Degrees of FMP in extensions of bi-intuitionistic logic

Anton Chernev

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Degrees of incompleteness

Purpose: "measuring" the gap between Kripke completeness and Kripke incompleteness.

Definition (Fine, 1974)

Given a normal modal logic L, we define the degree of incompleteness of L to be the number (cardinality) of normal modal logics with the same Kripke frames.

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Main result:

Theorem (Blok, 1978)

In the lattice of normal modal logics, every logic has degree of incompleteness 1 *or* $2^{\aleph_0}$.

Also known as Blok's dichotomy.

Degrees of FMP

Idea: measure the failure of the finite model property (FMP) in a similar way.

Introduced by G. Bezhanishvili, N. Bezhanishvili, T. Moraschini, 2021.

Definition

Given a logic L, which is an extension of L', we define the degree of FMP of L relative to L' to be the number (cardinality) of extensions of L' with the same finite Kripke frames as L.

Recent results

(G. Bezhanishvili, N. Bezhanishvili, T. Moraschini, 2021)

- Relative to **K**, the degree of FMP is either 1 or $2^{\aleph_0}$.
- Relative to **K4**, for every $\kappa \leq \aleph_0$ or $\kappa = 2^{\aleph_0}$ we can find a variety with degree of FMP κ.
- The latter holds also for **S4** and **IPC**.

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- The latter holds also for **S4** and **IPC**.

Question: what about bi-intuitionistic logic **bi-IPC**?

The logic bi-IPC

The logic **bi-IPC** is a conservative extension of **IPC** with an additional connective $\leftarrow$.

The set of validities in intuitionistic Kripke models, where $\leftarrow$ is interpreted as follows:

$$M, x \Vdash \varphi \leftarrow \psi \iff \exists y \leq x \, (M, y \vDash \varphi \text{ and } M, y \nvDash \psi).$$

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Thus $\leftarrow$ is dual to $\rightarrow$, which is interpreted as:

$$M, x \Vdash \varphi \rightarrow \psi \iff \forall y \geq x \, (M, y \nvDash \varphi \text{ or } M, y \vDash \psi).$$

The problem in algebraic terms

Algebraic semantics for **bi-IPC**: the variety bi-HA of bi-Heyting algebras.

The algebra $\langle A, 1, 0, \wedge, \vee, \rightarrow, \leftarrow \rangle$ is a bi-Heyting algebra when:

- $(A, 0, 1, \wedge, \vee, \rightarrow)$ is a Heyting algebra and
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We have an algebraic counterpart of the notion degree of FMP.

Definition

Given a variety U, which is a subvariety of a variety V, we define the degree of FMP of U relative to V to be the number (cardinality) of subvarieties of V with the same finite algebras as U.

Degrees of FMP relative to HA

We work towards characterising degrees of FMP relative to bi-HA by borrowing ideas from the characterisation relative to HA.

Relative to HA: for each $\kappa \leq \aleph_0$, we can construct a variety $V \subseteq \mathsf{HA}$ with degree of FMP κ.
These varieties are constructed inside the Kuznetsov-Gerčiu variety KG.

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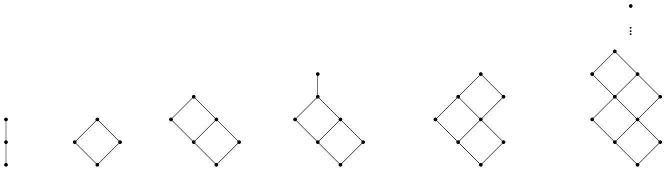

Figure: Some 1-generated Heyting algebras

Degrees of FMP relative to HA, continued

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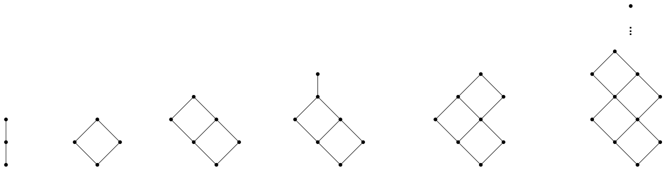

Figure: Some 1-generated Heyting algebras

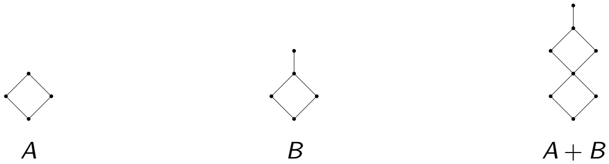

Figure: Example of a binary sum of Heyting algebras

Contributions

- Define a bi-Heyting counterpart of KG, which we call bi-KG.
- Describe all subdirectly irreducible members of bi-KG and prove that bi-KG is semi-simple.
- Characterise subvarieties of bi-KG with the FMP.
- Find all degrees of FMP in bi-KG.

Defining bi-KG

We use the same generators, but this time in the bi-Heyting signature.

Definition

Let $\mathcal{G}$ be the class of finite sums of 1-generated Heyting algebras equipped with the $\leftarrow$ operation. Define:

$$\text{bi-KG} := \mathbb{V}(\mathcal{G}).$$

Subdirectly irreducible members of bi-KG

Idea: use Jónsson's Lemma.

Theorem

Let V be a congruence-distributive variety such that $V = \mathbb{V}(\mathcal{K})$ for some class $\mathcal{K}$. Then $V_{SI} \subseteq \mathbb{HSP}_U(\mathcal{K})$.

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Can we find $\mathbb{SP}_U(\mathcal{G})$, i.e., the universal class generated by $\mathcal{G}$?

Finding the universal class of $\mathcal{G}$

Fact: $\mathbb{SP}_U(\mathcal{G})$ consists of the algebras that satisfy all universal sentences true in members of $\mathcal{G}$.

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Key observation: two-element anti-chains in members of $\mathcal{G}$ have a particular local structure that can be described with universal sentences.

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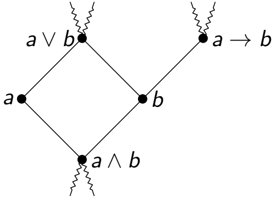
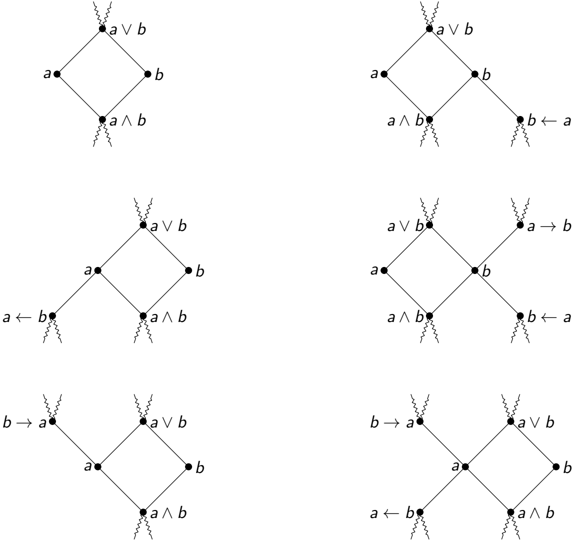
Example of a local anti-chain pattern:



Finding the universal class of $\mathcal{G}$, continued

Other possible patterns:



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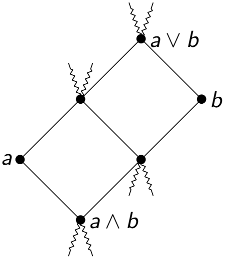
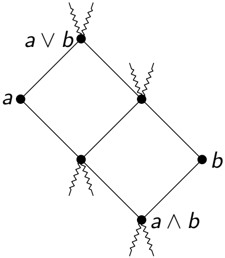
Last patterns:

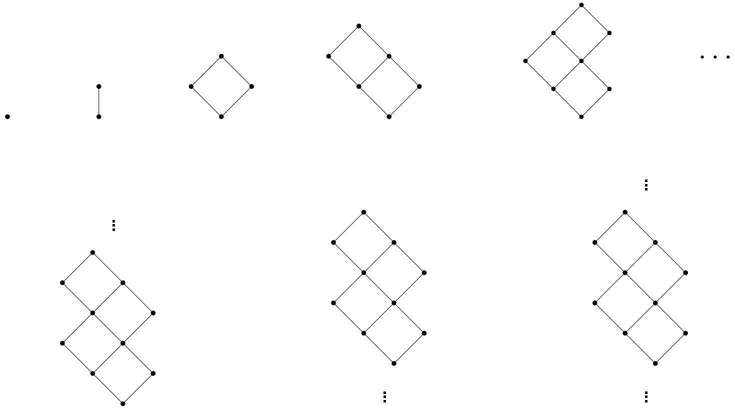




The universal class of $\mathcal{G}$

Theorem

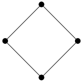
The class $\mathbb{SP}_U(\mathcal{G})$ consists of sums of the following prime algebras:



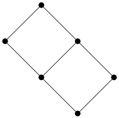
Semi-simplicity of bi-KG

By Jónsson's Lemma and the description of $\mathbb{SP}_U(\mathcal{G})$:

$$\text{bi-KG}_{SI} = \mathbb{SP}_U(\mathcal{G}) \setminus \{L_4, L_6\} = \text{bi-KG}_S$$



L_4



L_6

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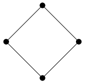
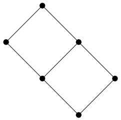


L_4



L_6

Theorem

The variety bi-KG is semi-simple. Consequently, every subvariety of bi-KG is determined by its finitely generated simple members.

Local embeddability

We can determine membership to subvarieties of bi-KG using local embeddability.

A local subgraph of an algebra A is a finite partial subalgebra of A. If A is an algebra and $\mathcal{K}$ is a class of algebras, we write $A \stackrel{loc}{\hookrightarrow} \mathcal{K}$ if every local subgraph of A embeds into a member of $\mathcal{K}$.

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In our case this leads to:

Theorem

If $\{A\} \cup \mathcal{K}$ is a class of finitely generated simple bi-KG algebras, then $A \in \mathbb{V}(\mathcal{K})$ if and only if $A \overset{loc}{\hookrightarrow} \mathcal{K}$.

The FMP in bi-KG

Some notation:

- $FGS(V)$ is the class of finitely generated simple algebras in V.
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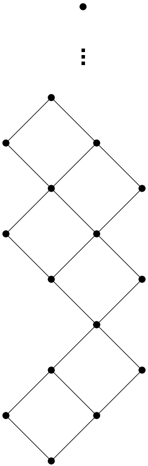
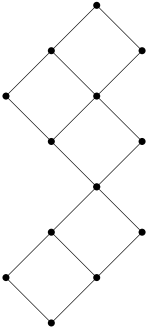
Theorem

A variety $V \subseteq$ bi-KG has the FMP if and only if for every $A \in FGS(V)$ and $m \in \mathbb{N}$, there exists an m-compression of A in V.

Example of an m-compression



A



An 8-compression of A

Corollaries of the FMP theorem

Corollary

The variety bi-KG has the FMP.

Corollary

The variety generated by the bi-Heyting Rieger-Nishimura lattice lacks the FMP.

Degrees of FMP relative to bi-KG

We have a dichotomy-style characterisation of degrees of FMP relative to bi-KG.

Theorem

Relative to bi-KG, all possible degrees of FMP are 1 *and* $2^{\aleph_0}$.

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Relative to bi-KG, all possible degrees of FMP are 1 *and* $2^{\aleph_0}$*.*

Note: stark contrast with KG, where every degree $\kappa \leq \aleph_0$ exists.

Proof idea for the degrees of FMP characterisation

1. Find a variety with degree of FMP 1 relative to bi-KG.
   - Relative to itself, bi-KG has degree of FMP 1.

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1. Find a variety with degree of FMP 1 relative to bi-KG.
    - Relative to itself, bi-KG has degree of FMP 1.
2. Find a variety with degree of FMP greater than 1.
    - Relative to bi-KG, the variety generated by the bi-Heyting Rieger-Nishimura lattice has degree of FMP greater that 1.

Proof idea for the degrees of FMP characterisation

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   - Relative to itself, bi-KG has degree of FMP 1.
2. Find a variety with degree of FMP greater than 1.
   - Relative to bi-KG, the variety generated by the bi-Heyting Rieger-Nishimura lattice has degree of FMP greater that 1.
3. Prove that every variety with degree of FMP relative to bi-KG greater than 1 has degree of FMP $2^{\aleph_0}$.
   - Given a variety V with degree of FMP greater than 1, build continuum many varieties with the same finite algebras as V.

Constructing continuum many varieties

Without loss of generality, there exists $A \in FGS(V)$ with $A \overset{loc}{\not\hookrightarrow} FinS(V)$.

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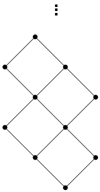
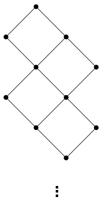
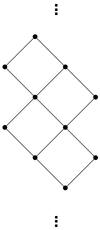
Thus A is infinite, i.e., A "contains" at least one infinite prime summand.



L^u



L_u



L_u^u

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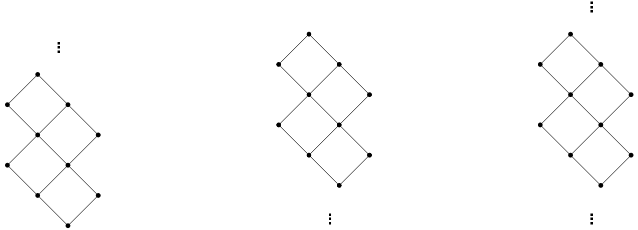


L^u $\quad$ L_u $\quad$ L^u_u

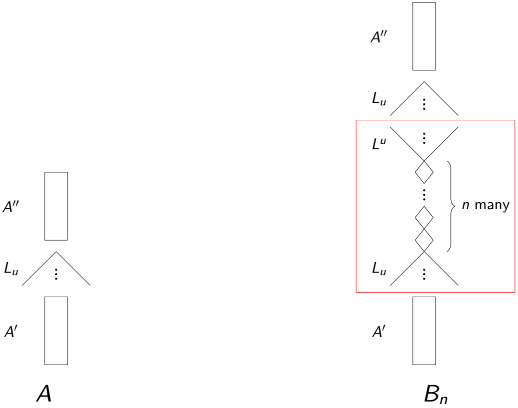
Suppose A contains L_u (the other cases are symmetric).

Constructing continuum many varieties, continued

Thus we have $A = A' + L_u + A''$.

Define for every $n \in \mathbb{N}$:

$$B_n := A' + L_u + \Sigma_{i \in \{1,\dots,n\}} L_4 + L^u + L_u + A''.$$



Constructing continuum many varieties, continued

For every $I \subseteq \mathbb{N}$, define:

$$V_I := \mathbb{V}(\{B_i \mid i \in I\} \cup FinS(V)).$$

Constructing continuum many varieties, continued

For every $I \subseteq \mathbb{N}$, define:

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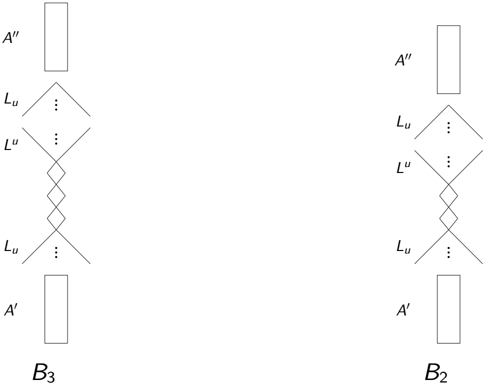
In order to show that $V_I \neq V_J$ for $I \neq J$, it suffices to show $B_n \overset{loc}{\not\hookrightarrow} B_m$ for $n \neq m$.



Directions for future work

- Characterisation of degrees of FMP in bi-HA.
- Adapting ideas from bi-intuitionistic logic to study degrees of FMP in temporal logic.

Thank you!