Algebras of Counterfactual Conditionals

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Motivations

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What?

Counterfactuals are subjunctive conditional statements about hypothetical situations of the form “If [antecedent] were the case, then [consequent] would be the case”.

They have many applications in the philosophy of language, linguistics, causal inference and AI.

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How?

We try to introduce an algebraic setting for counterfactual reasoning based on Boolean Algebras With Operators.

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"If the executioner had not fired, then the prisoner would not have died"

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Counterfactuals have many applications in the philosophy of language, linguistics, causal inference and AI.

Example

The executioner firing is the cause of the death of the prisoner if the corresponding counterfactuals *"If the executioner had (not) fired, then the prisoner would (not) have died" are true*.

(Counterfactual analysis of causation)

Lewis' Logics of Counterfactuals

Let $\mathcal{L}$ be a classical language in the signature $\vee, \wedge, \neg, \top, \bot$

Language

$\mathcal{L}^{\Box\!\!\rightarrow}$ is obtained by extending $\mathcal{L}$ with the binary connective $\Box\!\!\rightarrow$:

$$\varphi := p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \Box\!\!\rightarrow \varphi$$

The connective $\Box\!\!\rightarrow$ stands for the counterfactual conditional.

Lewis' Logic(s) of Counterfactuals

C1 *is the correct logic of counterfactual conditionals*

[Lewis, 1971]

Deductive System

Rules:

(TI) $\varphi_1, \ldots, \varphi_n \rhd \psi$ if $(\varphi_1, \ldots, \varphi_n) \to \psi$ is a tautology

(DWC) (i) $\psi \rhd \varphi \;\Box\!\!\to \psi$

(ii) $(\varphi_1 \wedge \cdots \wedge \varphi_n) \to \psi \rhd ((\delta \;\Box\!\!\to \varphi_1) \wedge \cdots \wedge (\delta \;\Box\!\!\to \varphi_n)) \to (\delta \;\Box\!\!\to \psi)$

Axioms:

(A) $\emptyset \rhd \varphi$ if φ is a classical tautology

(B) $\emptyset \rhd \varphi \;\Box\!\!\to \varphi$

(C) $\emptyset \rhd ((\varphi \;\Box\!\!\to \psi) \wedge (\psi \;\Box\!\!\to \varphi)) \to ((\varphi \;\Box\!\!\to \delta) \leftrightarrow (\psi \;\Box\!\!\to \delta))$

(D) $\emptyset \rhd ((\varphi \vee \psi) \;\Box\!\!\to \varphi) \vee ((\varphi \vee \psi) \;\Box\!\!\to \psi) \vee$
$(((\varphi \vee \psi) \;\Box\!\!\to \delta) \leftrightarrow ((\varphi \;\Box\!\!\to \delta) \wedge (\psi \;\Box\!\!\to \delta))$

(E) $\emptyset \rhd (\varphi \;\Box\!\!\to \psi) \to (\varphi \to \psi)$

(F) $\emptyset \rhd (\varphi \wedge \psi) \to (\varphi \;\Box\!\!\to \psi)$

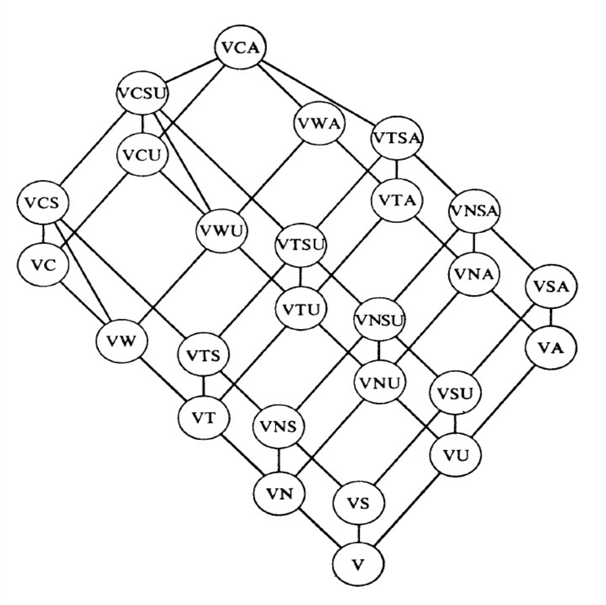
Lewis' Logic(s) of Counterfactuals

Remark

Strictly speaking, Lewis identifies **C1** with the smallest set Σ of formulas in $\mathcal{L}^{\Box\!\!\rightarrow}$ such that:

1. Σ contains all axioms (A)-(F)
2. Σ is closed under (DWC)
3. Σ is closed under (TI)
4. Σ is closed under substitution

Lewis' Logic(s) of Counterfactuals



*'If kangaroos had no tails, they would topple over' seems to mean something like this: in **any possible state of affairs** in which kangaroos have no tails, and **which resembles our actual state of affairs** as much as kangaroos having no tails permits it to, the kangaroos would topple over. I shall give a general analysis of counterfactual conditionals along these lines.* *[Lewis, 1973]*

Lewis' Logic of Counterfactuals - Semantics

> *'If kangaroos had no tails, they would topple over' seems to mean something like this: in* ***any possible state of affairs*** *in which kangaroos have no tails, and* ***which resembles our actual state of affairs*** *as much as kangaroos having no tails permits it to, the kangaroos would topple over. I shall give a general analysis of counterfactual conditionals along these lines.* *[Lewis, 1973]*

Th ingredients of Lewis' semantics for counterfactuals are:

1. possible worlds ("**any possible state of affairs**)
2. a relation of similarity among possible worlds ("**which resembles our actual state of affairs** ") represented in therms of *centered spheres*.

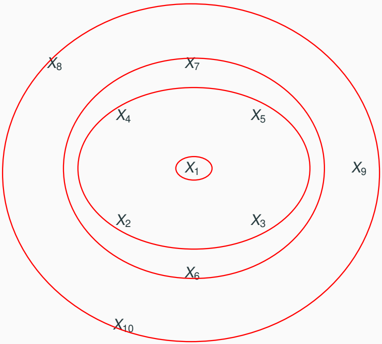
Lewis' Logic of Counterfactuals - Semantics

Definition: Sphere Model

A sphere model is a tuple $\Sigma = (I, \mathscr{S}, v)$ where:

- I is a non-empty set;
- $\mathscr{S}$ is a function $\mathscr{S} : I \to \wp(\wp(I))$ such that, for each $i \in I$, $\mathscr{S}_i \subseteq \wp(I)$, and moreover $\mathscr{S}(i)$ is:
  - (S1) nested: for all $S, T \in \mathscr{S}(i)$, either $S \subseteq T$ or $T \subseteq S$;
  - (S2) non-empty: for all $S \in \mathscr{S}(i)$, $i \in S$;
  - (S3) centered: either $\bigcup \mathscr{S}(i) = \emptyset$, or $\{i\} \in \mathscr{S}(i)$.
- v is a valuation function $v : \mathcal{P} \to \wp(I)$ that is extended to compound formulas as follows (we define $i \Vdash \varphi \Leftrightarrow i \in v(\varphi)$):
  - $v(\neg\Phi) = I \setminus v(\Phi)$, $v(\Phi \wedge \Psi) = v(\Phi) \cap v(\Psi)$, $v(\Phi \vee \Psi) = v(\Phi) \cup v(\Psi)$
  - $v(\psi \,\Box\!\!\rightarrow \varphi) = \{i \in I \mid v(\psi) \cap \bigcup \mathscr{S}(i) = \emptyset,$ or $\exists S \in \mathscr{S}(i)\,(\emptyset \neq (v(\psi) \cap S) \subseteq v(\varphi))\}$;

Sphere Model-Example

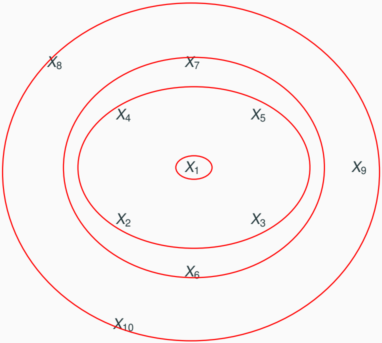


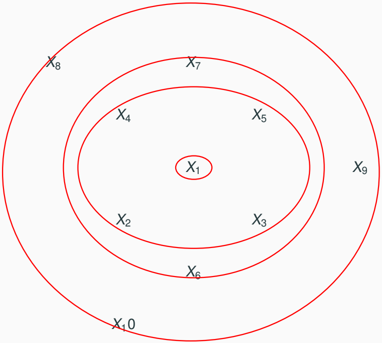
$\mathscr{S}(X_1) = \{$
$\{X_1\}$
$\{X_1, X_2, X_3, X_4, X_5\}$
$\{X_1, X_2, X_3, X_4, X_5, X_6, X_7\}$
$\{X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8, X_9, X_{10}\}$
$\}$

$\mathscr{S}_{X_1}$ is

- non-empty;
- centered: $\{X_1\}$ is included in all the other members of $\mathscr{S}_{X_1}$;
- nested: the members of $\mathscr{S}_{X_1}$ are totally ordered by set-inclusion.

Sphere Model-Example





if $v(p) = \{X_5\}$ and $v(q) = \{X_5, X_6\}$, then $X_1 \Vdash p \mathrel{\Box\!\!\rightarrow} q$ since $X_5 \Vdash p$ and $X_5 \Vdash q$

if $v(p) = \{X_5\}$ and $v(q) = \{X_6\}$, then $X_1 \nVdash p \mathrel{\Box\!\!\rightarrow} q$, since $X_5 \Vdash p$ but $X_5 \nVdash q$

Lewis' Logic of Counterfactuals

Validity

$$\models_{\mathbf{C1}} \psi \quad \Leftrightarrow \quad \text{for all sphere models } (I, \mathscr{S}, v),\ \text{for all } i \in I,\ i \Vdash \psi$$

Soundness and Completeness [Lewis, 1971]

$$\psi \in \mathbf{C1} \Leftrightarrow \models_{\mathbf{C1}} \psi$$

Global *vs* Local Consequence

Just like Modal Logic

Just like in modal logic, we can associate two logics with **C1**:

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1: The local consequence $\mathbf{C1}_l$

$\mathbf{C1_l}$- Axiomatic System

- $\emptyset \rhd \varphi$ if $\varphi \in \mathbf{C1}$
- $\varphi_1, \dots \varphi_n \rhd \psi$ if φ is a tautology

Example: $p \nvdash_{\mathbf{C1}_l} q \boxright p$

2: The global consequence $\mathbf{C1}_g$

$\mathbf{C1_g}$- Axiomatic System

- $\emptyset \rhd \varphi$ if $\varphi \in \mathbf{C1}$
- $\varphi_1, \dots \varphi_n \rhd \psi$ if φ is a tautology
- (DWC) (i) $\varphi \rhd \psi \boxright \varphi$
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Example: $p \vdash_{\mathbf{C1}_g} q \boxright p$

Soundness and Completeness

Local consequence - Semantics

$$\Gamma \models_{\mathbf{C1}} \psi \quad \Leftrightarrow \quad \text{for all sphere models } (I, \mathscr{S}, v),\ \text{for all } i \in I, \text{ if } i \Vdash \bigwedge \Gamma \text{ then } i \Vdash \psi$$

Soundness and Completeness - $\mathbf{C1}_l$

$$\Gamma \vdash_{\mathbf{C1}_l} \psi \Leftrightarrow \Gamma \models_{\mathbf{C1}_l} \psi$$

$\mathbf{C1}_l$ is the logic preserving local truth

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Global consequence - Semantics

$$\Gamma \models_{\mathbf{C1}_g} \psi \quad \Leftrightarrow \quad \text{for all sphere models } (I, \mathscr{S}, v), \text{ if for all } i \in I, i \Vdash \bigwedge \Gamma \text{ then for all } i \in I, i \Vdash \psi$$

Soundness and Completeness - $\mathbf{C1}_g$

$$\Gamma \vdash_{\mathbf{C1}_g} \psi \Leftrightarrow \Gamma \models_{\mathbf{C1}_g} \psi$$

$\mathbf{C1}_g$ is the logic preserving global truth

Global *vs* Local Consequence

Just like in modal logic, we can analyze the relations between the local and the global consequence.

Proposition

The following hold:

1. $\models_{\mathbf{C1}_g} \varphi \Leftrightarrow \models_{\mathbf{C1}_l} \varphi$
2. $\Gamma \models_{\mathbf{C1}_l} \varphi \Rightarrow \models \Gamma \models_{\mathbf{C1}_g} \varphi$

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We introduce a useful connective:

Notation

Let's define the unary connective $\boxdot$ in $\mathcal{L}^{\Box\!\!\rightarrow}$ as:

$$\boxdot\varphi := (\neg\varphi) \Box\!\!\rightarrow \varphi$$

and $\boxdot^n\varphi$ is inductively defined as: $\boxdot^0(\varphi) := \varphi$, $\boxdot^{n+1}(\varphi) := \boxdot(\boxdot^n(\varphi))$ (see [Lewis, 1973])

Global *vs* Local Consequence

Proposition: Global consequence via Local consequence

$$\Gamma \models_{\mathbf{C1}_g} \varphi \Leftrightarrow \{\boxdot^n \gamma \mid n \in \mathbb{N} \text{ and } \gamma \in \Gamma\} \models_{\mathbf{C1}_l} \varphi$$

Proof.

By employing a notion of generated submodel for sphere models □

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Deduction Theorem for $\mathbf{C1}_g$

$$\Gamma, \gamma \models_{\mathbf{C1}_g} \varphi \Leftrightarrow \Gamma \models_{\mathbf{C1}_g} (\gamma \wedge \boxdot\gamma \wedge \cdots \wedge \boxdot^n \gamma) \to \varphi \text{ for some } n \in \mathbb{N}$$

Algebras of Counterfactuals

Algebras of Counterfactuals

Definition: Counterfactual Algebra

An *Algebra of Counterfactuals* is a tuple of the form $\mathbf{C} = \langle C, \wedge, \vee, \neg, \boxright, \top, \bot \rangle$ where $\langle C, \wedge, \vee, \neg, \bot, \top \rangle$ is a Boolean algebra and $\boxright$ is a binary operator such that (for all $x, y, z \in C$):

1. $x \boxright x = \top$
2. $((x \boxright y) \wedge (y \boxright x)) \wedge ((x \boxright z) \leftrightarrow (y \boxright z)) = (x \boxright y) \wedge (y \boxright x)$
3. $((x \vee y) \boxright x) \vee ((x \vee y) \boxright y) \vee (((x \vee y) \boxright z) \leftrightarrow ((x \boxright z) \wedge (y \boxright z)) = \top$
4. $x \boxright (y \wedge z) = (x \boxright y) \wedge (x \boxright z)$
5. $(x \boxright (y \wedge z)) \rightarrow (x \boxright (y \vee z)) = \top$

Moreover, we set $x \diamondright y := \neg(x \boxright \neg y)$

Algebras of Counterfactuals - Equations

In every algebra of counterfactuals the following hold:

1. $(x \mathrel{\Box\!\!\rightarrow} z) \wedge (y \mathrel{\Box\!\!\rightarrow} z) \leq (x \vee y) \mathrel{\Box\!\!\rightarrow} z$
2. $x \mathrel{\Box\!\!\rightarrow} y \leq x \mathrel{\Box\!\!\rightarrow} (y \vee z)$
3. $(x \vee y) \mathrel{\Diamond\!\!\rightarrow} z \leq x \mathrel{\Diamond\!\!\rightarrow} z \vee y \mathrel{\Box\!\!\rightarrow} z$
4. $x \rightarrow y = \top$ iff $x \mathrel{\Box\!\!\rightarrow} y = \top$
5. $\bot \mathrel{\Box\!\!\rightarrow} x = \top$
6. $\bot \mathrel{\Diamond\!\!\rightarrow} x = \bot$
7. $\top \mathrel{\Box\!\!\rightarrow} x = x$

1. $\top \mathrel{\Diamond\!\!\rightarrow} x = x$
2. $x \mathrel{\Box\!\!\rightarrow} \top = \top$
3. $x \mathrel{\Box\!\!\rightarrow} \bot \leq \neg x$
4. $x \leq x \mathrel{\Diamond\!\!\rightarrow} \top$
5. $x \mathrel{\Diamond\!\!\rightarrow} \bot = \bot$
6. $\neg x \mathrel{\Box\!\!\rightarrow} x \leq y \mathrel{\Box\!\!\rightarrow} x$
7. $(x \mathrel{\Box\!\!\rightarrow} \neg y) \vee (((x \wedge y) \mathrel{\Box\!\!\rightarrow} z) \leftrightarrow (x \mathrel{\Box\!\!\rightarrow} (y \rightarrow z))) = 1$

Algebras of Counterfactuals - Algebraic Semantics

We have now all the ingredients to provide and algebrais semantics for $\mathbf{C1}_l$ and $\mathbf{C1}_g$

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Algebraic Semantics for $\mathbf{C1}_g$

For $\Gamma \cup \{\varphi\}$ a set of formulas in $\mathcal{L}^{\Box\!\!\rightarrow}$ and $\tau = \{x \approx 1\}$, we have that:

$$\Gamma \vdash_{\mathbf{C1}_g} \varphi \Leftrightarrow \tau[\Gamma] \models_{\mathfrak{CF}} \tau(\varphi)$$

Alegbrizability

Observe that for the set of formulas $\Delta = \{x \to y, y \to x\}$, it holds that $x \approx y$ $=\!\models_{\mathfrak{CF}} \{x \to y \approx 1, y \to x \approx 1\}$. So:

The logic $\mathbf{C1}_g$ is algebrizable with respect to algebras of counterfactuals

Algebras of Counterfactuals - Algebraic Semantics

What happens to $\mathbf{C1}_l$, i.e. the logic preserving local truth?

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Definition

For $\Gamma \cup \{\varphi\}$ set of formulas in $\mathcal{L}^{\Box\!\!\rightarrow}$, we write $\Gamma \models^{\leq}_{\mathfrak{CF}} \varphi$ iff for all counterfactual algebras $\mathbf{A}$, for all homomorphisms $h : For_{\mathcal{L}^{\Box\!\!\rightarrow}} \rightarrow \mathbf{A}$, for all $a \in \mathbf{A}$, if $a \leq h(\gamma)$, for every $\gamma \in \Gamma$, then $a \leq h(\varphi)$.

Notation: Recall

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Notation: Recall

$$\boxdot\varphi := (\neg\varphi) \Box\!\!\rightarrow \varphi$$

and $\boxdot^n\varphi$ is inductively defined as: $\boxdot^0(\varphi) := \varphi$, $\boxdot^{n+1}(\varphi) := \boxdot(\boxdot^n(\varphi))$

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Observe

$\boxdot$ is a normal modal operator (as in the modal logic **T**)

Algebras of Counterfactuals - Algebraic Semantics

Proposition

$\mathbf{C1}_l$ is the logic preserving degrees of truth over counterfactual algebras:

$$\Gamma \vdash_{\mathbf{C1}_l} \varphi \Leftrightarrow \Gamma \models^{\leq}_{\mathbf{C1}_l} \varphi$$

Proof.

using the Lindembaum-Tarski algebra, $For_{\mathcal{L}^{\Box\!\rightarrow}}/\theta$ where θ is the congruence relation defined as:

$$\theta \quad := \quad \{(\varphi, \psi) \in For_{\mathcal{L}^{\Box\!\rightarrow}} \times For_{\mathcal{L}^{\Box\!\rightarrow}} : \Gamma \vdash_{\mathbf{C1}} \boxdot^n(\varphi \rightarrow \psi) \text{ and } \Gamma \vdash_{\mathbf{C1}} \boxdot^n(\psi \rightarrow \varphi) \text{ for all } n \in \mathbb{N}\,\}$$

□

Observe: $\mathbf{C1}_\mathbf{l}$ is not algebraizable

Ongoing Research

Structure Theory

As a consequence of having a Boolean reduct, congruences of counterfactual algebras are 1-regular

Lemma

Let $\mathbf{A}$ be a counterfactual algebra and $\theta \in Con(\mathbf{A})$, then

$$(x, y) \in \theta \Leftrightarrow (x \leftrightarrow y, \top) \in \theta$$

Lemma

A congruence filter of a counterfactual algebra $\mathbf{A}$ is a lattice filter F such that if $x \leftrightarrow y \in F$, then:

1. $(z \mathrel{\Box\!\!\rightarrow} x) \to (z \mathrel{\Box\!\!\rightarrow} y)$
2. $(x \mathrel{\Box\!\!\rightarrow} z) \to (y \mathrel{\Box\!\!\rightarrow} z)$

Notation: we use $\mathfrak{CF}(\mathbf{A})$ to denote the set of congruence filters over a counterfactual algebra $\mathbf{A}$

Congruence Filters: Characterization

Proposition

A lattice filter F over a counterfactual algebra $\mathbf{A}$ is a congruence filter over $\mathbf{A}$ iff the following holds: for all $a \in \mathbf{A}$

$$\text{if } a \in F, \text{ then } \boxdot^n a \in F \text{ for all } n \in \mathbb{N}$$

Observe: $Con(\mathbf{A}) \approxeq \mathfrak{CF}(\mathbf{A})$

Recall: $\boxdot$ is a normal modal operator

Remark

Every counterfactual algebra $\langle A, \wedge, \vee, \neg, \bot, \top, \Box\!\!\rightarrow \rangle$ has a corresponding modal algebra reduct $\langle A, \wedge, \vee, \neg, \bot, \top, \boxdot \rangle$ where $\boxdot x := (\neg x) \Box\!\!\rightarrow x$

Observe: congruence filters over counterfactual algebras are characterized in terms of $\boxdot$

Remark

Congruence filters over a counterfactual algebra $\mathbf{A}$ are also congruence filters over the modal algebra reduct of $\mathbf{A}$

Remark

A counterfactual algebra $\mathbf{A}$ is subdirectly irreducible iff its corresponding modal algebra reduct is subdirectly irreducible

Conclusions

- Logical investigations of Lewis' counterfactuals/conditionals (global vs local)
- Algebraic Semantics
- Beginning of Structure Theory

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- Algebraic Semantics
- Beginning of Structure Theory
- Duality Theory
- Varieties of Counterfactual Algebras

Conclusions

Thank You!

Selection Functions - Dual Semantics

Definition: Function Model for C1

A **C1**-function model is a tuple $\Sigma = (I, , v)$ where:

- I is a non-empty set;
- f is a function $f : For_{\mathcal{L}^{\Box\rightarrow}} \times I \to \wp(I)$ assigning a subset ot I to each pair made of an element in I and a formulas in $\mathcal{L}^{\Box\rightarrow}$. is such that

  (F1) $f(\varphi, i) \subseteq v(\varphi)$;
  (F2) if $f(\varphi, i) \subseteq v(\psi)$ and $f(\psi, i) \subseteq v(\varphi)$
  (F3) either $f(\varphi \vee \psi, i) \subseteq v(\varphi)$ or $f(\varphi \vee \psi, i) \subseteq v(\psi)$ or $f(\varphi \vee \psi, i) = f(\varphi, i) \cup f(\psi, i)$
  (F4) if $i \in v(\varphi)$, then $i \in f(\varphi, i)$;
  (F5) if $i \in v(\varphi)$, then $j \in f(\varphi, i)$ only if $j = i$
- v is a valuation function $v : \mathcal{P} \to \wp(I)$ that is extended to compound formulas as follows (we define $i \Vdash \varphi \Leftrightarrow i \in v(\varphi)$):
  - $v(\neg\Phi) = I \setminus v(\Phi)$, $v(\Phi \wedge \Psi) = v(\Phi) \cap v(\Psi)$, $v(\Phi \vee \Psi) = v(\Phi) \cup v(\Psi)$
  - $v(\psi \Box\!\!\rightarrow \varphi) = \{i \in I \mid f(\psi, i) \subseteq v(\varphi)\}$;

Selection Functions - Dual Semantics

Definition

Global $\mathbf{C1}_g$ and local consequence $\mathbf{C1}_l$ are defined as usual over function models.

Proposition

$\mathbf{C1}_l$ and $\mathbf{C1}_g$ is sound and complete with respect to **C1**-function models:

Duality

The duality of counterfactual algebras can be investigated within the framework of Boolean algebras with operators.

Definition

For a finite counterfactual algebra $\mathbf{A} = \langle A, \wedge, \vee, \neg, \bot, \top, \Box\!\!\rightarrow \rangle$, we defined the corresponding Boolean algebra with operators: $\mathbf{A}^o = \langle A, \wedge, \vee, \neg, \bot, \top, \{\Box_a\}_{a \in A} \rangle$, one $\Box_a$ for each element $a \in A$, such that: for all $a, x \in A$

$$\Box_a x = a \Box\!\!\rightarrow x$$

Duality

For a finite counterfactual algebra $\mathbf{A}$, consider its corresponding Boolean Algebras with operators $\mathbf{A}^o$.

Definition

The dual relational structure of $\mathbf{A}^o$ is a tuple $\langle at(\mathbf{A}^o), \{R_a\}_{a \in A} \rangle$ where:

- $at(\mathbf{A}^o)$ is the set of atoms of $\mathbf{A}$
- for $a \in A$, $R_a = \{(x, y) \in at(\mathbf{A}^o) \times at(\mathbf{A}^o) \mid$ for all $w \in \mathbf{A}$, if $x \leq \Box_a w$ then $y \leq w\}$

Duality

For a finite counterfactual algebra $\mathbf{A}$, consider its corresponding Boolean Algebras with operators $\mathbf{A}^o$ and the relation structure of $\mathbf{A}^o$, $\langle at(\mathbf{A}^o), \{R_a\}_{a \in A} \rangle$.

Definition

From $\langle at(\mathbf{A}^o), \{R_a\}_{a \in A} \rangle$ we can define a function model $\langle at(\mathbf{A}^o), f \rangle$ where:

- $f : A \times at(\mathbf{A}) \to A$ such that, for $a \in A$ and $\alpha \in at(\mathbf{A})$, $f(a, \alpha) = R_a[\alpha]$ where $R_a[\alpha] = \{x \mid \alpha R_a x\}$

It is easy to prove that if the starting algebra is counterfactual algebra, then its dual relation structure is a function model.

Duality

Also the converse transformation is allowed:

Definition

For a function model $\langle I, f, v\rangle$, consider the Boolean algebra with operator $\langle \wp(I), \cup, \cap, \setminus, \{\Box_X\}_{X\subseteq I}, \emptyset, I\rangle$ where:

- $\Box_X Y = \{i \in I \mid f(X, i) \subseteq Y\}$

It is easy to prove that if the starting model is a function model, then its dual algebra is counterfactual algebra.

Conclusions

Thank You!