Bi-intermediate logics of trees and co-trees

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LATD 2022

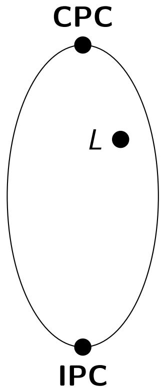
September 5, 2022

Overview

1. **bi-IPC**, bi-Heyting algebras and bi-Esakia spaces
2. The bi-intuitionistic linear calculus **bi-LC**
3. Jankov formulas for bi-Gödel algebras
4. Locally tabular extensions of **bi-LC**

Classical and intuitionistic propositional calculus

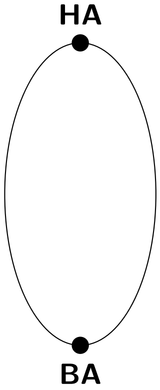
- **CPC** $:=$ classical propositional calculus.
- **IPC** $:=$ intuitionistic propositional calculus.
- **IPC** $\subseteq L \subseteq$ **CPC** is an intermediate logic.



The lattice $\Lambda($**IPC**$)$ of intermediate logics

Boolean and Heyting algebras

- **HA** := the class of Heyting algebras.
- **BA** := the class of Boolean algebras.



The lattice of non-trivial subvarieties of **HA**

Bi-intuitionistic propositional calculus

bi-IPC

The bi-intuitionistic propositional calculus **bi-IPC** is obtained by adding the binary connective $\leftarrow$ to the language of **IPC**, eight new axioms, and one additional inference rule.

The negation of a formula φ is defined by $\neg\varphi := \varphi \to \bot$.

The co-negation of a formula φ is defined by $\sim\varphi := \top \leftarrow \varphi$.

New axioms

- $p \to \big(q \vee (p \leftarrow q)\big)$
- $\neg(p \leftarrow q) \to (p \to q)$
- etc

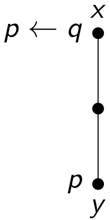
New inference rule

- Double negation: "from φ infer $\neg\sim\varphi$"

Bi-intuitionistic propositional calculus

The Kripke semantics of **bi-IPC** provides a transparent interpretation of co-implication:

$$\boxed{\mathfrak{M}, x \models p \leftarrow q \iff \exists y \leq x \, (\mathfrak{M}, y \models p \text{ and } \mathfrak{M}, y \not\models q).}$$

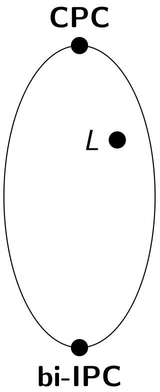


Bi-intermediate logics

- We have **bi-IPC** $\subsetneq$ **CPC**, since

$$(p \leftarrow q) \leftrightarrow (p \wedge \neg q) \in \textbf{CPC} \text{ and } p \vee \neg p \notin \textbf{bi-IPC}.$$

- **bi-IPC** $\subseteq L \subseteq$ **CPC** is a bi-intermediate logic.



The lattice $\Lambda(\textbf{bi-IPC})$ of bi-intermediate logics

Bi-Heyting algebras

Bi-Heyting algebra

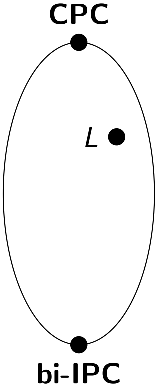
A Heyting algebra $\mathfrak{A} = (A, \vee, \wedge, \rightarrow, \leftarrow, 0, 1)$ equipped with an additional binary operation $\leftarrow$, satisfying, for all $a, b, c \in A$:

$$a \leftarrow b \leq c \iff a \leq b \vee c.$$

In other words, a Heyting algebra $\mathfrak{A}$ whose order-dual is also a Heyting algebra.

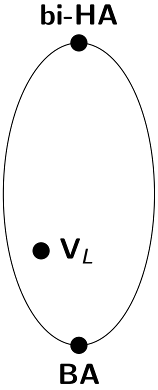
- **bi-HA** $:=$ the class of bi-Heyting algebras $= \{\mathfrak{A} \in \textbf{bi-HA} : \mathfrak{A} \models \textbf{bi-IPC}\}$.
- **bi-HA** $\models \varphi \iff$ **bi-IPC** $\vdash \varphi$.

Algebraic completeness of **bi-IPC**



The lattice Λ(**bi-IPC**) of bi-intermediate logics

$$\xrightarrow{\sim}_{op}$$



The lattice of non-trivial subvarieties of **bi-HA**

Bi-Esakia spaces

Bi-Esakia space

A topological space $\mathcal{X} = (X, \tau, \leq)$ equipped with a partial order satisfying:

- $\mathcal{X}$ is compact;
- PSA: $\forall x, y \in X \ \big(x \nleq y \implies \exists U \in \mathit{ClopUp}(\mathcal{X}) \ (x \in U \text{ and } y \notin U)\big)$;
- $\forall U \in \mathit{Clop}(\mathcal{X}) \ \big(\downarrow U \in \mathit{Clop}(\mathcal{X})\big)$;
- $\forall U \in \mathit{Clop}(\mathcal{X}) \ \big(\uparrow U \in \mathit{Clop}(\mathcal{X})\big)$.

Bi-Esakia morphisms

Bi-p-morphism

A map between posets $f\colon (X,\leq) \to (W,\leq)$ satisfying the following conditions:

- Order-preserving: $\forall x, y \in X \ \big(x \leq y \implies f(x) \leq f(y)\big)$;
- Up: $\forall x \in X, \forall w \in W \ \big(f(x) \leq w \implies \exists y \in {\uparrow}x \ (f(y) = w)\big)$;
- Down: $\forall x \in X, \forall w \in W \ \big(w \leq f(x) \implies \exists y \in {\downarrow}x \ (f(y) = w)\big)$.

Bi-Esakia morphism

A continuous bi-p-morphism $f\colon \mathcal{X} \to \mathcal{Y}$ between bi-Esakia spaces.

Bi-Esakia duality

Theorem (Esakia)

The categories of bi-Heyting algebras and bi-Esakia spaces are dually equivalent.

- Given $\mathfrak{A} \in \mathbf{bi\text{-}HA}$, we call $\mathfrak{A}_* := (\mathit{Spec}(\mathfrak{A}), \tau, \subseteq)$ the bi-Esakia dual of $\mathfrak{A}$.

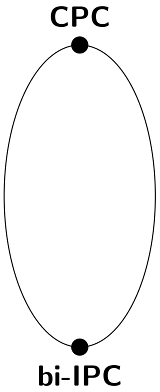
- Given a bi-Esakia space $\mathcal{X}$, we call $\mathcal{X}^* := (\mathit{ClopUp}(\mathcal{X}), \cup, \cap, \to, \leftarrow, \emptyset, X) \in \mathbf{bi\text{-}HA}$ the algebraic dual of $\mathcal{X}$, where

$$U \to V := X \setminus {\downarrow}(U \setminus V) \text{ and } U \leftarrow V := {\uparrow}(U \setminus V).$$

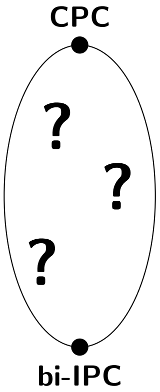
- For all $\mathfrak{A} \in \mathbf{bi\text{-}HA}$ and all formulas φ, we have

$$\mathfrak{A} \models \varphi \iff \mathfrak{A}_* \models \varphi.$$

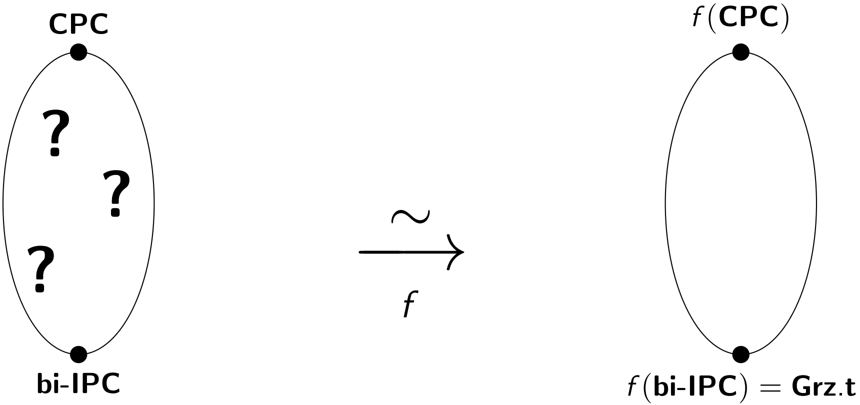
The lattice $\Lambda(\textbf{bi-IPC})$ of bi-intermediate logics



The lattice $\Lambda(\mathbf{bi\text{-}IPC})$ of bi-intermediate logics



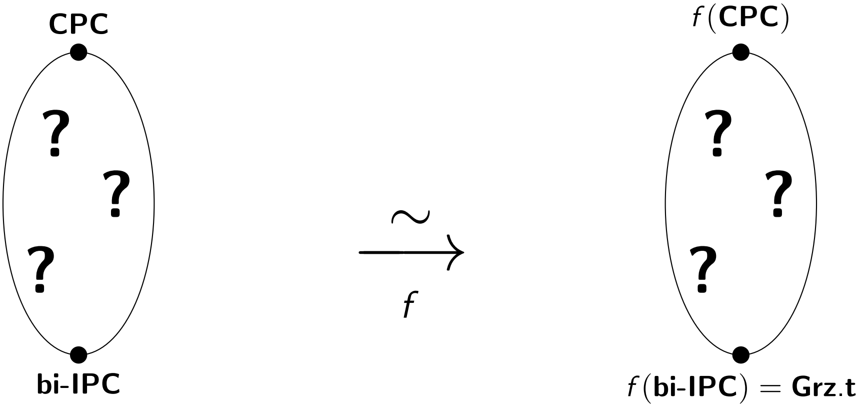
Wolter's translation



The lattice Λ(**bi-IPC**) of bi-intermediate logics

The lattice of consistent normal tense logics containing **Grz.t**

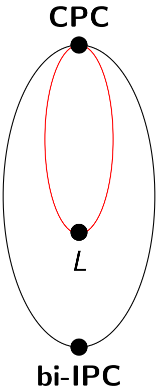
Wolter's translation



The lattice $\Lambda(\textbf{bi-IPC})$ of bi-intermediate logics

The lattice of consistent normal tense logics containing **Grz.t**

The sublattice $\Lambda(L)$



red $:= \Lambda(L) =$ the sublattice of consistent extensions of the bi-intermediate logic L.

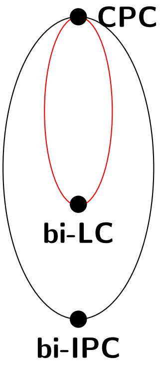
The bi-intuitionistic linear calculus bi-LC

Bi-intuitionistic linear calculus

bi-LC

The bi-intuitionistic linear calculus is the bi-intermediate logic axiomatized by Gödel's prelinearity axiom.

$$\mathbf{bi\text{-}LC} := \mathbf{bi\text{-}IPC} + (p \to q) \vee (q \to p).$$



Bi-intuitionistic linear calculus

bi-LC

The bi-intuitionistic linear calculus is the bi-intermediate logic axiomatized by the Gödel -Dummett axiom.

$$\mathbf{bi\text{-}LC} := \mathbf{bi\text{-}IPC} + (p \to q) \vee (q \to p).$$

- $\mathbf{LC} := \mathbf{IPC} + (p \to q) \vee (q \to p)$ has been thoroughly investigated.
- **bi-LC** is the (bi-intuitionistic) logic of co-trees.
- If $L \in \Lambda(\mathbf{bi\text{-}LC})$, then in the deductive system $\vdash_L$, we can reason by reductio ad absurdum, since

$$\Gamma \vdash_L \varphi \iff \Gamma \cup \{\sim\neg\sim\varphi\} \text{ is inconsistent.}$$

Bi-Gödel algebras

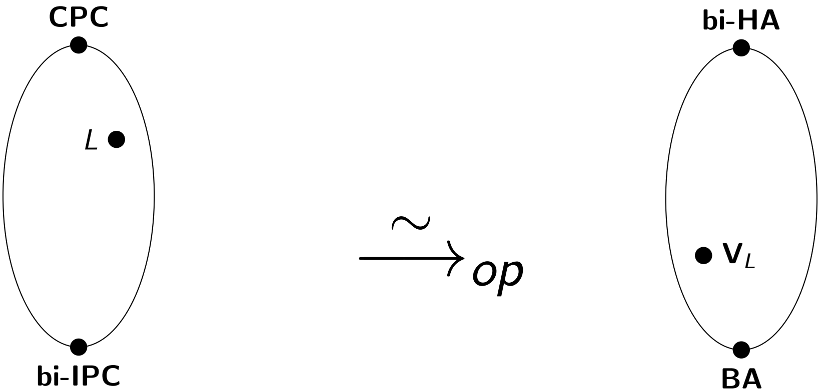
Bi-Gödel algebras

The bi-Heyting algebras axiomatized by $\mathbf{bi\text{-}LC} = \mathbf{bi\text{-}IPC} + (p \to q) \vee (q \to p)$.

$$\mathbf{bi\text{-}GA} := \mathbf{V}_{\mathbf{bi\text{-}LC}} = \{\mathfrak{A} \in \mathbf{bi\text{-}HA} : \mathfrak{A} \models \mathbf{bi\text{-}LC}\}.$$

bi-GA is a semi-simple discriminator variety with EDPC.

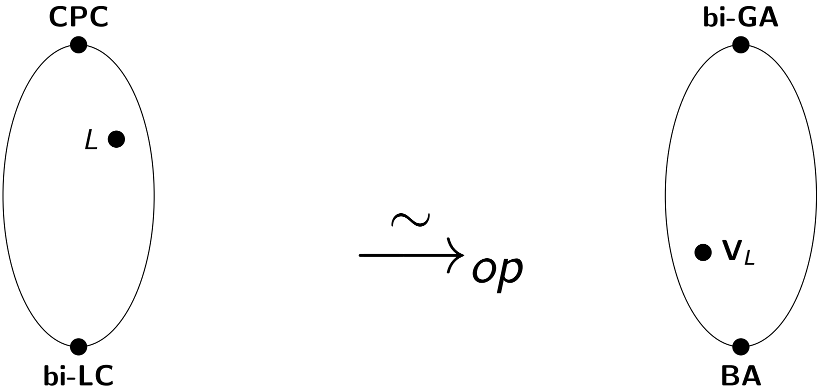
Algebraic completeness of **bi-IPC**



The lattice Λ(**bi-IPC**) of bi-intermediate logics

The lattice of non-trivial subvarieties of **bi-HA**

Algebraic completeness of **bi-LC**



The lattice Λ(**bi-LC**) of consistent extensions of **bi-LC**

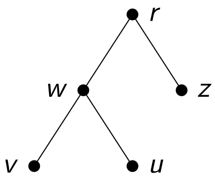
The lattice of non-trivial subvarieties of **bi-GA**

Co-trees and co-forests

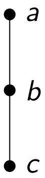
In a poset $(W, \leq)$, sets of the form $\uparrow w := \{u \in W : w \leq u\}$ are called principal upsets.

Co-tree

A poset $(W, \leq)$ with a greatest element, called the co-root, and whose principal upsets are chains.



The co-tree $\mathfrak{F}_1$

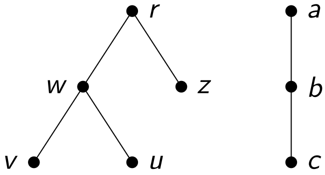


The co-tree $\mathfrak{F}_2$

Co-trees and co-forests

Co-forest

A disjoint union of co-trees.



The co-forest $\mathfrak{F} = \mathfrak{F}_1 \uplus \mathfrak{F}_2$

Bi-Gödel algebras and co-forests

Bi-Gödel algebras

The bi-Heyting algebras axiomatized by **bi-LC** = **bi-IPC** + $(p \to q) \vee (q \to p)$.

$$\mathbf{bi\text{-}GA} := \mathbf{V}_{\mathbf{bi\text{-}LC}} = \{\mathfrak{A} \in \mathbf{bi\text{-}HA} \colon \mathfrak{A} \models \mathbf{bi\text{-}LC}\}.$$

Theorem

- $\mathfrak{A}$ *is a bi-Gödel algebra* $\iff$ $\mathfrak{A}_*$ *is a bi-Esakia co-forest.*
- $\mathfrak{A}$ *is a subdirectly irreducible (SI) bi-Gödel algebra* $\iff$ $\mathfrak{A}_*$ *is a bi-Esakia co-tree.*

Finite model property

Theorem

bi-LC *has the finite model property, that is, a formula* φ *is a theorem of* **bi-LC** *iff* φ *is valid in all finite (bi-Esakia) co-trees.*

The proof of this theorem makes crucial use of the fact that the **HA**-reduct of **bi-GA**,

$$\mathbf{GA} := \{\mathfrak{A} \in \mathbf{HA} \colon \mathfrak{A} \models \mathbf{IPC} + (p \to q) \vee (q \to p)\}$$

is locally finite, i.e., that every finitely generated Gödel algebra is finite.

Jankov formulas for bi-Gödel algebras

Jankov formulas

If $\mathfrak{A} \in$ **bi-GA** is finite and SI, then the Jankov formula of $\mathfrak{A}$ encodes the full bi-Heyting structure of $\mathfrak{A}$.

$$\boxed{\mathcal{J}(\mathfrak{A}) := \neg \sim\Gamma \to \neg \bigwedge \left\{p_a \leftarrow p_b \colon a, b \in A \text{ and } a \nleq b\right\}}$$

$$\begin{aligned}\Gamma := \bigwedge\{p_{a\vee b} \leftrightarrow (p_a \vee p_b) \colon (a,b) \in A^2\} \wedge \bigwedge\{p_{a\wedge b} \leftrightarrow (p_a \wedge p_b) \colon (a,b) \in A^2\} \wedge \\ \bigwedge\{p_{a\to b} \leftrightarrow (p_a \to p_b) \colon (a,b) \in A^2\} \wedge \bigwedge\{p_{a\leftarrow b} \leftrightarrow (p_a \leftarrow p_b) \colon (a,b) \in A^2\} \wedge \\ \wedge \{p_0 \leftrightarrow \bot\} \wedge \{p_1 \leftrightarrow \top\}.\end{aligned}$$

Jankov formulas

Jankov Lemma

Let $\mathfrak{A}, \mathfrak{B} \in$ **bi-GA**. If $\mathfrak{A}$ is finite and SI, then

$$\mathfrak{B} \not\models \mathcal{J}(\mathfrak{A}) \iff \mathfrak{A} \in \mathbb{SH}(\mathfrak{B}).$$

Consequently, if **V** is a variety of bi-Gödel algebras, then

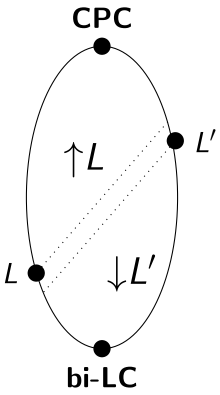
$$\mathbf{V} \not\models \mathcal{J}(\mathfrak{A}) \iff \mathfrak{A} \in \mathbf{V}.$$

In other words, $\mathcal{J}(\mathfrak{A})$ axiomatizes (relative to **bi-LC**) the least variety of bi-Gödel algebras that does not contain $\mathfrak{A}$.

Applications of Jankov formulas

Splitting logics of $\Lambda(\textbf{bi-LC})$

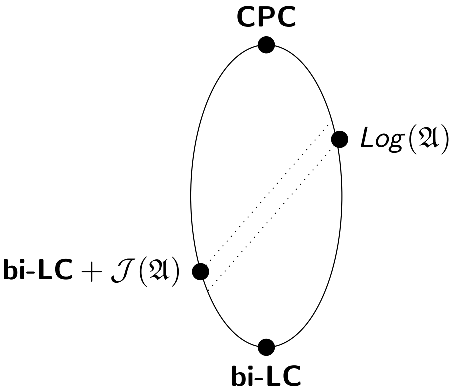
Given $L, L' \in \Lambda(\textbf{bi-LC})$, we call (L, L') a splitting pair for $\Lambda(\textbf{bi-LC})$ if $\Lambda(\textbf{bi-LC}) = {\uparrow}L \uplus {\downarrow}L'$. In this case, we say that L is a splitting logic of $\Lambda(\textbf{bi-LC})$.



Applications of Jankov formulas

Splitting Theorem

L is a splitting logic of $\Lambda(\textbf{bi-LC}) \iff L$ is axiomatized by a single Jankov formula.



Applications of Jankov formulas

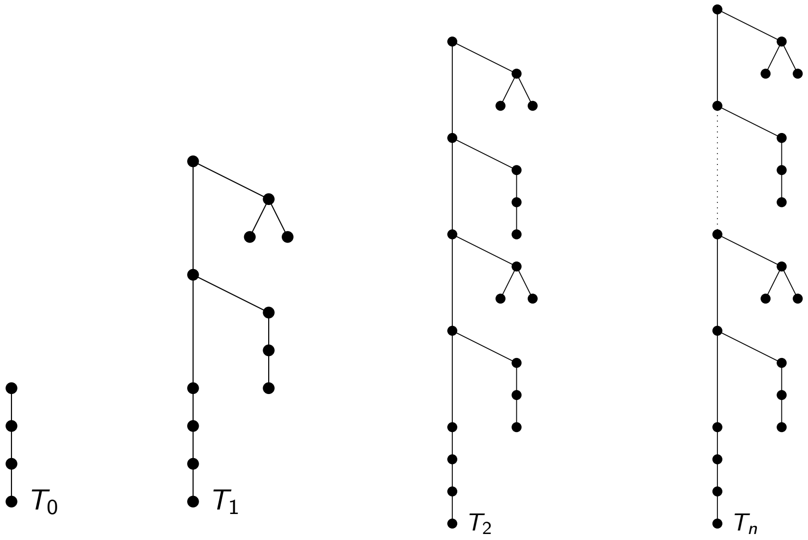
Theorem

The cardinality of $\Lambda(\mathbf{bi\text{-}LC})$ *is* $2^{\aleph_0}$.

This result can be derived from the existence of a countably infinite $\leq$-antichain of finite co-trees, where

$$\mathfrak{F} \leq \mathfrak{G} \iff \exists f : \mathfrak{G} \twoheadrightarrow \mathfrak{F}.$$

Applications of Jankov formulas



Locally tabular extensions of bi-LC

Local tabularity and the finite combs

Locally tabular

A logic $L \in \Lambda(\textbf{bi-LC})$ is *locally tabular* iff $\mathbf{V}_L = \{\mathfrak{A} \in \textbf{bi-GA} \colon \mathfrak{A} \models L\}$ is *locally finite*, that is, if every finitely generated algebra of $\mathbf{V}_L$ is finite.

Local tabularity and the finite combs

Locally tabular

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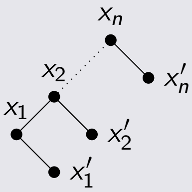
Finite combs

For each positive $n \in \omega$, we define the *n*-comb as the co-tree $\mathfrak{C}_n := (C_n, \leq_n)$ depicted below.



A criterion for local tabularity

Theorem

If $L \in \Lambda(\mathbf{bi\text{-}LC})$, then L is locally tabular $\iff$ $\mathcal{J}(\mathfrak{C}_n^*) \in L$, for some positive $n \in \omega$.

Finite combs

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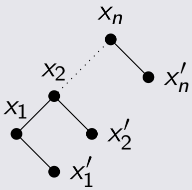

Lemma 1

Lemma 1

Let $\mathfrak{A} \in$ **bi-GA**. For all positive $n \in \omega$, we have

$$\mathfrak{A}_* \text{ does not admit } \mathfrak{C}_n \text{ as a subposet} \iff \mathfrak{A} \models \mathcal{J}(\mathfrak{C}_n^*).$$

While the proof of $\implies$ is straightforward, the reverse implication requires a very lengthy and technical proof.

Lemma 2

Lemma 2

If a variety $\mathbf{V} \subseteq \mathbf{bi\text{-}GA}$ contains all the algebraic duals of the finite combs, then $\mathbf{V}$ is not locally finite.

We proved that for each positive $n \in \omega$, $\mathfrak{C}_n^*$ is 1-generated as a bi-Gödel algebra.

Since there are arbitrarily large algebraic duals of finite combs contained in $\mathbf{V}$, the 1-generated free $\mathbf{V}$-algebra must be infinite, and therefore $\mathbf{V}$ cannot be locally finite. $\square$

Lemma 3

Lemma 3

For all positive $n, m \in \omega$, there exists $k(n, m) \in \omega$ such that if $\mathfrak{A} \in \mathbf{bi\text{-}GA}_{SI}$, then

$$\mathfrak{A} \text{ is } m\text{-generated and } \mathfrak{A}_* \text{ does not admit } \mathfrak{C}_n \text{ as a subposet} \implies |\mathfrak{A}| \leq k(n, m).$$

The first step is finding a bound for the depth of $\mathfrak{A}_*$.

As $\mathfrak{A}$ is an m-generated bi-Gödel algebra, there are distinguished upsets of $\mathfrak{A}_*$, $B_1, \ldots, B_{m+1}$, such that a bound for the depth of $\mathfrak{A}_*$ can be derived from a bound for the depths of the B_i.

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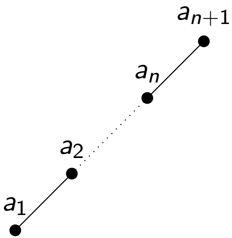
Let $i \leq m+1$ and suppose there exists a chain $a_1 < a_2 < \cdots < a_{n+1} \in B_i$.



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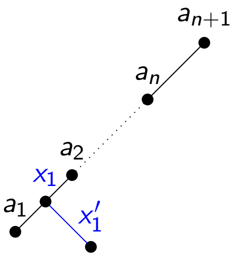
B_i is a distinguished upset.



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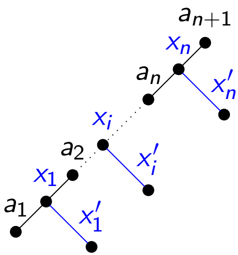

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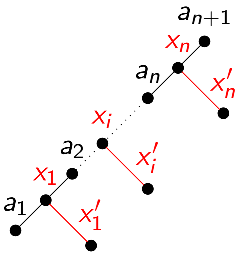

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Therefore, there is no chain of size $n + 1$ in B_i, i.e., $dp(B_i) \leq n$, as desired.

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Lemma 1

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For all positive $n, m \in \omega$, there exists $k(n,m) \in \omega$ such that if $\mathfrak{A} \in \mathbf{bi\text{-}GA}_{SI}$, then

$$\mathfrak{A} \text{ is } m\text{-generated and } \mathfrak{A} \models \mathcal{J}(\mathfrak{C}_n^*) \implies |\mathfrak{A}| \leq k(n,m).$$

Lemma 1

Let $\mathfrak{A} \in \mathbf{bi\text{-}GA}$. For all positive $n \in \omega$, we have

$$\mathfrak{A}_* \text{ does not admit } \mathfrak{C}_n \text{ as a subposet} \iff \mathfrak{A} \models \mathcal{J}(\mathfrak{C}_n^*).$$

A criterion for local tabularity

Theorem

If $L \in \Lambda(\mathbf{bi\text{-}LC})$, then L is locally tabular $\iff \mathcal{J}(\mathfrak{C}_n^*) \in L$, for some positive $n \in \omega$.

A criterion for local tabularity

Theorem

If $L \in \Lambda(\mathbf{bi\text{-}LC})$, then L is locally tabular $\iff$ $\mathcal{J}(\mathfrak{C}_n^*) \in L$, for some positive $n \in \omega$.

$\boxed{\Longrightarrow}$ We prove the contrapositive. Suppose $\mathcal{J}(\mathfrak{C}_n^*) \notin L$, for all positive $n \in \omega$.

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$\boxed{\Longrightarrow}$ We prove the contrapositive. Suppose $\mathcal{J}(\mathfrak{C}_n^*) \notin L$, for all positive $n \in \omega$.

By the Jankov Lemma, we have $\mathfrak{C}_n^* \in \mathbf{V}_L$ for all positive $n \in \omega$.

Jankov Lemma

$$\mathcal{J}(\mathfrak{A}) \notin L \iff \mathbf{V}_L \not\models \mathcal{J}(\mathfrak{A}) \iff \mathfrak{A} \in \mathbf{V}_L$$

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Theorem

If $L \in \Lambda(\textbf{bi-LC})$, then L is locally tabular $\iff$ $\mathcal{J}(\mathfrak{C}_n^*) \in L$, for some positive $n \in \omega$.

$\boxed{\Longrightarrow}$ We prove the contrapositive. Suppose $\mathcal{J}(\mathfrak{C}_n^*) \notin L$, for all positive $n \in \omega$.

By the Jankov Lemma, we have $\mathfrak{C}_n^* \in \mathbf{V}_L$ for all positive $n \in \omega$.

By Lemma 2, $\mathbf{V}_L$ is not locally finite, i.e., L is not locally tabular, as desired. $\square$

Lemma 2

If a variety $\mathbf{V} \subseteq \textbf{bi-GA}$ contains all the algebraic duals of the finite combs, then $\mathbf{V}$ is not locally finite.

A criterion for local tabularity

Theorem

If $L \in \Lambda(\mathbf{bi\text{-}LC})$, then L is locally tabular $\iff$ $\mathcal{J}(\mathfrak{C}_n^*) \in L$, for some positive $n \in \omega$.

$\boxed{\Longleftarrow}$ Suppose $\mathcal{J}(\mathfrak{C}_n^*) \in L$, for some positive $n \in \omega$. By duality, $\mathbf{V}_L \models \mathcal{J}(\mathfrak{C}_n^*)$.

A criterion for local tabularity

Theorem

If $L \in \Lambda(\mathbf{bi\text{-}LC})$, then L is locally tabular $\iff$ $\mathcal{J}(\mathfrak{C}_n^*) \in L$, for some positive $n \in \omega$.

$\boxed{\Longleftarrow}$ Suppose $\mathcal{J}(\mathfrak{C}_n^*) \in L$, for some positive $n \in \omega$. By duality, $\mathbf{V}_L \models \mathcal{J}(\mathfrak{C}_n^*)$.

In particular, for each positive $m \in \omega$, if $\mathfrak{A} \in \mathbf{V}_L$ is SI and m-generated, then $\mathfrak{A} \models \mathcal{J}(\mathfrak{C}_n^*)$.

A criterion for local tabularity

Theorem

If $L \in \Lambda(\mathbf{bi\text{-}LC})$, then L is locally tabular $\iff \mathcal{J}(\mathfrak{C}_n^*) \in L$, for some positive $n \in \omega$.

$\boxed{\Longleftarrow}$ Suppose $\mathcal{J}(\mathfrak{C}_n^*) \in L$, for some positive $n \in \omega$. By duality, $\mathbf{V}_L \models \mathcal{J}(\mathfrak{C}_n^*)$.

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By Lemma 3, $|\mathfrak{A}| \leq k(n, m)$.

Lemma 3

For all positive $n, m \in \omega$, there exists $k(n, m) \in \omega$ such that if $\mathfrak{A} \in \mathbf{bi\text{-}GA}_{SI}$, then

$$\mathfrak{A} \text{ is } m\text{-generated and } \mathfrak{A} \models \mathcal{J}(\mathfrak{C}_n^*) \implies |\mathfrak{A}| \leq k(n, m).$$

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Theorem

If $L \in \Lambda(\mathbf{bi\text{-}LC})$, then L is locally tabular $\iff \mathcal{J}(\mathfrak{C}_n^*) \in L$, for some positive $n \in \omega$.

$\boxed{\Longleftarrow}$ Suppose $\mathcal{J}(\mathfrak{C}_n^*) \in L$, for some positive $n \in \omega$. By duality, $\mathbf{V}_L \models \mathcal{J}(\mathfrak{C}_n^*)$.

In particular, for each positive $m \in \omega$, if $\mathfrak{A} \in \mathbf{V}_L$ is SI and m-generated, then $\mathfrak{A} \models \mathcal{J}(\mathfrak{C}_n^*)$.

By Lemma 3, $|\mathfrak{A}| \leq k(n, m)$. As m and $\mathfrak{A}$ were arbitrary, the Theorem below yields that L is locally tabular. □

Theorem

A variety $\mathbf{V}$ (of a finite type) is locally finite iff

$$\forall m \in \omega, \exists k(m) \in \omega, \forall \mathfrak{A} \in \mathbf{V}_{SI} \ (\mathfrak{A} \text{ is } m\text{-generated} \implies |\mathfrak{A}| \leq k(m)).$$

A criterion for local tabularity

Theorem

If $L \in \Lambda(\textbf{bi-LC})$, then L is locally tabular $\iff$ $\mathcal{J}(\mathfrak{C}_n^*) \in L$, for some positive $n \in \omega$.

Corollary

If $L \in \Lambda(\textbf{bi-LC})$, then L is locally tabular $\iff$ $L \nsubseteq Log(\text{Finite combs})$.

Consequently, the logic of the finite combs is the only pre-locally tabular extension of **bi-LC**, i.e., $Log(\text{Finite combs})$ is not locally tabular, but all of its proper extensions are so.

LC VS bi-LC

LC $=$ **IPC** $+ (p \to q) \vee (q \to p)$	**bi-LC** $=$ **bi-IPC** $+ (p \to q) \vee (q \to p)$
$\mathfrak{A} \in \mathbf{GA}_{SI} \iff \mathfrak{A}_*$ is a SRC	$\mathfrak{A} \in \mathbf{bi\text{-}GA}_{SI} \iff \mathfrak{A}_*$ is a co-tree
LC has the FMP	**bi-LC** has the FMP
LC is locally tabular	**bi-LC** is not locally tabular
All extensions of **LC** have the FMP	??
$\Lambda($**LC**$)$ is a chain of order-type $(\omega + 1)^\partial$	$\Lambda($**bi-LC**$)$ is of size $2^{\aleph_0}$ and is not a chain

Thank You!