$ in the arena ;
- each **O**-move is justified by *the immediately preceding* **P**-move.
- each **P**-move w has the same label as the immediately preceding **O**-move: if v is labeled by a so is w.

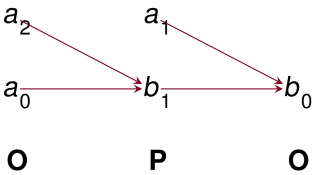
Let's play on $[\![((a \wedge a) \supset b) \supset a \supset b]\!]$



ϵ

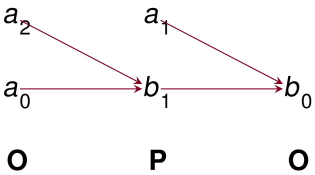
$\epsilon \quad b_0^{\mathbf{O}}$

Let's play on $[\![((a \wedge a) \supset b) \supset a \supset b]\!]$



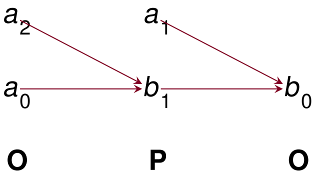
$$\epsilon$$
$$\epsilon \quad b_0^{\mathbf{O}}$$
$$\epsilon \quad b_0^{\mathbf{O}} \quad b_1^{\mathbf{P}}$$

Let's play on $[\![((a \wedge a) \supset b) \supset a \supset b]\!]$



$$\epsilon$$
$$\epsilon \quad b_0^{\mathbf{O}}$$
$$\epsilon \quad b_0^{\mathbf{O}} \quad b_1^{\mathbf{P}}$$
$$\epsilon \quad b_0^{\mathbf{O}} \quad b_1^{\mathbf{P}} \quad a_2^{\mathbf{O}}$$

Let's play on $[\![((a \wedge a) \supset b) \supset a \supset b]\!]$



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$\epsilon \quad b_0^{\mathbf{O}} \quad b_1^{\mathbf{P}} \quad a_2^{\mathbf{O}} \quad a_1^{\mathbf{P}}$

A strategy is a **plan of action**.

For any move that my Opponent can make, **there is a move** I can make that will eventually led me to victory.

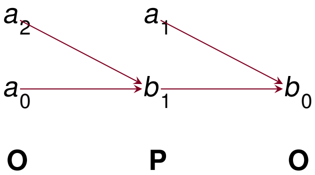
Winning Strategy

If σ and ρ are two plays over $[\![A]\!]$, we say that ρ is a **successor** of σ iff $\rho = \sigma v$ for some $v \in [\![A]\!]$.

A **Winning Strategy** $\mathcal{S}$ for F is a non-empty finite prefix-closed set of plays over $[\![F]\!]$ such that :

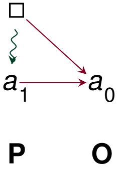
- **O**-completeness: if p $\in \mathcal{S}$ has even length, then **any** successor of p belongs to $\mathcal{S}$;
- **P**-determinism and totality: if p $\in \mathcal{S}$ has odd length, then **exactly one** successor of p belongs to $\mathcal{S}$.

A strategy for $((a \wedge a) \supset b) \supset a \supset b$



$$\mathcal{S} = \left\{ \quad \epsilon\, b_0\, b_1\, a_0\, a_1 \qquad \epsilon\, b_0\, b_1\, a_2\, a_1 \right\}$$

consider the following strategy over $[\![\Box a \supset a]\!]$



$$\mathcal{S} = \{\epsilon\, a_0^{\mathbf{O}}\, a_1^{\mathbf{P}}\}$$

… this formula is not a theorem of CK !

Well batched strategies

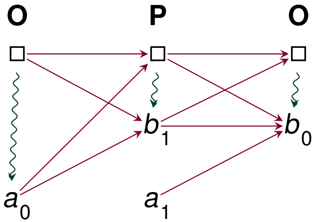
the address of a vertex $v \in [\![A]\!]$ is the sequence of modalities $\mathrm{add}_v = m_1, \dots m_k$ in the path in the formula tree of F connecting the node v to the root of F.

A play p is **well batched** whenever it respects the following:

- every move of p is either a $\Diamond$-modality or a propositional variable.
- if $\mathrm{p} = \sigma\, v^{\mathbf{O}}\, w^{\mathbf{P}}$ then $|\mathrm{add}_w| = |\mathrm{add}_v|$;

A winning strategy is well batched iff any of its plays is well batched.

. . . it is not enough, consider $[\![(\Box a \supset \Box b) \supset \Box(a \supset b)]\!]$



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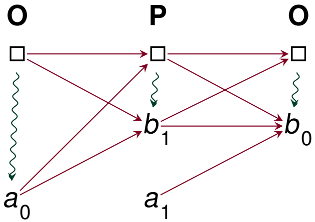


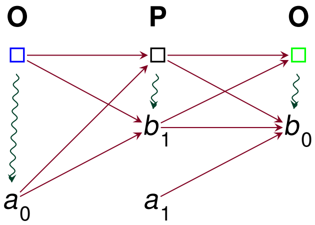
$$\mathcal{S} = \{b_0\, b_1\, a_0\, a_1\}$$

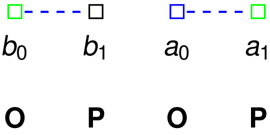
this is a well batched winning strategy

$$\frac{A_1^\bullet, \ldots, A_n^\bullet \vdash C^\circ}{(\Box_1 A_1)^\bullet, \ldots, (\Box_n A_n)^\bullet \vdash (\Box C)^\circ}\ \mathsf{K}^\Box \qquad \frac{B_1^\bullet, \ldots, B_m^\bullet, D^\bullet \vdash F^\circ}{(\Box_1 B_1)^\bullet, \ldots, (\Box_n B_m)^\bullet, (\Diamond D)^\bullet \vdash (\Diamond F)^\circ}\ \mathsf{K}^\Diamond$$

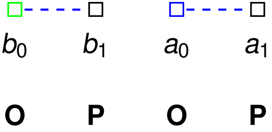
$$\frac{A_1^{\mathbf{P}}, \ldots, A_n^{\mathbf{P}} \vdash C^{\mathbf{O}}}{(\Box_1 A_1)^{\mathbf{P}}, \ldots (\Box_n A_n)^{\mathbf{P}} \vdash (\Box C)^{\mathbf{O}}}\ \mathrm{K}^{\Box} \qquad \frac{B_1^{\mathbf{P}}, \ldots, B_m^{\mathbf{P}}, D^{\mathbf{P}} \vdash F^{\mathbf{O}}}{(\Box_1 B_1)^{\mathbf{P}}, \ldots, (\Box_n B_m)^{\mathbf{P}}, (\Diamond D)^{\mathbf{P}} \vdash (\Diamond F)^{\mathbf{O}}}\ \mathrm{K}^{\Diamond}$$

...again on $[\![(\Box a \supset \Box b) \supset \Box(a \supset b)]\!]$





given two modalities m and m' and a play p, we write $m \overset{\mathrm{p}}{\sim} m'$ whenever $m = \mathrm{add}_k^{v}$, $m' = \mathrm{add}_k^{v'}$ where v and v' are two consecutive moves in p and v' is a **P**-move.



the reflexive, transitive and symmetric closure of the relation $\overset{\mathrm{p}}{\sim}$ contains two positive modalities .

Winning modal strategies

Let $\mathcal{S}$ be a winning, well batched strategy. We say that $\mathcal{S}$ is **well framed** iff for any p $\in \mathcal{S}$, any $\overset{p}{\sim}$-class is of the form $\{m_1^{\mathbf{P}}, \ldots m_n^{\mathbf{P}}, m^{\mathbf{O}}\}$

A winning well framed strategy $\mathcal{S}$ is a **modal** strategy iff for any $\sigma \in \mathcal{S}$ for any modal node $m^{\mathbf{O}}$ appearing in the address of some move v of σ

1. if $m = \Box$ then $m' = \Box$ for any $m' \overset{p}{\sim} m$;
2. if $m = \Diamond$ then there is a unique $m'^{\mathbf{P}} = \Diamond$ such that $m \overset{p}{\sim} m'$.

Results

Theorem

Given two modal strategies $\mathcal{S}$ for $A \supset B$ and $\mathcal{T}$ for $B \supset C$ we can define their composition $\mathcal{S}\,;\mathcal{T}$ which is a modal strategy for $B \supset C$. Moreover $(\mathcal{S};\mathcal{T});\mathcal{R} = \mathcal{S};(\mathcal{T};\mathcal{R})$.

Theorem

There is a function $\{\!\{-\}\!\}$ mapping any derivation $\mathscr{D}$ of $\vdash A$ to a winning strategy $\{\!\{\mathscr{D}\}\!\}$ for A dubbed its interpretation. Moreover:

1. *If $\mathscr{D}$ reduces to $\mathscr{D}'$ in 0 or more steps of cut elimination, then $\{\!\{\mathscr{D}\}\!\} = \{\!\{\mathscr{D}'\}\!\}$.*
2. *for any winning strategy $\mathcal{S}$, there is a proof $\mathscr{D}$ such that $\mathcal{S} = \{\!\{\mathscr{D}\}\!\}$.*

Perspectives

Déduction naturelle pour CK

$$\dfrac{\begin{matrix} \vdots & & \vdots & [A_1 \cdots A_n] \\ & & & \vdots \\ \Box A_1 & \cdots & \Box A_n & C \end{matrix}}{\Box C}\ \Box\mathrm{K}$$

$$\dfrac{\begin{matrix} \vdots & & \vdots & \vdots & [A_1 \cdots A_n\ A] \\ & & & & \vdots \\ \Box A_1 & \cdots & \Box A_n & \Diamond A & C \end{matrix}}{\Diamond C}\ \Diamond\mathrm{K}$$

$$\mathsf{CT} = \mathsf{CK} \cup \{(\Box A \supset A) \land (A \supset \Diamond A) \mid \text{for any } A \in \mathcal{F}\}$$

$$\mathsf{CS4} = \mathsf{CT} \cup \{(\Box A \supset \Box\Box A) \land (\Diamond\Diamond A \supset \Diamond A) \mid \text{for any } A \in \mathcal{F}\}$$

- Matteo Acclavio, Davide Catta et Lutz Straßburger (2021), *Game Semantics for Constructive Modal Logics*. In: Das A., Negri S. (eds) Automated Reasoning with Analytic Tableaux and Related Methods. TABLEAUX 2021. Lecture Notes in Computer Science, vol 12842. Springer, Cham.https://doi.org/10.1007/978-3-030-86059-2_25
- My PhD thesis. *Proofs as games and games as proofs: dialogical semantics for logic and natural language.*

Thank You !