Polyhedral Completeness in Intermediate and Modal Logics: Convex Polyhedra

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Logic Algebra and Truth Degrees 2022

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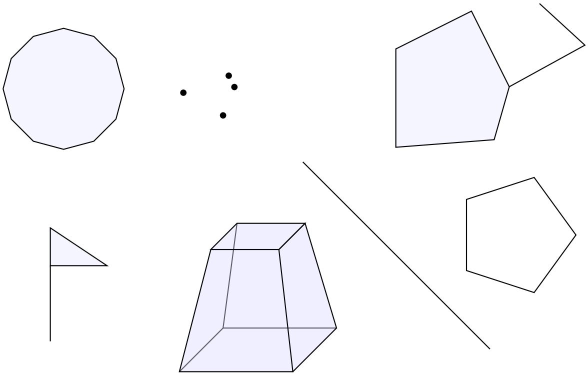
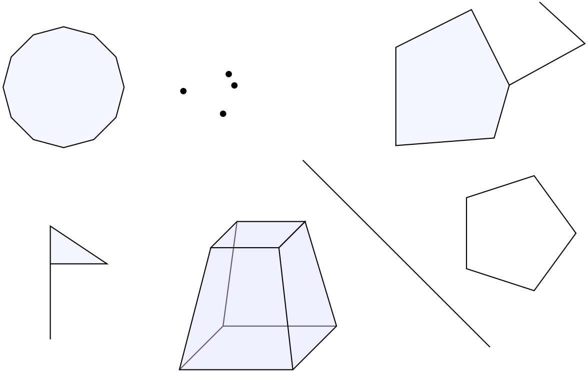
Polyhedra



- Polyhedra can be of any dimension, and need not be convex nor connected.

Polyhedra



- Polyhedra can be of any dimension, and need not be convex nor connected.
- Our polyhedra are always compact.
- Formally: finite unions of convex hulls of finite sets.

Polyhedral Semantics

Definition

Let P be a polyhedron. An *open subpolyhedron* of P is the complement in P of a subpolyhedron. $\mathrm{Sub_o}P$ is the set all of open subpolyhedra.

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Theorem (Bezhanishvili, Marra, Mcneill and Pedrini[1])

$\mathrm{Sub_o}P$ *is a Heyting algebra.*

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Theorem (Bezhanishvili, Marra, Mcneill and Pedrini)

Polyhedra provide a complete semantics for **IPC**.

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Convex Polyhedra

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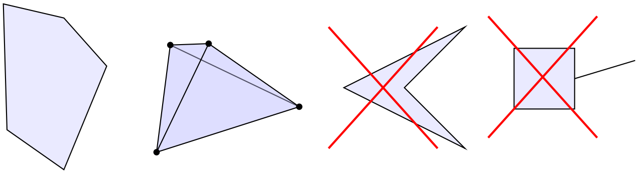
A polyhedron P is *convex* if whenever $x, y \in P$, the straight line from x to y is also in P.



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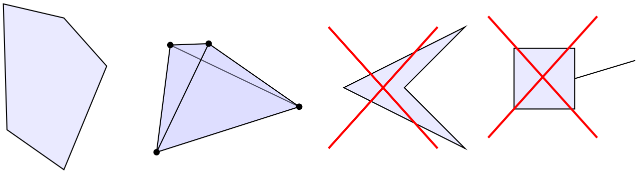
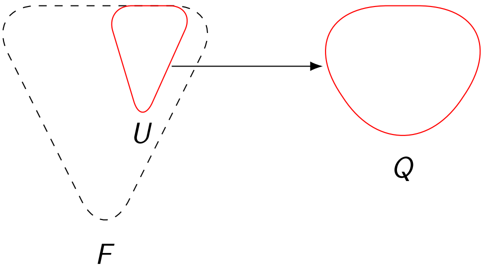


- What is the logic of the class of convex polyhedra?
- We give an axiomatisation.

Jankov-Fine Formulas

Theorem

For every finite frame Q, there is a formula $\chi(Q)$, the Jankov-Fine formula of Q, such that for any frame F, we have $F \not\models \chi(Q)$ if and only if there is $U \subseteq F$ upwards closed and a surjective p-morphism $U \to Q$.



Encodes the idea of ‘forbidden configurations’.

Main Theorem

Theorem (Adam-Day, Bezhanishvili, Gabelaia, Marra)

The logic of convex polyhedra is axiomatised by:

$\mathbf{PL} = \mathbf{IPC} + \chi($  $) + \chi($  $)$

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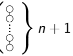
$\mathbf{PL}_n = \mathbf{IPC} + \chi\Big($  $\Big) + \chi($  $) + \chi($  $)$

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Lemma (Zakharyaschev 1993[2])

PL *has the finite model property.*

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Soundness

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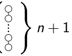
$\mathbf{PL}_n = \mathbf{IPC} + \chi\Big($  $\Big) + \chi($  $) + \chi($  $)$

Theorem (Soundness)

$\mathbf{PL}_n$ *is valid on every convex polyhedron of dimension at most* n*.*

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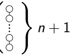
$\mathbf{PL}_n = \mathbf{IPC} + \chi\Big($$\Big) + \chi($$) + \chi($$)$

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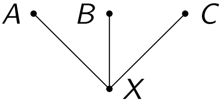
Proof idea.

For $\chi($$)$ we show that no convex polyhedron be partitioned into non-empty sets A, B, C, X such that A, B, C are open subpolyhedra and $X = \overline{A} \cap \overline{B} \cap \overline{C}$.



There is also a combinatorial argument for this fact. □

Completeness: Combinatorial Step I

Theorem (Completeness)

If $\mathbf{PL}_n \nvDash \phi$ then there is P convex, n-dimensional with $P \nvDash \phi$.

- The proof has two steps: combinatorial and geometric.

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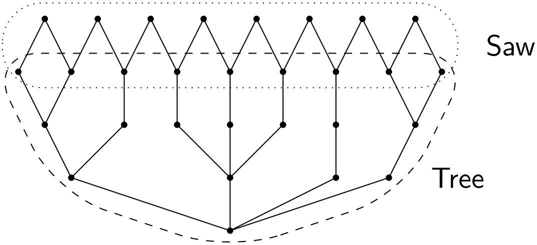
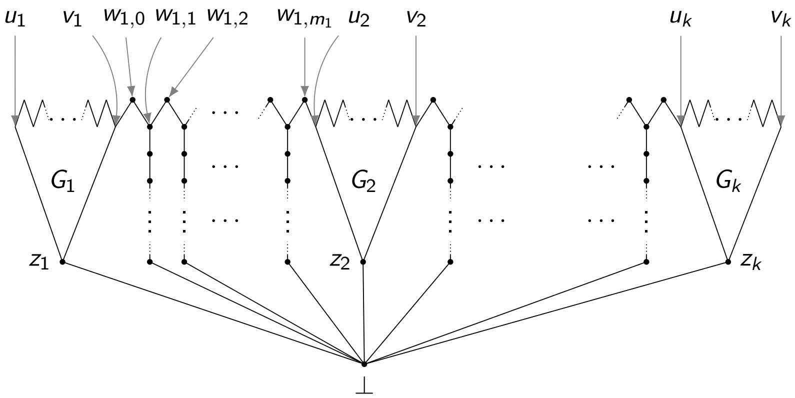
Theorem (Completeness)

If $\mathbf{PL}_n \nvDash \phi$ then there is P convex, n-dimensional with $P \nvDash \phi$.

- The proof has two steps: combinatorial and geometric.
- First step: give a nice general form for the finite posets of $\mathbf{PL}_n$.
- Each such frame is the p-morphic image of a *sawed tree*.



Completeness: Combinatorial Step II

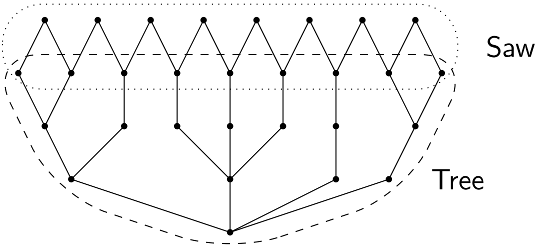


Completeness: Convex Geometric Realisation

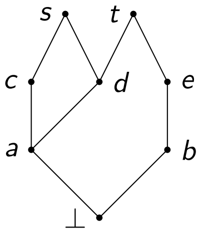
- Show that every height-n sawed tree has a geometric realisation in a dimension-n convex polyhedron.

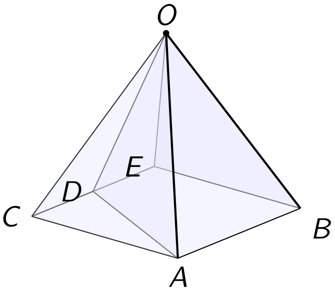
Completeness: Convex Geometric Realisation

- Show that every height-n sawed tree has a geometric realisation in a dimension-n convex polyhedron.
- Use the standard construction to realise the tree part.
- Then realise the top nodes as n-dimensional polyhedra to glue things up into a convex polyhedron.



A 3-dimensional Example





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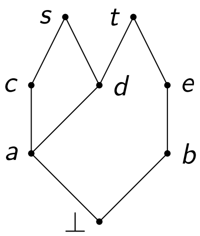
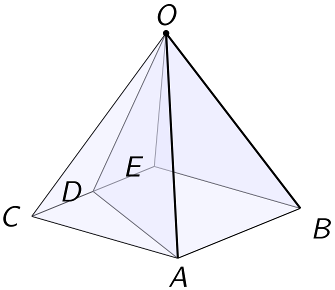
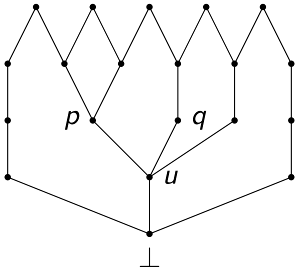
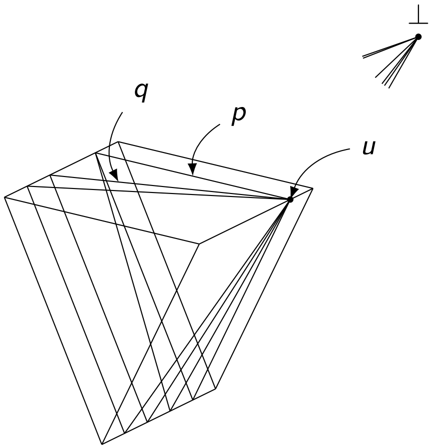




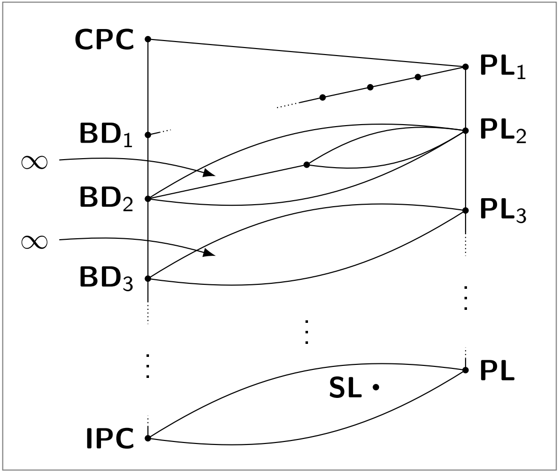
$$\{O\} \mapsto \bot, \quad OA \setminus \{O\} \mapsto a, \quad OB \setminus \{O\} \mapsto b,$$
$$OAC \setminus OA \mapsto c, \quad OAD \setminus OA \mapsto d, \quad OBE \setminus OB \mapsto d,$$
$$OACD \setminus (OAC \cup OAD) \mapsto s, \quad OABED \setminus (OAD \cup OBE) \mapsto t$$

A 4-dimensional Example





Conclusion



Thanks for listening!

Adam-Day, S., N. Bezhanishvili, D. Gabelaia and V. Marra (2021). *Polyhedral completeness of intermediate logics: the Nerve Criterion*. DOI: 10.48550/ARXIV.2112.07518. URL: https://arxiv.org/abs/2112.07518.

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