Gödel temporal logic

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(with Juan Pablo Aguilera, Martín Diéguez, and David Fernández-Duque)

Language

$\mathcal{L}$

Propositional logic:

- Variables $p, q, r, \ldots$
- Connectives $\wedge, \vee, \Rightarrow, \perp$

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- Eventually $\Diamond\phi$
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$\Leftarrow$ is not necessary for first part of talk

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A **real valuation** on $\mathcal{T}$ is a $V\colon \mathcal{L} \times T \to [0,1]$ such that, for all $t \in T$:

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\begin{aligned}
V(\bot, t) &= 0 \\
V(\varphi \wedge \psi, t) &= \min\{V(\varphi, t), V(\psi, t)\} \\
V(\varphi \vee \psi, t) &= \max\{V(\varphi, t), V(\psi, t)\} \\
V(\varphi \Rightarrow \psi, t) &= \begin{cases} 1 & \text{if } V(\varphi, t) \leq V(\psi, t) \\ V(\psi, t) & \text{otherwise} \end{cases} \\
V(\varphi \Leftarrow \psi, t) &= \begin{cases} 0 & \text{if } V(\varphi, t) \leq V(\psi, t) \\ V(\varphi, t) & \text{otherwise} \end{cases} \\
V(\bigcirc\varphi, t) &= V(\varphi, S(t)) \\
V(\Diamond\varphi, t) &= \sup_{n<\omega} V(\varphi, S^n(t)) \\
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A flow equipped with a valuation is a **real (Gödel temporal) model**.

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A **Kripke valuation** on $\mathcal{F}$ is a function $[\![\cdot]\!] : \mathcal{L} \to 2^{W \times T}$ such that, for each $p \in \mathbb{P}$, the set $[\![p]\!]$ is *downward closed* in its first coordinate, and

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\begin{aligned}
[\![\bot]\!] &= \varnothing \\
[\![\varphi \wedge \psi]\!] &= [\![\varphi]\!] \cap [\![\psi]\!] \\
[\![\varphi \vee \psi]\!] &= [\![\varphi]\!] \cup [\![\psi]\!] \\
[\![\varphi \Rightarrow \psi]\!] &= \{(w, t) \in W \times T \mid \forall v \leq w((v, t) \in [\![\varphi]\!] \\
&\qquad\qquad \text{implies } (v, t) \in [\![\psi]\!])\} \\
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[\![\bigcirc\varphi]\!] &= (\mathrm{id}_W \times S)^{-1} [\![\varphi]\!] \\
[\![\Diamond\varphi]\!] &= \bigcup_{n<\omega} (\mathrm{id}_W \times S)^{-n} [\![\varphi]\!] \\
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A Kripke frame equipped with a valuation is a **(Gödel temporal) Kripke model**.

Example

$\Diamond(p \Rightarrow \bigcirc p)$

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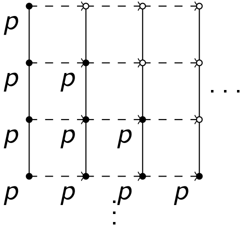

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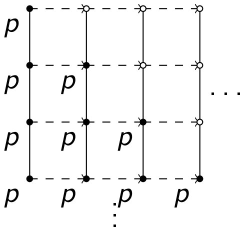

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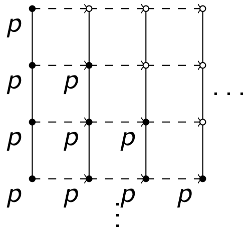
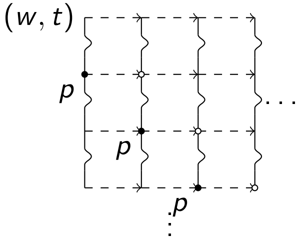


$\Diamond(p \Rightarrow \bigcirc p)$ is falsified

Requires an infinite model to falsify—no finite model property



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Theorem (Aguilera, Diéguez, Fernández-Duque, McLean)

Denote the set of $\mathcal{L}$-formulas valid on all real models by $\mathsf{GTL}_{\mathbb{R}}$.
Denote the set of $\mathcal{L}$-formulas valid on all Kripke models by $\mathsf{GTL}_{\mathrm{K}}$.
Then

$$\mathsf{GTL}_{\mathbb{R}} = \mathsf{GTL}_{\mathrm{K}}.$$

Decidability: quasimodels

Σ a subformula-closed set of formulas
(think sub(ϕ) for ϕ we want to decide)

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$\Phi \subseteq \Sigma$ is a Σ-**type** if:

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2. If $\varphi \wedge \psi \in \Sigma$, then $\varphi \wedge \psi \in \Phi$ if and only if $\varphi, \psi \in \Phi$.
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    1. $\varphi \Rightarrow \psi \in \Phi$ implies that $\varphi \notin \Phi$ or $\psi \in \Phi$,
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5. If $\varphi \Leftarrow \psi \in \Sigma$, then
    1. $\varphi \Leftarrow \psi \in \Phi$ implies $\varphi \in \Phi$,
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The set of Σ-types will be denoted by $\mathbb{T}_\Sigma$.

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A **convex relation** between posets $(A, \leq_A)$ and $(B, \leq_B)$ is an $R \subseteq A \times B$ such that:
for each $x \in A$ the image set $\{y \in B \mid x \, R \, y\}$ is convex with respect to $\leq_B$
for each $y \in B$ the preimage set $\{x \in A \mid x \, R \, y\}$ is convex w.r.t. $\leq_A$.

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The relation R is **fully confluent** if:

forth-down if $x \leq_A x' \, R \, y'$ there is y such that $x \, R \, y \leq_B y'$,

forth-up if $x' \geq_A x \, R \, y$ there is y' such that $x' \, R \, y' \geq_B y$,

back-down if $x' \, R \, y' \geq_B y$ there is x such that $x' \geq_A x \, R \, y$,

back-up if $x \, R \, y \leq_B y'$ there is x' such that $x \leq_A x' \, R \, y'$.

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Let $\Phi, \Psi \in \mathbb{T}_\Sigma$. The pair (Φ, Ψ) is **sensible** if

1. for all $\bigcirc\varphi \in \Sigma$: $\bigcirc\varphi \in \Phi$ if and only if $\varphi \in \Psi$,
2. for all $\Diamond\varphi \in \Sigma$: $\Diamond\varphi \in \Phi$ if and only if $\varphi \in \Phi$ or $\Diamond\varphi \in \Psi$,
3. for all $\Box\varphi \in \Sigma$: $\Box\varphi \in \Phi$ if and only if $\varphi \in \Phi$ and $\Box\varphi \in \Psi$.

A pair (w, v) of worlds in a labelled space $\mathcal{W}$ is **sensible** if $(\ell(w), \ell(v))$ is sensible. A *relation* $R \subseteq |\mathcal{W}| \times |\mathcal{W}|$ is **sensible** if every pair in R is sensible.

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A sensible R is ω-**sensible** if it is serial and

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A Σ-**quasimodel** is a Σ-labelled space equipped with a fully confluent, convex, ω-sensible relation.

Decidability: quasimodel $\to$ model

From a quasimodel $\mathcal{Q}$ and $w \in \mathcal{Q}$ with $\phi \in \Sigma \setminus \ell(w)$:
construct a model falsifying ϕ

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Each extension of the grid removes a 'defect', with every defect eventually removed.

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From a Kripke model $\mathcal{X}$ and $x \in \mathcal{X}$ falsifying ϕ:

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where $\pi_2 \colon W \times T \to T$ is the projection $(w, t) \mapsto t$.

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Now define a partial order $\leq_{\mathcal{Q}}$ on the equivalence classes $(W \times T)/\sim$ of $\sim$ by

$$[x] \leq_{\mathcal{Q}} [y] \iff L(x) = L(y) \text{ and } \ell(x) \supseteq \ell(y),$$

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Relation $R_{\mathcal{Q}}$ containing all $([(w, t)], [(w, S(t))])$.

Decidability: model $\to$ finite quasimodel

From a Kripke model $\mathcal{X}$ and $x \in \mathcal{X}$ falsifying ϕ:
construct a *finite* Σ-quasimodel model 'falsifying' ϕ $\quad (\Sigma = \text{sub}(\phi))$

Let $\mathcal{X} = (W, T, \leq, S, \llbracket\cdot\rrbracket)$.

For $x \in W \times T$, define

$$\ell(x) = \{\psi \in \Sigma \mid x \in \llbracket\psi\rrbracket\},$$

$$L(x) = \{\ell(y) \mid \pi_2(x) = \pi_2(y)\},$$

where $\pi_2\colon W \times T \to T$ is the projection $(w, t) \mapsto t$.

Define $\sim$ on $W \times T$ by

$$x \sim y \iff (\ell(x), L(x)) = (\ell(y), L(y)).$$

Now define a partial order $\leq_{\mathcal{Q}}$ on the equivalence classes $(W \times T)/{\sim}$ of $\sim$ by

$$[x] \leq_{\mathcal{Q}} [y] \iff L(x) = L(y) \text{ and } \ell(x) \supseteq \ell(y),$$

noting that this is well-defined and is indeed a partial order.
Relation $R_{\mathcal{Q}}$ containing all $([(w,t)], [(w, S(t))])$. Check everything

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By Savitch's theorem: problem is PSPACE.

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  - $\bigcirc(\varphi \vee \psi) \Rightarrow (\bigcirc\varphi \vee \bigcirc\psi)$
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- **Standard modal rules:**
  - $\dfrac{\varphi, \ \varphi \Rightarrow \psi}{\psi}$
  - $\dfrac{\varphi}{\bigcirc\varphi}$
  - $\dfrac{\varphi}{\Box\varphi}$

GTL := formulas generated by deductive system

Completeness: canonical structure

Points are pairs $\Phi = (\Phi^+, \Phi^-)$ partitioning $\mathcal{L}$ such that for all finite $\Delta^+ \subseteq \Phi^+$ and $\Delta^- \subseteq \Phi^-$

$$(\bigwedge \Delta^+ \Rightarrow \bigvee \Delta^-) \notin \mathsf{GTL}$$

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Verify everything except ω-sensibility.

Completeness: canonical structure $\rightarrow$ quasimodel

Choose $\Sigma \subseteq \mathcal{L}$ finite and subformula closed (e.g. $\mathrm{sub}(\phi)$).

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Same quotient construction as before:

$$L(x) = \{\ell(y) \mid x \text{ and } y \text{ in same linear component } \}.$$

Define $\sim$

$$x \sim y \iff (\ell(x), L(x)) = (\ell(y), L(y)).$$

Define a partial order $\leq$ on the equivalence classes by

$$[x] \leq [y] \iff L(x) = L(y) \text{ and } \ell(x) \supseteq \ell(y).$$

Relation R containing all $([x], [S(x)])$.

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Check everything except possibly ω-sensibility still holds.

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—"I'm not above a point of type Δ"

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$$\chi^0(w) := \bigwedge_{\Delta \in L(w)} \sim\overrightarrow{\Delta} \wedge \bigwedge_{\Delta \notin L(w)} \neg\overleftarrow{\Delta}$$

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$$\chi^+(w) = \overleftarrow{\ell(w)} \wedge \chi^0(w)$$

—"I'm below w"

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To prove:
if all $v \in R^*(w)$ have φ in their label
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Similar argument for formulas of form $\Diamond\varphi$

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