One-sorted Program Algebras[1]

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Kleene algebra

Definition

A Kleene algebra [Koz94] is a structure $\mathscr{K} = (K, \vee, \cdot, {}^*, 1, 0)$ such that

- $(K, \vee, \cdot, 1, 0)$ is an *idempotent semiring*, i.e.,
  - $(K, \cdot, 1)$ is a monoid,
  - $(K, \vee, 0)$ is an idempotent commutative monoid (hence, a join-semilattice),
  - $x(y \vee z) = xy \vee xz$, $(y \vee z)x = yx \vee zx$, and
  - $x0 = 0 = 0x$, and
- ${}^* : K \to K$ such that

$$1 \vee a \vee a^*a^* \leq a^* \qquad ax \leq x \Rightarrow a^*x \leq x \qquad xa \leq x \Rightarrow xa^* \leq x$$

Examples: Kleene algebras of regular languages, Kleene algebra of paths...

Examples

Example

The relational Kleene algebra over a set X is $\mathscr{R}(X) = (2^{X \times X}, \cup, \circ, {}^*, \mathrm{id}, \emptyset)$;

- $\circ$ denotes composition, and
- $R^* = \bigcup_{i \geq 0} R^i$, where $R^0 = \mathrm{id}$ and $R^{i+1} = R \circ R^i$.

Example

The tropical Kleene algebra is defined over $0 > -1 > -2 > \cdots > -\omega$, where

- $+$ is multiplication,
- 0 is the *multiplicative* unit, and
- $-\omega$ the $\vee$ unit.

Kleene algebra with tests

Definition

A Kleene algebra with tests [Koz97] is $\mathscr{B} = (K, B, \vee, \cdot, ^*, 1, 0, ^-)$ where

- $(K, \vee, \cdot, ^*, 1, 0)$ is a Kleene algebra
- $B \subseteq K$
- $(B, \vee, \cdot, ^-, 1, 0)$ is a Boolean algebra.

Prop. Every KA is a KAT, where the test subalgebra is $B = \{0, 1\}$.

Examples

Example

The relational KAT over a set X is $\mathscr{R}(X)$ together with the Boolean test subalgebra 2^{id}.

Prop. [KS97] The equational theory of KAT is identical with the equational theory of rKAT.

Example

The only possible test subalgebra of the tropical Kleene algebra is $\{-\omega, 0\}$.

Propositional while programs

$$\text{Tests} \quad \beta := \mathsf{b} \mid \bar{\beta} \mid \beta \wedge \beta \mid \beta \vee \beta$$

$$\text{Programs} \quad \pi := \textbf{skip} \mid \mathsf{p} \mid \pi;\pi \mid \textbf{if } \beta \textbf{ then } \pi \textbf{ else } \pi \mid \textbf{while } \beta \textbf{ do } \pi$$

In KAT:

$$\textbf{skip} := b \vee \overline{b}$$
$$\textbf{if } b \textbf{ then } p \textbf{ else } q := (bp) \vee (\overline{b}q)$$
$$\textbf{while } b \textbf{ do } p := (bp)^*\overline{b}$$

Partial correctness: $bp = bpc$.

Kleene algebra with domain

Definition

A Kleene algebra with domain [DS11] is $\mathscr{D} = (K, \vee, \cdot, ^*, 1, 0, d)$ where $d : K \to K$ such that:

$$x = d(x)x$$
$$d(xy) = d(xd(y))$$
$$d(x) \leq 1$$
$$d(0) = 0$$
$$d(x \vee y) = d(x) \vee d(y)$$

(Similarly codomain c with $x \leq xc(x)$ and $c(xy) = c(c(x)y)$.)

Prop. $(d(K), \vee, \cdot, 1, 0)$ and $(c(K), \vee, \cdot, 1, 0)$ are bounded distr. lattices.

Open Prob. When is $(d(K), \vee, \cdot, 1, 0)$ a Heyting algebra?

Example

Example

Extend a relational Kleene algebra with

$$d(R) = \{(s, s) \mid \exists t.(s, t) \in R\}.$$

Intuitively, $d(x)$ should be the least left preserver of x under 1:

$$\text{if } y \leq 1, \text{ then } x \leq yx \iff d(x) \leq y \tag{1}$$

The equational theory of domain semirings (delete * and the corresponding axioms from KAD) coincides with the equational theory of relation algebras in the signature $(\cup, \circ, \emptyset, \mathrm{id}, d)$ [McL20].

Open Prob. What about the full signature with *?

Kleene algebra with antidomain

Definition

A Kleene algebra with antidomain [DS11] is $\mathscr{A} = (K, \vee, \cdot, ^*, 1, 0, a)$ where $a : K \to K$ such that

$$a(x)x = 0$$
$$a(xy) = a(xa^2(y))$$
$$a^2(x) \vee a(x) = 1$$

A domain operation is then defined by $d(x) := a^2(x)$.

Prop. $(d(K), \vee, \cdot, 1, 0)$ is a Boolean algebra where $a(x)$ is the complement of $x \in d(K)$.

Thm. The domain subalgebra of a KAAD is the maximal Boolean subalgebra of the semiring of elements $x \leq 1$.

Kleene algebra with (anti)domain

Example

Take a relational Kleene algebra and define

$$a(R) = \{(s,s) \mid \neg\exists t.(s,t) \in R\},$$

then $a(a(R)) = d(R)$.

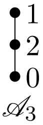
- The equational theory of relation algebras in the $(\circ, a, \emptyset, \mathrm{id})$ signature has been finitely axiomatized by equations [Hol97]
- The equational theory of KAAD is EXPTIME [MS06]

Thm. The equational theory of KAAD with $*$-*continuity* and *seperability* is EXPTIME-complete

Conj. The equational theory of KAAD coincides with that of rKAAD and the class above

Problem

However: **Prop.** Some finite KA cannot be extended with a domain operation.



$\cdot$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	0

$*$	0	1	2
	1	1	1

The culprit is the locality axiom $d(xy) = d(xd(y))$.

Can one find a one-sorted alternative ALT to KAT that satisfies

1. ALT expands Kleene algebras by additional operations t and t'.
2. Every Kleene algebra extends to an ALT.
3. The test algebra $t(\mathscr{A})$ need not be the maximal Boolean subalgebra of elements $x \leq 1$.
4. The equational theory of KAT embeds into the equational theory of ALT.

One-sorted Kleene algebras with tests

Definition

A KAt is $\mathscr{K} = (K, \vee, \cdot, ^*, 1, 0, t, t')$ where $t, t' : K \to K$ such that

$$t(0) = 0 \tag{2}$$

$$t(1) = 1 \tag{3}$$

$$t(t(x) \vee t(y)) = t(x) \vee t(y) \tag{4}$$

$$t(t(x)t(y)) = t(x)\, t(y) \tag{5}$$

$$t(x)t(x) = t(x) \tag{6}$$

$$t(x) \leq 1 \tag{7}$$

$$1 \leq t'(t(x)) \vee t(x) \tag{8}$$

$$t'(t(x))\, t(x) \leq 0 \tag{9}$$

$$t'(t(x)) = t(t'(t(x))) \tag{10}$$

Examples

Example

Relational Kleene algebra with $t := d$ and $t' = a$.

Theorem 1

Every KAT $\mathscr{K} = (K, B, \vee, \cdot, ^*, 1, 0, ^-)$ *expands to a KAt* $\mathscr{K} = (K, \vee, \cdot, ^*, 1, 0, t, t')$, *i.e.,* $B = t(K)$.

In particular, we take

$$t(x) = \begin{cases} x & \text{if } x \in B \\ 1 & \text{otherwise.} \end{cases} \qquad t'(x) = \begin{cases} \bar{x} & \text{if } x \in B \\ x & \text{otherwise.} \end{cases}$$

Prop. Every Kleene algebra extends to a KAt, so KAt is a conservative extension of KAt.

Embedding result

Theorem 2

We show that equational theory of KAT embeds into the equational theory of ALT provided that

- *ALT expands Kleene algebras by additional operations t and t'.*
- *The test algebras $t(\mathscr{A})$ is a Boolean algebra for each ALT $\mathscr{A}$.*
- *Every KAT expands to an ALT.*

Proof sketch.
$Tr(\mathsf{p_n}) = \mathsf{x}_{2n}$, $Tr(\mathsf{b_n}) = t(\mathsf{x}_{2n+1})$, $Tr(\bar{b}) = t'(Tr(b))$;
Tr commutes with 1, 0, $\cdot$, $\vee$ and *.

If KAT $\not\models p \approx q$, which give rise to an expansion. Conversely, each ALT induces a Boolean algebra of tests, and hence a KAT. $\square$

Prop. The equational theory of KAT embeds into that of KAt.

So, KAt exhibits all the required properties (1-4).

Embedding result 2

Theorem 3

We show that equational theory of KAT embeds into the equational theory of ALT provided that

- *ALT expands Kleene algebras by additional operations t and t'.*
- *The test algebras $t(\mathscr{A})$ is a Boolean algebra for each ALT $\mathscr{A}$.*
- *Every relational KAT expands to an ALT.*

Proof sketch.

As before...

If KAT $\not\models p \approx q$, then $p \approx q$ fails in an rKAT, which gives rise to an expansion. Conversely, each ALT induces a Boolean algebra of tests, and hence a KAT. □

Prop. The equational theory of KAT embeds into that of KAD, but properties 2 and 3 fail.

strong KAt

Extending KAt with all of the following axioms retains properties (1-4)

$$t(x \vee y) = t(x) \vee t(y) \tag{11}$$

$$x \leq t(x)x \tag{12}$$

$$t(t(x)y) \leq t(x) \tag{13}$$

$$t(xy) \leq t(xt(y)) \tag{14}$$

(11) entails that t is monotonic; (12) says that $t(x)$ is a left preserver of x;
(13) entails that $t(x)$ is the *least* left preserver among tests;
(14) is called sublocality, and we can not add the reverse inequality.

One operation

Definition

In [AGS16] a 1KAT is $\mathscr{K} = (K, \vee, \cdot, ^*, 1, 0, a)$ where $a : K \to K$ such that

$$a^2(1) = 1 \tag{15}$$

$$a^2(a^2(x)\, a^2(y)) = a^2(y)\, a^2(x) \tag{16}$$

$$a(x)\, a^2(x) = 0 \tag{17}$$

$$a(x) \vee a(y) = a(a^2(x)\, a^2(y)) \tag{18}$$

- A 1KAT is a KAt where $t' = a$ and $t = a^2$
- 1KAT satisfies the conditions of Theorem 2
- 1KAT has properties 1-4

Residuated program algebras

Definition

A residuated KAt $\mathscr{P} = (K, \vee, \cdot, \to, \leftarrow, ^*, 1, 0, t)$ is a strong KAt (ignoring the t' axioms) that is extended with residuals for $\cdot$ satisfying

$$t(x \to y) \leq x \to xt(y) \tag{19}$$

$$1 \leq t(x) \vee (t(x) \to 0). \tag{20}$$

One can define $t'(x) := t(x \to 0)$, to obtain a KAt reduct.

Prop.: Every relational KAT expands to a residuated KAt, so the equational theory of KAT embeds into that of residuated KAt.

Another result: The substructural logic of partial correctness by Kozen and Tiuryn [KT03] embeds into *-continuous residuated KAt expanded by e such that $t(x) \leq y \iff x \leq e(y)$.

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