Nelson conuclei and nuclei: the twist construction beyond involutivity

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(joint work with Manuela Busaniche)

Twist-algebras

- The twist-algebra construction is used (since at least Kalman in the 1950s) to represent an algebra $\mathbf{A}$ as a subalgebra of a special binary power of some other algebra $\mathbf{L}$ (we'll write $\mathbf{A} \leq \mathbf{L}^{\bowtie}$).

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  - Various classes of residuated lattices, e.g. Kalman lattices, Sugihara monoids ($\mathbf{L}$ is a residuated lattice).
- Usually the twist construction produces an algebra $\mathbf{A}$ carrying an involutive negation, but in this talk we'll see how to go beyond the involutive setting.

Twist-algebras

The involutive case

Let $\mathbf{L} = \langle L; \wedge, \vee, \to, 0, 1\rangle$ be (e.g.) a Heyting algebra. The full twist-algebra over $\mathbf{L}$ is the algebra $\mathbf{L}^{\bowtie} = \langle L \times L; \wedge, \vee, \to, *, \Rightarrow, \sim, 0, 1\rangle$ with operations given, for all $\langle a_1, a_2\rangle, \langle b_1, b_2\rangle \in L \times L$, by:

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\begin{aligned}
\langle a_1, a_2\rangle \wedge \langle b_1, b_2\rangle &:= \langle a_1 \wedge b_1, a_2 \vee b_2\rangle \\
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\langle a_1, a_2\rangle \to \langle b_1, b_2\rangle &= \langle a_1 \to b_1, a_1 \wedge b_2\rangle \\
\langle a_1, a_2\rangle * \langle b_1, b_2\rangle &= \langle a_1 \wedge b_1, (a_1 \to b_2) \wedge (b_1 \to a_2)\rangle \\
\langle a_1, a_2\rangle \Rightarrow \langle b_1, b_2\rangle &:= \langle (a_1 \to b_1) \wedge (b_2 \to a_2), a_1 \wedge b_2\rangle \\
\sim\langle a_1, a_2\rangle &:= \langle a_2, a_1\rangle \\
1 &:= \langle 1, 0\rangle \\
0 &:= \langle 0, 1\rangle.
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A twist-algebra over $\mathbf{L}$ is any subalgebra $\mathbf{A} \leq \mathbf{L}^{\bowtie}$ satisfying $\pi_1[A] = L$ and $a_1 \wedge a_2 = 0$ for all $\langle a_1, a_2\rangle \in A$.

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- A classic result (from the 1970s) on Nelson algebras states that every Nelson algebra is isomorphic to a twist-algebra over a Heyting algebra. This has several applications (subvarieties, congruences, logical calculi etc.).

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  - Restricting the subsets of $L \times L$ allowed as universes of twist-algebras, e.g., given a lattice filter $F \subseteq L$ and an ideal $I \subseteq L$:

$$Tw(\mathbf{L}, F, I) := \{\langle a_1, a_2 \rangle \in L \times L : a_1 \wedge a_2 \in I, \; a_1 \vee a_2 \in F\}.$$

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$$Tw(\mathbf{L}, F, I) := \{\langle a_1, a_2 \rangle \in L \times L : a_1 \wedge a_2 \in I,\ a_1 \vee a_2 \in F\}.$$

- In all the above cases, the involutive negation that swaps components remains a key feature of the construction: $\langle a_1, a_2 \rangle \mapsto \langle a_2, a_1 \rangle$.

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$$\sim\sim x = x$$

  where $\sim x := x \Rightarrow 0$.

- the (Nelson) identity:

$$(x \to y) \wedge (\sim y \to \sim x) = x \Rightarrow y$$

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Notice that (Nelson) entails that $\Rightarrow$ is definable in the language $\{\wedge \to, \sim\}$.

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For every Nelson algebra $\mathbf{A} = \langle A; \wedge, \vee, *, \Rightarrow, 0, 1 \rangle$, letting:

$$\theta := \{\langle a, b \rangle \in A \times A : a * a = b * b\}$$

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- $\mathrm{Con}(\mathbf{A}) \cong \mathrm{Con}(\mathbf{L_A})$.
- A filter $F \subseteq \mathbf{L_A}$ describes precisely the subalgebras or $(\mathbf{L_A})^{\bowtie}$ that are isomorphic to Nelson algebras.

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Remark

- The term $x * x$ may be viewed more abstractly as an instance of a unary term $\nu(x)$ satisfying suitable properties.

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- This allows us to define a more general representation for structures $\langle \mathbf{A}, \nu \rangle$, dubbed Nelson conucleus algebras in recent work by Busaniche, Galatos & Marcos.
- This path, explored by B., G. & M. in the involutive setting, appears to be easily generalizable to the non-involutive one (ongoing research).

Twist-algebras

HOW to drop involutivity

Recent work (by A. Jung, M. Spinks and myself) has shown that the involutivity requirement can be dropped:

- By replacing $\mathbf{L}$ with two algebras $\mathbf{L}_1, \mathbf{L}_2$ (related by back-and-forth maps).

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Recent work (by A. Jung, M. Spinks and myself) has shown that the involutivity requirement can be dropped:

- By replacing $\mathbf{L}$ with two algebras $\mathbf{L}_1, \mathbf{L}_2$ (related by back-and-forth maps).
- As a special case of the former, in some cases $\mathbf{L}_2$ may be taken to be the image of $\mathbf{L}_1$ by a nucleus operator.

  (Today we'll explore the applicability of the latter route.)

Twist-algebras

WHY to drop involutivity: a motivating example

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Twist-algebras

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WHY to drop involutivity: a motivating example

The domains of positive and negative evidence may not have the same structure. Consider whether a given Turing machine will stop:

- Positive evidence is the observation that the machine has stopped. Until this happens, we do not have any positive evidence, so the lattice of positive evidence has just two elements: "unknown" and "has stopped".
- Negative evidence should be treated differently, since we cannot observe non-halting behaviour directly. Instead, we employ the lattice of natural numbers together with a top element $\top$, where each $n \in \mathbb{N}$ indicates that we have observed that the machine has been running for n steps (or units of time) and has not yet stopped.
  The element $\top$ means non-termination, but it is an "ideal" value that cannot be observed directly but is the supremum of the infinite set of propositions "has not stopped after n steps".

Nuclei

Definition 1

Given an algebra with a bounded meet-semilattice reduct $\langle L; \leq, \wedge, 0\rangle$, a (dense) nucleus on L is a unary operation $\Box$ satisfying:

(NS1) $x \leq \Box x = \Box\Box x$

(NS2) $\Box(x \wedge y) = \Box x \wedge \Box y$

(NS3) $\Box 0 = 0$. (density)

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Definition 2

Given an algebra with a bounded Hilbert algebra reduct $\langle L; \to, 0\rangle$, a (dense) nucleus on L is a unary operation $\Box$ satisfying:

(NH1) $x \to \Box y = \Box x \to \Box y$

(NH2) $\Box 0 = 0.$ (density)

Obs.: Definitions 1 and 2 coincide on Heyting algebras and on bounded implicative meet-semilattices.

Nuclei

Definition 2 can be further generalized as follows:

Definition 3

Given an algebra with a residuated semigroup reduct $\langle L, *, \backslash, / \rangle$, a nucleus on L is a unary operation $\Box$ satisfying:

(NR1) $x \backslash \Box y = \Box x \backslash \Box y$

(NR2) $\Box x / y = \Box x / \Box y$.

Non-involutive twist-algebras

Over Heyting algebras

Let $\mathbf{L} = \langle L; \wedge, \vee, \rightarrow, \Box, 0, 1 \rangle$ be (e.g.) a Heyting algebra with a nucleus. The full twist-algebra over $\mathbf{L}$ is the algebra $\mathbf{L}^{\bowtie} = \langle L \times \Box L; \wedge, \vee, \rightarrow, *, \Rightarrow, \sim, 0, 1 \rangle$, where $\Box L := \{\Box a : a \in L\}$, with operations given, for all $\langle a_1, a_2 \rangle, \langle b_1, b_2 \rangle \in L \times \Box L$, by:

$$\begin{aligned}
\langle a_1, a_2 \rangle \wedge \langle b_1, b_2 \rangle &:= \langle a_1 \wedge b_1, \Box(a_2 \vee b_2) \rangle \\
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Non-involutive twist-algebras

Over residuated lattices

Let $\mathbf{L} = \langle L, *, \wedge, \vee, \backslash, /, \Box, 1\rangle$ be a residuated lattice with a nucleus, and let $\iota \in L$ be an element such that $\Box\iota = \iota$. The full twist-algebra over $\mathbf{L}$ is the algebra $\mathbf{L}^{\bowtie} = \langle L \times \Box L; \wedge, \vee, *, \Rightarrow, \Leftarrow, \sim, 1, 0\rangle$, where $\Box L := \{\Box a : a \in L\}$, with operations given, for all $\langle a_1, a_2\rangle, \langle b_1, b_2\rangle \in L \times \Box L$, by:

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\langle a_1, a_2\rangle * \langle b_1, b_2\rangle &:= \langle a_1 * b_1, (b_2/a_1) \wedge (b_1\backslash a_2)\rangle \\
\langle a_1, a_2\rangle \Rightarrow \langle b_1, b_2\rangle &:= \langle (a_1\backslash b_1) \wedge (a_2/b_2), \Box(b_2 * a_1)\rangle \\
\langle a_1, a_2\rangle \Leftarrow \langle b_1, b_2\rangle &:= \langle (a_1/b_1) \wedge (a_2\backslash b_2), \Box(b_1 * a_2)\rangle \\
1 &:= \langle 1, \iota\rangle \\
0 &:= \langle \iota, \Box 1\rangle.
\end{aligned}$$

A twist-algebra over $\mathbf{L}$ is any subalgebra $\mathbf{A} \leq \mathbf{L}^{\bowtie}$ satisfying $\pi_1[A] = L$ and $(a_1 * a_2) \vee (a_2 * a_1) \leq \iota$ for all $\langle a_1, a_2\rangle \in A$.

Non-involutive twist-algebras

New representations

By means of the preceding constructions we can obtain twist representations for:

- Quasi-Nelson algebras, i.e. commutative integral bounded (not-necessarily involutive) residuated lattices that satisfy the (Nelson) identity.

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- Subreducts of the above-mentioned classes of algebras (examples below).

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New representations: subreducts of quasi-Nelson

- $\{*, \Rightarrow, 0\}$-subreducts of QNAs correspond to twist-algebras over implicative semilattices with a nucleus.

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- $\{\wedge, \vee, \sim, \neg\}$-subreducts of QNAs (where $\neg x := x \to 0$) correspond to twist-algebras over pseudo-complemented lattices with a nucleus.

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- The $\{\to, \sim\}$-, $\{*, \sim\}$- and $\{\wedge, *, \Rightarrow, 0\}$-subreducts of QNAs admit similar characterizations.

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- The $\{\to, \sim\}$-, $\{*, \sim\}$- and $\{\wedge, *, \Rightarrow, 0\}$-subreducts of QNAs admit similar characterizations.
- Other subreducts proved to be harder: e.g. $\{\Rightarrow\}$, $\{\Rightarrow, \sim\}$ and the $\{\sim\}$-free subreducts.

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Every $\{\rightarrow, \sim\}$-subreduct of a QNA corresponds to a twist-algebra over an algebra $\mathbf{L} = \langle L; \odot, \rightarrow, 0, 1 \rangle$ such that:

1. $\langle L; \rightarrow, 0, 1 \rangle$ is a bounded Hilbert algebra.
2. $\langle L; \odot \rangle$ is a commutative semigroup.
3. The operation $\Box x := x \odot x$ is a dense nucleus on $\langle L; \rightarrow, 0, 1 \rangle$.
4. $x \odot y = x \odot (x \rightarrow y)$.
5. $\Box x \rightarrow (\Box y \rightarrow z) = (x \odot y) \rightarrow z$.
6. $x \odot 0 = 0$.
7. $x \odot 1 = \Box x$.

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6. $x \odot 0 = 0$.
7. $x \odot 1 = \Box x$.

Example: on a bounded Hilbert algebra $\langle A; \to, 0, 1\rangle$, letting $\neg x := x \to 0$ and $x \odot y := \neg(x \to \neg y)$, we obtain an algebra in the above-defined class (in this case $\Box x = \neg\neg x$).

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Every $\{*, \sim\}$-subreduct of a QNA corresponds to a twist-algebra over an algebra $\mathbf{L} = \langle L; \wedge, \rightharpoonup, 0, 1 \rangle$ such that (letting $\Box x := 1 \rightharpoonup x$):

1. $\langle L; \wedge, \Box, 0, 1 \rangle$ is a bounded semilattice with a dense nucleus.
2. $x \rightharpoonup (y \rightharpoonup z) = (x \wedge y) \rightharpoonup z$.
3. $x \rightharpoonup (y \wedge z) = (x \rightharpoonup y) \wedge (x \rightharpoonup z)$.
4. $x \wedge \Box y = x \wedge (x \rightharpoonup y)$.
5. $x \leq y \rightharpoonup z$ if and only if $x \wedge y \leq \Box z$.
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Example: on a pseudo-complemented semilattice $\langle S, \wedge, \neg, 0, 1\rangle$, letting $x \rightharpoonup y := \neg(x \wedge \neg y)$, we obtain an algebra in the above-defined class.

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- Characterizations of subreducts of QNAs specialize to subreducts of Nelson algebras (among which only the 'two-negations' subreducts had been previously characterized, by Sendlewski).
- As in the involutive case, some representations – those of type $Tw(\mathbf{L}, F, I)$ – can be upgraded to category equivalences.

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- The new representations are very smooth generalizations of their involutive counterparts and have similar applications, but the factor algebras are more exotic classes of 'modal' algebras.
- The preceding observations suggest that it may be fruitful to further study these modal algebras (e.g., from a universal algebraic and a duality point of view).

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