From implicative reducts to Mundici's functor

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Introduction

- In algebraic logic, one of the most relevant theorems connecting different structures is **Mundici's theorem**, which links MV-algebras and abelian ℓ-groups whit strong unit.
- **Dvurečenskij** extended Mundici's approach to the non commutative case.

Alternative algebraic approaches to Mundici's functor have been proposed by other authors.

- Galatos and Tsinakis extended both results to the non integral and non commutative setting of generalized MV-algebras, using a truncation construction, based on the work of Bosbach on cone algebras.
- Rump, also inspired by Bosbach, generalized Mundici's result starting from structures having only $\rightarrow$ and 1.

Introduction

In the present talk we re-elaborate Rump's work.

- We find conditions that are different, but equivalent to the ones found by Rump;
- we extend some of Rump's constructions to categorical equivalences of the algebraic categories involved.
- With some adjustments, we use the same construction for the non commutative case.

Residuated lattices

A **residuated lattice** is an algebra $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \to, \rightsquigarrow, 1\rangle$ such that:

- $\langle A, \wedge, \vee\rangle$ is a lattice;
- $\langle A, \cdot, 1\rangle$ is a monoid;
- for all $a, b, c \in A$

$$ab \leq c \text{ if and only if } a \leq b \to c \text{ if and only if } b \leq a \rightsquigarrow c$$

An ***FL*-algebra** is a residuated lattice with an additional constant operation 0.

Some important properties for residuated lattices are:

- **integrality**: for all x, $x \leq 1$;
- **representability**: for all x, y, z, w, $[z \rightsquigarrow (x \to (x \vee y)) \wedge 1] \vee [w \to w(y \to (x \vee y)) \wedge 1] = 1$
- **divisibility**: for all x, y, $x(x \to y) = x \wedge y$;
- **commutativity**: the monoidal operation $\cdot$ is commutative. Because of that $\to = \rightsquigarrow$.

Residuated lattices

The residuated lattices that we are going to consider are:

- A **Wajsberg pseudohoop** is a divisible, integral, representable residuated lattice satisfying
  1. $(a \to b) \rightsquigarrow b = (b \to a) \rightsquigarrow a$;
  2. $(a \rightsquigarrow b) \to b = (b \rightsquigarrow a) \to a$.
- A **Wajsberg hoop** is a commutative Wajsberg pseudohoop.
- An **ℓ-group** **G** can be seen as an integral residuated lattice satisfying $1 = x(x \rightsquigarrow 1)$.
- The **negative cone of an ℓ-group** is a cancellative, integral, divisible residuated lattice.

Residuated lattices

- A **pseudo MV-algebra** is an *FL*-algebra satisfying:
  1. divisibility;
  2. for any x we have both $0 \leq x$ and $x \leq 1$;
  3. $x \vee y = (y \to x) \rightsquigarrow x = (y \rightsquigarrow x) \to x$.
- A **MV-algebra** is a commutative pseudo MV-algebra.

Magms and H-algebras

A **unital magma** is a structure $\mathbf{A} = \langle A, \to, 1 \rangle$ such that:

UM1 $x \to x \approx 1$

UM2 $x \to 1 \approx 1$

UM3 $1 \to x \approx x$

UM4 $(x \to y \approx y \to x) \Rightarrow x \approx y$

In particular an **H-algebra** is a unital magma that satisfies also

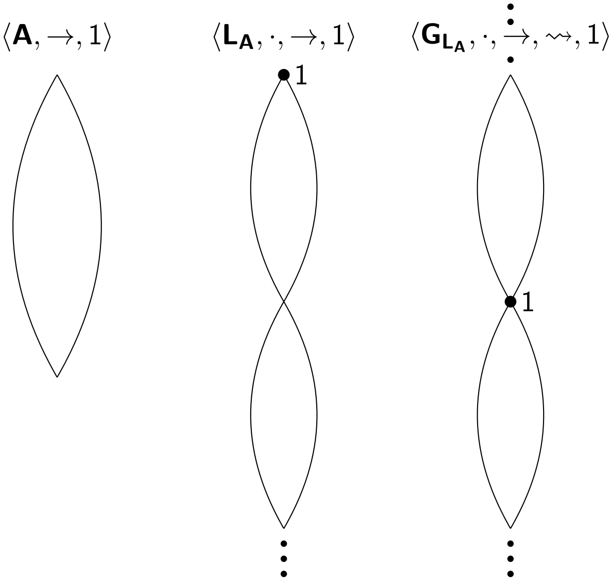
(H) $(x \to y) \to (x \to z) \approx (y \to x) \to (y \to z)$

(K) $x \to (y \to x) \approx 1.$

H-algebras has a partial order defined as

$$x \leq y \text{ if and only if } x \to y = 1$$

A graphic idea



$\mathbf{M_A}$

Now we would like to extend $\to$ to structures having the product. In order to do this, we take an H-algebra $\mathbf{A}$ and construct $\mathbf{M_A}$, the free monoid over $\mathbf{A}$, with unit u.

Lemma

It is possible to define an operation $\to$ over $\mathbf{M_A}$ which extends the operation $\to$ on $\mathbf{A}$ such that:

1. *for all $a \in M_A$, $a \to u = u$ and $u \to a = a$;*
2. *for all $a, b, c \in M_A$, $ab \to c = a \to (b \to c)$;*
3. *for all $a, b, c \in M_A$*

$$(a \to b) \to (a \to c) = (b \to a) \to (b \to c).$$

Left complemented monoid

Now we would like to obtain a residuated structure, in particular: a **left complemented monoid** is an algebra $\mathbf{A} = \langle A, \cdot, \to, 1\rangle$ such that:

LM1 $x \to x \approx 1$;

LM2 $(x \to y)x \approx (y \to x)y$;

LM3 $xy \to z \approx x \to (y \to z)$.

L_A

We define the following relation γ over $\langle M_A, \cdot, \to, u\rangle$: $(a, b) \in \gamma$ if and only if for all $c, d \in M_A$,

$$(c \to a) \to d = (c \to b) \to d.$$

We define $\mathbf{L_A} = \mathbf{M_A}/\gamma$.

Proposition

$\mathbf{L_A} = \mathbf{M_A}/\gamma$ *is a right cancellative left complemented monoid.*

Moreover the map

$$x \longmapsto x/\gamma$$

is an embedding from $\mathbf{A}$ to $\mathbf{L_A}$.

$\equiv$ and right reversibility

Taken an H-algebra $\mathbf{A}$, we would like to show when it embeds in the negative cone of an ℓ-group.

So, starting from $\mathbf{L_A}$, we want to construct a group.

Given a monoid $\mathbf{M}$, let's define the following equivalence relation on $\mathbf{M} \times \mathbf{M}$:

$(a, b) \equiv (c, d)$ if and only if there exists $u, v \in M$ such that $ua = vc$ and $ub = vd$.

A monoid $\mathbf{M}$ is called **right reversible** if for all $a, b \in M$, there exists $u, v \in M$ such that $ua = vb$.

G_M

We can prove that given **M**, a right cancellative, right reversible monoid, we can equip $\mathbf{M} \times \mathbf{M}/_{\equiv}$ with a group structure defining

$$[(a, b)/_{\equiv} \cdot (c, d)/_{\equiv}] = (ua, vd)/_{\equiv},$$

where $ub = vc$.

Moreover the map

$$a \longmapsto (1, a)/_{\equiv}$$

is a monoidal homomorphism.

The generated group is denoted by $\mathbf{G_M}$.

Isomorphism

- Since every left complemented monoid is right reversible, starting from $\mathbf{L_A}$, we can construct the generated group $\mathbf{G_{L_A}}$.
- Moreover, we can show that the cancellativity of $\mathbf{L_A}$ is a necessary and sufficient condition to embed $\mathbf{L_A}$, and so $\mathbf{A}$, into the negative cone of an ℓ-group.

Isomorphism

An H-algebra **A** is said to be **regular** if it satisfies

$$((x \to y) \to z) \to ((y \to x) \to z) \le ((x \to y) \to z) \to z$$

and it is said to be **full** if for all $b, c \in A$, if $b \le c$, then there exists $a \in A$ such that $a \to b = c$.

If an H-algebra is regular, the associated group is an ℓ-group. Moreover

Lemma

A cancellative left complemented monoid is isomorphic to the negative cone of an ℓ-group if and only if it is full and regular.

As a consequence,we can show that there is a categorical equivalence between the category of full,regular,cancellative left complemented monoids and the category of ℓ-groups.

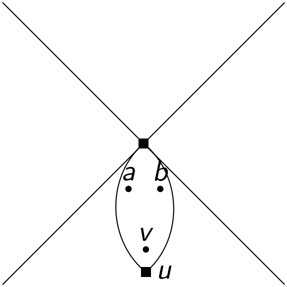
Bounded H-algebras

We need to focus our attention on bounded H-algebras and characterize their embeddings into an ℓ-group.

Lemma

A bounded H-algebra is $\to$-isomorphic to an interval $[u, 1]$ of a given ℓ-group $\mathbf{G}$ if and only if:

- *for all $a, b \in A$, if there exists $v \in L_A$, $v \leq a, b$, such that $a \to v = b \to v$, then $a = b$;*
- *$\mathbf{A}$ is full and regular.*



Bounded H-algebras

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Lemma

A bounded H-algebra is $\rightarrow$-isomorphic to an interval $[u, 1]$ of a given ℓ-group $\mathbf{G}$ if and only if:

- *for all $a, b \in A$, if there exists $v \in L_A$, $v \leq a, b$, such that $a \rightarrow v = b \rightarrow v$, then $a = b$;*
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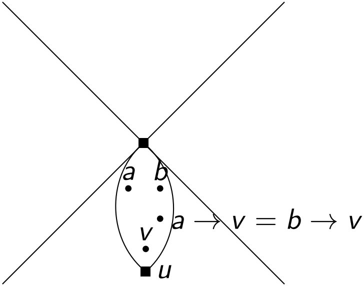

Bounded H-algebras

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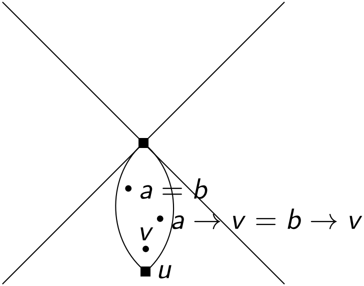
Lemma

A bounded H-algebra is $\rightarrow$-isomorphic to an interval $[u, 1]$ of a given ℓ-group $\mathbf{G}$ *if and only if:*

- *for all* $a, b \in A$*, if there exists* $v \in L_A$, $v \leq a, b$ *, such that* $a \rightarrow v = b \rightarrow v$*, then* $a = b$*;*
- $\mathbf{A}$ *is full and regular.*



A $\rightarrow$-isomorphism for bounded H-algebras

We can prove that if an H-algebra satisfies Tanaka's equation

$$(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$$

then $\mathbf{L_A}$ is commutative, has a lattice order and satifies the first condition of the previous lemma and so

Corollary

Given a full, regular and bounded H-algebra $\mathbf{A}$*, satisfying Tanaka's equation, then it is* $\rightarrow$*-isomorphic to the interval* $[u, 1]$ *of an* ℓ*-group, where u is the strong unit.*

Towards Mundici's Theorem

Consider $\mathbf{A}$ a MV-algebra.

- First of all, we can observe that $\mathbf{A}$ is full and regular left complemented monoid.
- Moreover it satisfies Tanaka's equation, so by the previous result, it is $\rightarrow$-isomorphic with the interval $[u, 1]$ of an ℓ-group, in the reduced signature.
- We would like to extend the $\rightarrow$-isomorphism to the whole structure.

Towards Mundici's Theorem

An *MV*-algebra can be seen as an algebra having signature $\{0, \to, 1\}$,defining:

$$\neg x = x \to 0$$
$$x \cdot y = \neg(x \to \neg y)$$
$$1 = \neg 0.$$

Thanks to this we gain

Lemma

A MV-algebra is isomorphic with the interval $[u, 1]$ of a given ℓ-group seen in the signature $\{0, \to, 1\}$.

Mundici's Theorem

Building on the previous result, Mundici's theorem follows:

Theorem

The category of MV-algebras and the category of ℓ-groups with strong unit are equivalent.

The non commutative case

- The same construction, with some changes, can be used also to prove Dvurečenskij's categorical equivalence.
- We need to be careful since we also have $\rightsquigarrow$, so we need to extend some results, considering properties given by $\rightsquigarrow$.

The non commutative case

Consider $\mathbf{A} = \langle A, \cdot, \rightarrow, \rightsquigarrow, 1 \rangle$ to be a pseudo MV-algebra.

First we prove that we can extend $\rightsquigarrow$ on $\mathbf{M_A}$ in a way which is dual to the case of $\rightarrow$.

Lemma

It is possible to define an operation $\rightsquigarrow$ on $\mathbf{M_A}$ which extends the operation $\rightsquigarrow$ of $\mathbf{A}$ such that:

1. *for all $a \in M_A$, $u \rightsquigarrow a = a$ and $a \rightsquigarrow u = u$;*
2. *for all $a, b, c \in M_A$, $ab \rightsquigarrow c = b \rightsquigarrow (a \rightsquigarrow c)$;*
3. *for all $a, b, c \in M_A$,*

$$(a \rightsquigarrow b) \rightsquigarrow (a \rightsquigarrow c) = (b \rightsquigarrow a) \rightsquigarrow (b \rightsquigarrow c)$$

Moreover, we can prove that γ respect $\rightsquigarrow$ and so, by construction, $\rightsquigarrow$ becomes an operation of $\mathbf{L_A}$

The non commutative case

$\mathbf{L_A}$ is a left complemented monoid and we prove that it is cancellative, so it is embeddable in the negative cone of an ℓ-group.

Thanks to the embedding, we can define the $\vee$ on $\mathbf{L_A}$ as

$$a \vee b = (a \to b) \rightsquigarrow b = (b \to a) \rightsquigarrow a.$$

We observe that :

1. **A** is full and regular;
2. thanks to the definition of the $\vee$, also the first condition of the technical lemma is satisfied.

A is $\to$-isomorphic to the interval $[u, 1]$ of an ℓ-group, where u is the strong unit.

The non commutative case

As before, we would like to extend the isomorphism to the whole structure.

We observe that the pseudo MV-algebras can be seen as algebras having signature $\{0, \rightarrow, \rightsquigarrow, 1\}$, and so we can prove

Theorem

A pseudo MV-algebra is isomorphic to the interval $[u, 1]$ *of a given* ℓ*-group seen in the signature* $\{0, \rightarrow, \rightsquigarrow, 1\}$.

The non commutative case

Building on the previous result, we can show Dvurečenskij's theorem:

Theorem

The category of pseudo MV-algebras and the category of ℓ-groups with strong unit are equivalent.

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Thank you for your attention