Connexive implication in substructural logics

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Connexive logic

The basic ideas of connexive logic have been traced back to Aristotle's *Prior Analytics* and to Boethius' *De hypotheticis syllogismis* (see McCall 1975).

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Connexive principles reflect a *connection*, or *compatibility*, between the antecedent and consequent of sound conditionals.

Specifically, they establish that a conditional

"if A then B"

is sound provided that the negation of B is incompatible with A.

Connexive logic, cont.

Connexitivty can be expressed in a language containing a unary (negation) connective ~ and a binary (implication) connective ⇒:

Aristotle's Theses:

- $\sim(A \Rightarrow \sim A)$
- $\sim(\sim A \Rightarrow A)$

Boethius' Theses:

- $(A \Rightarrow B) \Rightarrow \sim(A \Rightarrow \sim B)$
- $(A \Rightarrow \sim B) \Rightarrow \sim(A \Rightarrow B)$

Connexive logic, cont.

Connexitivty can be expressed in a language containing a unary (negation) connective $\sim$ and a binary (implication) connective $\Rightarrow$:

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All the above theses fail in classical logic for material implication and negation, i.e., $\sim = \neg$ and $\Rightarrow = \rightarrow$.

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They are satisfied for material bi-implication in classical logic,
i.e., $\sim\, = \neg$ and $\Rightarrow\, = \leftrightarrow$,
however bi-implication is obviously symmetric.

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A connexive logic is nothing but a logic having the above formulas as theorems w.r.t. a negation $\sim$, and a non-symmetric implication $\Rightarrow$.

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A systematic (programmatic) investigation of connexive implication connectives which are *term-definable* within well known systems of non-classical logic has not been offered yet.

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We base our work in the context of *Substructural logics*, and investigate them through the lens of their algebraic semantics of residuated lattices.

Residuated lattices and FL-algebras

A **commutative residuated lattice** is an algebraic structure $\mathbf{R} = (R, \vee, \wedge, \cdot, \to, 1)$, such that

- $(R, \vee, \wedge)$ is a lattice
- $(R, \cdot, 1)$ is a commutative monoid
- For all $x, y, z \in R$

$$x \cdot y \le z \iff y \le x \to z,$$

where $\le$ is the induced lattice order: $x \le y \iff x \wedge y = x$

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An $\mathsf{FL_e}$-*algebra* is simply a 0-pointed commutative residuated lattice, i.e., an expansion of a CRL by an arbitrary constant 0.

Double negation image

In the language of $\mathsf{FL_e}$, we define the unary operation $\neg$ via:

$$\neg x := x \to 0$$

The following (quasi-) identites hold in $\mathsf{FL_e}$:

- $x \leq \neg\neg x$
- $x \leq y \implies \neg y \leq \neg x$ and $\neg\neg x \leq \neg\neg y$
- $\neg\neg\neg x \approx \neg x$ and so $\neg\neg(\neg\neg x) \approx \neg\neg x$
- $\neg\neg x \cdot \neg\neg y \leq \neg\neg(xy)$ [equivalently, $\neg\neg(\neg\neg x \cdot \neg\neg y) \approx \neg\neg(xy)$]

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Nuclear image of $\neg\neg$

Let $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \to, 0, 1\rangle$ be an $\mathsf{FL_e}$-algebra. Then $\neg\neg\mathbf{A} := \langle \neg\neg A, \wedge, \vee_{\neg\neg}, \cdot_{\neg\neg}, \to, 0, \neg\neg 1\rangle$ is an $\mathsf{FL_e}$-algebra where

$\neg\neg A := \{\neg\neg a : a \in A\}$ $\quad x \cdot_{\neg\neg} y := \neg\neg(x \cdot y)$ $\quad x \vee_{\neg\neg} y := \neg\neg(x \vee y)$

Algebrbraization

Theorem

- *For every substructural logic* $\mathbf{L}$, $\mathbf{L}$ *is algebraizable with*
$$\mathsf{V}(\mathbf{L}) := \mathsf{FL_e} \cap \mathrm{Mod}(\{1 \leq \varphi : \ \vdash_{\mathbf{L}} \varphi\}).$$
as its equivalent algebraic semantics. In particular, for any set of formulas $\Phi \cup \{\gamma\}$.
$$\Phi \vdash_{\mathbf{L}} \psi \textit{ iff } \{1 \leq \varphi : \varphi \in \Phi\} \vDash_{\mathsf{V}(\mathbf{L})} 1 \leq \psi,$$
- *For any subvariety* $\mathsf{V} \subseteq \mathsf{FL_e}$, V *is the equivalent algebraic semantics of the logic*
$$\mathbf{L}(\mathsf{V}) := \{\varphi \in Fm_{\mathcal{L}} : \mathsf{V} \vDash 1 \leq \varphi\}, \textit{ and}$$
Specifically, for any set of equations $E \cup \{s \approx t\}$ *one has:*
$$E \vDash_{\mathsf{V}} s \approx t \textit{ iff } \{u \to v, v \to u : (u \approx v) \in E\} \vdash_{\mathbf{L}(\mathsf{V})} (s \to t) \wedge (t \to s).$$

Equational Connexive Laws

Let $\mathbf{A}$ an $\mathsf{FL_e}$-algebra, and define $\neg x := x \to 0$.
Suppose $\Rightarrow$ is binary operation that is term-definable in the language of $\mathsf{FL_e}$.

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We say $(\mathbf{A}, \Rightarrow)$ is proto-connexive if:

- *Equational Aristotle Theses*:
$$1 \leq \neg(x \Rightarrow \neg x) \tag{AT}$$
$$1 \leq \neg(\neg x \Rightarrow x) \tag{AT'}$$
- *Equational Boethius Theses*:
$$1 \leq (x \Rightarrow y) \Rightarrow \neg(x \Rightarrow \neg y) \tag{BT}$$
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We say $(\mathbf{A}, \Rightarrow)$ is connexive if it further satisfies

- *Non-symmetry*: $\Rightarrow$ is not a symmetric relation on A, i.e.,
$$(\mathbf{A}, \Rightarrow) \not\models x \Rightarrow y \approx y \Rightarrow x \tag{NS}$$

A useful proposition

Proposition

Let $\mathbf{A}$ be an $\mathsf{FL_e}$-algebra with and $\Rightarrow$ some binary operation. If the following hold in $\mathbf{A}$:

$$1 \leq x \Rightarrow \neg\neg x \tag{P1}$$

$$\neg\neg(x \Rightarrow y) \approx \neg(x \Rightarrow \neg y), \tag{P2}$$

then $(\mathbf{A}, \Rightarrow)$ is proto-connexive.

Definitions of connexivity

Definition

Let $\Rightarrow$ be a binary operation term-definable in the language $\{\wedge, \vee, \cdot, \rightarrow, 0, 1\}$. Let K be a class of $\mathsf{FL_e}$-algebras and $\mathbf{L}$ a substructural logic.

- We say $(\mathsf{K}, \Rightarrow)$ is proto-connexive if every member of K is proto-connexive.

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- We say $(\mathbf{L}, \Rightarrow)$ is (proto-) connexive if $(\mathsf{V}(\mathbf{L}), \Rightarrow)$ is (proto-) connexive.

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- We say $(\mathbf{L}, \Rightarrow)$ is (proto-) connexive if $(\mathsf{V}(\mathbf{L}), \Rightarrow)$ is (proto-) connexive.

Note

In our setting, a substructural logic being connexive exactly corresponds to the conventional one. Namely, both Aristotle's and Boethius' theses hold and the connective $\Rightarrow$ is not symmetric.

The obvious place to start: $\Rightarrow = \to$

Proposition

The following are equivalent for $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \to, 0, 1 \rangle \in \mathsf{FL_e}$:

- $(\mathbf{A}, \to)$ is proto-connexive.
- $(\mathbf{A}, \to) \models$ (AT).
- 0 is the greatest element in $\mathbf{A}$.

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Proposition

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- $(\mathbf{A}, \rightarrow)$ is proto-connexive.
- $(\mathbf{A}, \rightarrow) \vDash$ (AT).
- 0 is the greatest element in $\mathbf{A}$.

Proposition

Let $\mathsf{V}^0 := \mathsf{FL_e} + (x \leq 0)$. Then $(\mathsf{V}^0, \Rightarrow)$ is connexive.

Note: $\mathsf{V}^0 \vDash \neg x \approx 0$.

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The following are equivalent for $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \to, 0, 1 \rangle \in \mathsf{FL_e}$:

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Let $\mathsf{V}^0 := \mathsf{FL_e} + (x \leq 0)$. Then $(\mathsf{V}^0, \Rightarrow)$ is connexive.

Note: $\mathsf{V}^0 \vDash \neg x \approx 0$.

But this is unsatisfying...

Let $\mathbf{L}^0 := \mathbf{L}(\mathsf{V}^0)$. Then for any formula φ, $\vdash_{\mathbf{L}^0} \varphi \to \neg\varphi$.

The connective $\Rightarrow_\wedge$

We define the connective $\Rightarrow_\wedge$ via

$$x \Rightarrow_\wedge y := (x \to y) \wedge (y \to \neg\neg x),$$

where $\neg x := x \to 0$.

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$$x \Rightarrow_\wedge y := (x \rightarrow y) \wedge (y \rightarrow \neg\neg x),$$

where $\neg x := x \rightarrow 0$.

$\Rightarrow_\wedge$ can be considered as the best candidate for embodying the adage:

"No plausible statement can be implied by an implausible one,"

whenever "it is plausible that x" is formalized as $\neg\neg x$.

The connective $\Rightarrow_\wedge$

We define the connective $\Rightarrow_\wedge$ via

$$x \Rightarrow_\wedge y := (x \to y) \wedge (y \to \neg\neg x),$$

where $\neg x := x \to 0$.

Note, for any $\mathsf{FL_e}$-algebra $\mathbf{A}$, one has that

- $1 \leq x \Rightarrow_\wedge y$ iff $x \leq y$ and $\neg x = \neg y$.
- $x \Rightarrow_\wedge y$ is the largest $c \in A$ such that $x \cdot c \leq y$ and $y \cdot c \leq \neg\neg x$.

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We define the connective $\Rightarrow_\wedge$ via

$$x \Rightarrow_\wedge y := (x \to y) \wedge (y \to \neg\neg x),$$

where $\neg x := x \to 0$.

In fact:

Proposition

For an $\mathsf{FL_e}$-algebra $\mathbf{A}$, $(\mathbf{A}, \Rightarrow_\wedge) \models$ (P1): $1 \leq x \Rightarrow_\wedge \neg\neg x$,
and the following are equivalent:

- $(\mathbf{A}, \Rightarrow_\wedge)$ is proto-connexive.
- $(\mathbf{A}, \Rightarrow_\wedge) \models$ (BT): $1 \leq (x \Rightarrow y) \Rightarrow \neg(x \Rightarrow \neg y)$.
- $(\mathbf{A}, \Rightarrow_\wedge) \models$ (BT'): $1 \leq (x \Rightarrow \neg y) \Rightarrow \neg(x \Rightarrow y)$.
- $(\mathbf{A}, \Rightarrow_\wedge) \models$ (P2) : $\neg\neg(x \Rightarrow y) \approx \neg(x \Rightarrow \neg y)$.
- $(\mathbf{A}, \Rightarrow_\wedge) \models$ (P3) : $\neg(x \Rightarrow y) \approx x \Rightarrow \neg y$.

$\Rightarrow_\wedge$ and connexivity.

Lemma

Let $\mathbf{A}$ *be an* $\mathsf{FL_e}$*-algebra.*
If $(\mathbf{A}, \Rightarrow_\wedge)$ *is proto-connexive then the following hold:*

- $\mathbf{A}$ *satisfies the identity* $x{\to}y \approx x{\Rightarrow_\wedge}(\neg\neg x \wedge y)$.

$\Rightarrow_\wedge$ and connexivity.

Lemma

Let $\mathbf{A}$ *be an* $\mathsf{FL_e}$*-algebra.*
If $(\mathbf{A}, \Rightarrow_\wedge)$ *is proto-connexive then the following hold:*

- $\mathbf{A}$ *satisfies the identity* $x{\to}y \approx x{\Rightarrow_\wedge}(\neg\neg x \wedge y)$.
- $\mathbf{A}$ *has largest element* $\neg\neg 1$, *i.e.,* $\neg\neg\mathbf{A}$ *is integral.*
- $\mathbf{A}$ *is pseudo-complemented, i.e., satisfies* $x \wedge \neg x \leq 0$.
- $\mathbf{A}$ *satisfies* $1 \leq \neg\neg(\neg\neg x{\to}x)$.

The Glivenko Property

Let $\mathbf{L}$ and $\mathbf{K}$ be substructural logics. We say the *Glivenko property holds for* $\mathbf{K}$ *relative to* $\mathbf{L}$ iff

$$\vdash_{\mathbf{L}} \varphi \iff \vdash_{\mathbf{K}} \neg\neg\varphi,$$

for every formula φ.

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Equivalently, this holds iff

$$\mathsf{V}(\mathbf{L}) \vDash 1 \leq t \iff \mathsf{V}(\mathbf{K}) \vDash 1 \leq \neg\neg t$$

for every $\mathsf{FL_e}$-term t.

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for every formula φ.

Which is further equivalent to

$$\mathsf{V}(\mathbf{L}) \models s \approx t \iff \mathsf{V}(\mathbf{K}) \models \neg s \approx \neg t$$

for any $\mathsf{FL_e}$-terms s, t.

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$$\vdash_{\mathbf{L}} \varphi \iff \vdash_{\mathbf{K}} \neg\neg\varphi,$$

for every formula φ.

Let V and W be subvarieties of $\mathsf{FL_e}$. We say the *equational Glivenko property holds for* W *relative to* V iff

$$\mathsf{V} \vDash s \approx t \iff \mathsf{W} \vDash \neg s \approx \neg t$$

for every equation $s \approx t$ in the language of $\mathsf{FL_e}$.

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$$\mathsf{V} \vDash s \approx t \iff \mathsf{W} \vDash \neg s \approx \neg t$$

for every equation $s \approx t$ in the language of $\mathsf{FL_e}$.

- By $\mathbf{G}_{\mathsf{FL_e}}(\mathsf{V})$ we denote the largest subvariety W such that the equational Glivenko property holds for W relative to V.
- $\mathbf{G}_{\mathsf{U}}(\mathsf{V}) := \mathbf{G}_{\mathsf{FL_e}}(\mathsf{V}) \cap \mathsf{U}$, where U is a subvariety of $\mathsf{FL_e}$.

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$$\mathsf{V} \vDash s \approx t \iff \mathsf{W} \vDash \neg s \approx \neg t$$

for every equation $s \approx t$ in the language of $\mathsf{FL_e}$.

- By $\mathbf{G_{FL_e}}(\mathbf{L})$ we denote the weakest substructural logic $\mathbf{K}$ such that the Glivenko property holds for $\mathbf{K}$ relative to $\mathbf{L}$.
- $\mathbf{G_K}(\mathbf{L}) := \mathbf{G_{FL_e}}(\mathbf{L}) \vee \mathbf{K}$, where $\mathbf{K}$ is an extension of $\mathbf{FL_e}$.

Theorem (Galatos & Ono 2006)

$\mathbf{G}_{\mathsf{FL_e}}(\mathsf{BA})$ *is axiomatized relative to* $\mathsf{FL_e}$ *by the following identities:*

(a) $\neg(x \cdot y) \approx \neg(x \wedge y)$

(b) $\neg(x \to y) \approx \neg(\neg x \vee y)$

Alternatively, (b) can be replaced by $1 \leq \neg\neg(\neg\neg x \to x)$.

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Alternatively, (b) can be replaced by $1 \leq \neg\neg(\neg\neg x{\to}x)$.

Lemma

For an $\mathsf{FL_e}$*-algebra* $\mathbf{A}$*, the following are equivalent:*

- $\mathbf{A} \in \mathbf{G}_{\mathsf{FL_e}}(\mathsf{BA})$.
- $\mathbf{A}$ *is pseudo-complemented, satisfies* $1 \leq \neg\neg(\neg\neg x{\to}x)$*, and has* $\neg\neg 1$ *as its greatest element (i.e.,* $\neg\neg\mathbf{A}$ *is integral).*
- $\neg\neg\mathbf{A}$ *is Boolean and* $\mathbf{A} \vDash 1 \leq \neg\neg(\neg\neg x{\to}x)$.

Characterizing when $\Rightarrow_{\wedge}$ is connexive

Recall

For an $\mathsf{FL_e}$-algebra $\mathbf{A}$, if $(\mathbf{A}, \Rightarrow_{\wedge})$ is proto-connexive then:

- $\neg\neg 1$ is the greatest element of $\mathbf{A}$.
- $\mathbf{A}$ is pseudo-complemented.
- $\mathbf{A} \models 1 \leq \neg\neg(\neg\neg x \to x)$.

Characterizing when $\Rightarrow_\wedge$ is connexive

Corollary

Let $\mathbf{A}$ *be an* $\mathsf{FL_e}$*-algebra.*
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Lemma

If $\mathbf{A} \in \mathbf{G}_{\mathsf{FL_e}}(\mathsf{BA})$ *then* $(\mathbf{A}, \Rightarrow_\wedge)$ *is proto-connexive.*

Proof.

Boolean algebras are involutive and have $\wedge = \cdot$, so it follows that

$$\mathbf{G}_{\mathsf{FL_e}}(\mathsf{BA}) \vDash \neg[x \Rightarrow_\wedge y] \approx \neg[(x \to y) \wedge (y \to x)].$$

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Furthermore, it is easily verified that Boolean algebras satisfy the identity $\neg[(x \to y) \wedge (y \to x)] \approx (x \to \neg y) \wedge (\neg y \to x)$.

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Furthermore, it is easily verified that Boolean algebras satisfy the identity $\neg[(x \to y) \wedge (y \to x)] \approx (x \to \neg y) \wedge (\neg y \to x)$. Hence it follows that $\mathbf{G}_{\mathsf{FL_e}}(\mathsf{BA})$ satisfies $\neg(x \Rightarrow_\wedge y) \approx x \Rightarrow_\wedge \neg y$. $\square$

Theorem

Let V *be a subvariety of* $\mathsf{FL_e}$*-algebras. Then the following are equivalent:*

- $(\mathsf{V}, \Rightarrow_\wedge)$ *is proto-connexive.*
- $(\mathsf{V}, \Rightarrow_\wedge) \models$ (BT), (BT'), (P2), *or* (P3).
- $\mathsf{V} \subseteq \mathbf{G}_{\mathsf{FL_e}}(\mathsf{BA})$

Consequently, the largest variety of $\mathsf{FL_e}$*-algebras for which* $\Rightarrow_\wedge$ *is proto-connexive is exactly* $\mathbf{G}_{\mathsf{FL_e}}(\mathsf{BA})$*.*

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- $(\mathsf{V}, \Rightarrow_\wedge) \models$ (BT), (BT'), (P2), *or* (P3).
- $\mathsf{V} \subseteq \mathbf{G}_{\mathsf{FL_e}}(\mathsf{BA})$

Consequently, the largest variety of $\mathsf{FL_e}$-algebras for which $\Rightarrow_\wedge$ is proto-connexive is exactly $\mathbf{G}_{\mathsf{FL_e}}(\mathsf{BA})$.

Facts

Let $\mathbf{A} \in \mathsf{FL_e}$ and suppose $(\mathbf{A}, \Rightarrow_\wedge)$ is proto-connexive. Then

- $\Rightarrow_\wedge$ is material bi-implication in $\neg\neg A$.
- $(\mathbf{A}, \Rightarrow_\circ)$ is also proto-connexive, where $x \Rightarrow_\circ y := (x \to y) \cdot (y \to \neg\neg x)$ [The converse does not hold]. However....

Theorem

Let V *be a subvariety of* $\mathsf{FL_e}$*-algebras. Then the following are equivalent:*

- $(\mathsf{V}, \Rightarrow_\wedge)$ *is proto-connexive.*
- $(\mathsf{V}, \Rightarrow_\wedge) \models \text{(BT)}, \text{(BT')}, \text{(P2)},$ *or* (P3).
- $\mathsf{V} \subseteq \mathbf{G}_{\mathsf{FL_e}}(\mathsf{BA})$

Consequently, the largest variety of $\mathsf{FL_e}$*-algebras for which* $\Rightarrow_\wedge$ *is proto-connexive is exactly* $\mathbf{G}_{\mathsf{FL_e}}(\mathsf{BA})$*.*

Theorem

Let V *be a subvariety of* FL_e*-algebras. And define* $\Rightarrow_\circ$ *via:*

$$x \Rightarrow_\circ y := (x \to y) \cdot (y \to \neg\neg x).$$

Then,

$(\mathsf{V}, \Rightarrow_\wedge)$ *is proto-connexive* *iff* $(\mathsf{V}, \Rightarrow_\circ)$ *is proto-connexive and* $\mathsf{V} \models x \leq \neg\neg 1$.

Theorem

Let V *be a subvariety of* $\mathsf{FL_e}$*-algebras. Then the following are equivalent:*

- $(\mathsf{V}, \Rightarrow_\wedge)$ *is proto-connexive.*
- $(\mathsf{V}, \Rightarrow_\wedge) \vDash$ (BT), (BT'), (P2), *or* (P3).
- $\mathsf{V} \subseteq \mathbf{G}_{\mathsf{FL_e}}(\mathsf{BA})$

Consequently, the largest variety of $\mathsf{FL_e}$*-algebras for which* $\Rightarrow_\wedge$ *is proto-connexive is exactly* $\mathbf{G}_{\mathsf{FL_e}}(\mathsf{BA})$.

Theorem

Let $\mathbf{A} \in \mathbf{G}_{\mathsf{FL_e}}(\mathsf{BA})$ *and* $\Rightarrow \in \{\Rightarrow_\circ, \Rightarrow_\wedge\}$*. Then*

$(A, \Rightarrow)$ *proto-connexive but* $\Rightarrow$ *is symmetric* *iff* $\mathbf{A}$ *is a Boolean algebra.*

The integral case: $\mathsf{FL_{ei}}$

Recall:

$$x \Rightarrow_{\circ} y := (x \to y) \cdot (y \to \neg\neg x) \quad \text{and} \quad x \Rightarrow_{\wedge} y := (x \to y) \wedge (y \to \neg\neg x)$$

The integral case: $\mathsf{FL_{ei}}$

Recall:

$$x \Rightarrow_{\circ} y := (x \to y) \cdot (y \to \neg\neg x) \quad \text{and} \quad x \Rightarrow_{\wedge} y := (x \to y) \wedge (y \to \neg\neg x)$$

Let $\mathsf{FL_{ei}}$ denote the variety of integral $\mathsf{FL_e}$-algebras, i.e., those satisfying the equation $x \leq 1$.

Fact

$\mathsf{FL_{ei}} \vDash x \cdot y \leq x \wedge y$, and hence $\mathsf{FL_{ei}} \vDash x \Rightarrow_{\circ} y \leq x \Rightarrow_{\wedge} y$

The integral case: $\mathsf{FL_{ei}}$

Recall:

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$\mathsf{FL_{ei}} \vDash x \cdot y \leq x \wedge y$, and hence $\mathsf{FL_{ei}} \vDash x \Rightarrow_\circ y \leq x \Rightarrow_\wedge y$

Let $\mathbf{A}$ be an integral $\mathsf{FL_e}$-algebra. By $[\Rightarrow_\circ, \Rightarrow_\wedge]$ we denote the (nonempty) interval of binary operators between $\Rightarrow_\circ$ and $\Rightarrow_\wedge$, i.e.,

$$f \in [\Rightarrow_\circ, \Rightarrow_\wedge] \iff (\forall x, y \in A)[x \Rightarrow_\circ y \leq f(x,y) \leq x \Rightarrow_\wedge y]$$

The integral case: continued

Theorem

Let $\mathbf{A}$ be an integral $\mathsf{FL_e}$-algebra. Then the following are equivalent:

- $\mathbf{A} \in \mathbf{G}_{\mathsf{FL_{ei}}}(\mathsf{BA})$.
- *For all* $\Rightarrow \in [\Rightarrow_\circ, \Rightarrow_\wedge]$, $(\mathbf{A}, \Rightarrow)$ *is proto-connexive.*
- *There exists* $\Rightarrow \in [\Rightarrow_\circ, \Rightarrow_\wedge]$ *such that* $(\mathbf{A}, \Rightarrow) \models (\mathsf{BT})$.

The integral case: continued

Theorem

Let $\mathbf{A}$ *be an integral* $\mathsf{FL_e}$*-algebra. Then the following are equivalent:*

- $\mathbf{A} \in \mathbf{G}_{\mathsf{FL_{ei}}}(\mathsf{BA})$.
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Theorem

Let $\mathbf{A}$ *be an integral* $\mathsf{FL_e}$*-algebra and* $\Rightarrow \in [\Rightarrow_\circ, \Rightarrow_\wedge]$. *Then*

$(A, \Rightarrow)$ *proto-connexive but* $\Rightarrow$ *is symmetric* *iff* $\mathbf{A}$ *is a Boolean algebra.*

The case with weakening: $\mathsf{FL_{ew}}$

Let $\mathsf{FL_{ew}}$ denote the variety of 0-bounded integral $\mathsf{FL_e}$-algebras.

Fact

$\mathbf{G}_{\mathsf{FL_{ew}}}(\mathsf{BA})$ is exactly the variety of pseudo-complemented $\mathsf{FL_{ew}}$-algebras.

The case with weakening: $\mathsf{FL_{ew}}$

Let $\mathsf{FL_{ew}}$ denote the variety of 0-bounded integral $\mathsf{FL_e}$-algebras.

Fact

$\mathbf{G}_{\mathsf{FL_{ew}}}(\mathsf{BA})$ is exactly the variety of pseudo-complemented $\mathsf{FL_{ew}}$-algebras.

Corollary

Let $\mathbf{A} \in \mathsf{FL_{ew}}$. *Then the following are equivalent.*

- $\mathbf{A} \in \mathbf{G}_{\mathsf{FL_{ew}}}(\mathsf{BA})$ *(i.e.,* $\mathbf{A}$ *is pseudo-complemented).*
- *For all* $\Rightarrow \in [\Rightarrow_\circ, \Rightarrow_\wedge]$, $(\mathbf{A}, \Rightarrow)$ *is proto-connexive.*
- *There exists* $\Rightarrow \in [\Rightarrow_\circ, \Rightarrow_\wedge]$ *such that* $(\mathbf{A}, \Rightarrow) \vDash (\mathsf{AT})$.

The case with weakening: $\mathsf{FL_{ew}}$

Let $\mathsf{FL_{ew}}$ denote the variety of 0-bounded integral $\mathsf{FL_e}$-algebras.

Fact

$\mathbf{G}_{\mathsf{FL_{ew}}}(\mathsf{BA})$ is exactly the variety of pseudo-complemented $\mathsf{FL_{ew}}$-algebras.

Corollary

Let $\mathbf{A} \in \mathsf{FL_{ew}}$*. Then the following are equivalent.*

- $\mathbf{A} \in \mathbf{G}_{\mathsf{FL_{ew}}}(\mathsf{BA})$ *(i.e.,* $\mathbf{A}$ *is pseudo-complemented).*
- *For all* $\Rightarrow \in [\Rightarrow_{\circ}, \Rightarrow_{\wedge}]$, $(\mathbf{A}, \Rightarrow)$ *is proto-connexive.*
- *There exists* $\Rightarrow \in [\Rightarrow_{\circ}, \Rightarrow_{\wedge}]$ *such that* $(\mathbf{A}, \Rightarrow) \vDash (\mathsf{AT})$.

Corollary

Let $\Rightarrow \in \{\Rightarrow_{\circ}, \Rightarrow_{\wedge}\}$*. Then* $(\mathsf{HA}, \Rightarrow)$ *is connexive.*

Main results

Theorem

Let $\Rightarrow \in \{\Rightarrow_\circ, \Rightarrow_\wedge\}$*. Then for any variety* $\mathsf{V} \subseteq \mathsf{FL_e}$*,*
if $\mathsf{HA} \subseteq \mathsf{V} \subseteq \mathbf{G}_{\mathsf{FL_e}}(\mathsf{BA})$*, then* $(\mathsf{V}, \Rightarrow)$ *is connexive.*

Main results

Theorem

Let $\Rightarrow \in \{\Rightarrow_\circ, \Rightarrow_\wedge\}$*. Then for any variety* $\mathsf{V} \subseteq \mathsf{FL_e}$*,*
if $\mathsf{HA} \subseteq \mathsf{V} \subseteq \mathbf{G}_{\mathsf{FL_e}}(\mathsf{BA})$*, then* $(\mathsf{V}, \Rightarrow)$ *is connexive.*

Let $\mathbf{CPL}$ and $\mathbf{IPL}$ denote classical and intuitionistic propositional logic, respectively.

Theorem

Let $\Rightarrow \in \{\Rightarrow_\circ, \Rightarrow_\wedge\}$*. Then for any logic* $\mathbf{L}$ *in the interval between* $\mathbf{G}_{\mathbf{FL_e}}(\mathbf{CPL})$ *and* $\mathbf{IPL}$*,* $(\mathbf{L}, \Rightarrow)$ *is connexive.*

Notes on weak and strong Connexitivity

Definition (Wansing & Unterhuber (2019))

A logic called *weakly connexive* if it satisfies Aristotle's theses and the following two *weak* versions of Boethius theses:

$$A \Rightarrow B \vdash {\sim}(A \Rightarrow {\sim}B)$$
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Definition (Kapsner (2012))

A logic called *strongly connexive* is $\Rightarrow$ is non-symmetric and the following hold:

- In no model, $A \Rightarrow \neg A$ is satisfiable nor $\neg A \Rightarrow A$ is satisfiable.
- In no model, $A \Rightarrow B$ and $A \Rightarrow \neg B$ are simultaneously satisfiable.

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Theorem

The conditions of weakly connexive, proto-connexive, connexive, strongly connexive, are equivalent to Aristotle's thesis for any logic in the interval $\mathbf{FL_{ew}}$ *to* $\mathbf{IPL}$.

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- In no model, $A \Rightarrow B$ and $A \Rightarrow \neg B$ are simultaneously satisfiable.

Theorem

The conditions of weakly connexive, proto-connexive, connexive, strongly connexive, are equivalent to Aristotle's thesis for any logic in the interval $\mathbf{FL_{ew}}$ *to* $\mathbf{IPL}$*. Furthermore, they all hold in the interval* $\mathbf{G_{FL_{ew}}}(\mathbf{CPL})$ *to* $\mathbf{IPL}$*.*

Thank you!

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