

Probability via Łukasiewicz logic: A multi-type semantic and proof theoretical account

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joint work (in progress) with
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Motivation: A probabilistic 2-layer logic

e.g. [Hájek, Godo, & Esteva 1995; Flaminio & Montagna 2005; Flaminio & Godo 2007; Flaminio & Montagna 2009; Baldi, Cintula, & Noguera 2020]

$$A ::= p \mid \top \mid \perp \mid \neg A \mid A \sqcap A \mid A \sqcup A$$

$$\phi ::= \mu(A) \mid 1 \mid 0 \mid \sim \phi \mid \phi \wedge \phi \mid \phi \vee \phi \mid \phi \oplus \phi \mid \phi \ominus \phi$$

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- ▶ Classical logic axioms for the non-modal formula fragment
- ▶ Łukasiewicz axioms for the modal formula fragment

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- A1.** From $A \vdash B$ infer $\mu(A) \vdash \mu(B)$;
- A2.** $\mu(\neg A) \dashv\vdash \sim \mu(A)$;
- A3.** $(\mu(A) \ominus \mu(A \wedge B)) \oplus \mu(B) \dashv\vdash \mu(A \vee B)$;
- Nec.** from $\top \vdash A$ infer $1 \vdash \mu(A)$.

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► Semantic framework:

- ▶ Classical formulas are interpreted in a Boolean algebra \mathbb{B} .
- ▶ Probability formulas are interpreted on an (MV-)algebra \mathbb{C} .
- ▶ $\mu : \mathbb{B} \rightarrow \mathbb{C}$, a monotone map.

Proper Multi-type Display Calculi

- ▶ **Display property:**

$$\frac{\frac{Y \vdash X > Z}{X ; Y \vdash Z}}{X \vdash Z < Y}$$

display rules semantically justified by **adjunction/residuation**

- ▶ **Multi-type:** Separate **syntactic types** for different types of semantic objects
- ▶ **Proper:** Rules closed under **uniform substitution** (Wansing '98) **within each type**
- ▶ **Canonical proof of cut elimination (via metatheorem)**

Display calculi and correspondence

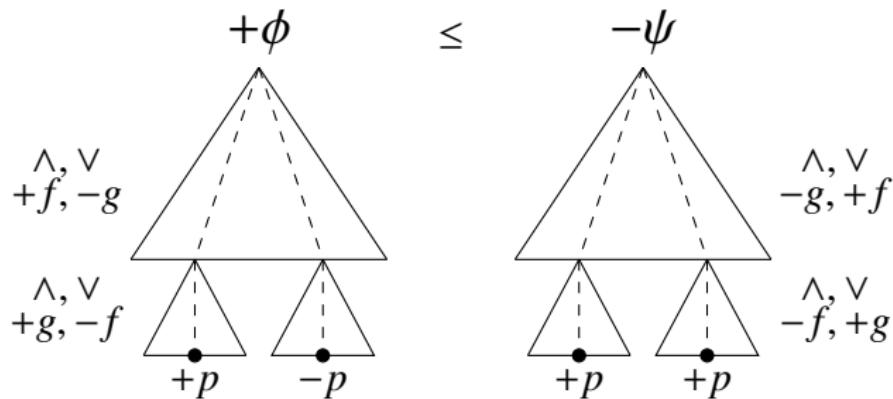
- ▶ The algorithm **ALBA** (properly adjusted) can transform an analytic inductive inequality into primitive quasi-inequalities.
- ▶ Analytic rules in display calculi semantically correspond to primitive quasi-inequalities.

Which logics are properly displayable?

[Kracht 1996], [Ciabattoni & Ramanayake 2015], [Greco⁺ 2016]

Complete characterization:

1. the logics of any **basic** normal (D)LE;
2. axiomatic extensions of these with **analytic inductive inequalities**:



The gaps

1. Many-sorted signature and heterogeneous connectives.
2. The connective μ is monotone not normal.
3. The {connective \oplus is regular} (for join preservation).

Monotone modal logic as a 2-sorted frame

A monotone neighbourhood frame [Chellas 80], [Herzig⁺ 96], [Hansen 03]

$$\mathbb{N} := (W, \nu : W \rightarrow \mathcal{PP}(W))$$

can be represented as a **2-sorted n-frame**:

$$\mathbb{K} := (X, Y, R_\nu, R_\exists, R_{\nu^c}, R_\nexists) \quad \text{where}$$

- ▶ $X := W$ and $Y := \mathcal{P}(W)$;
- ▶ $R_\nu \subseteq X \times Y \quad w R_\nu Z \text{ iff } Z \in \nu(w)$;
- ▶ $R_\exists \subseteq Y \times X \quad Z R_\exists w \text{ iff } w \in Z$ for all $x \in X$ and $Z \in Y$.

$$\nabla\varphi := \langle\nu\rangle[\exists]\varphi$$

$$\mathbb{N}, w \Vdash \nabla\varphi$$

- iff $\exists Z (Z \in \nu(w) \& Z \subseteq \varphi^\mathbb{N})$
- iff $\exists Z (w R_\nu Z \& \forall z (z \in Z \Rightarrow z \Vdash \varphi))$
- iff $\exists Z (w R_\nu Z \& \forall z (Z R_\exists z \Rightarrow z \Vdash \varphi))$
- iff $\exists Z (w R_\nu Z \& Z \Vdash [\exists]\varphi)$
- iff $\mathbb{K}, w \Vdash \langle\nu\rangle[\exists]\varphi$

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can be represented as a **2-sorted n-frame**:

$$\mathbb{K} := (X, Y, R_\nu, R_\exists, \textcolor{red}{R_{\nu^c}}, \textcolor{red}{R_\nexists}) \quad \text{where}$$

- ▶ $X := W$ and $Y := \mathcal{P}(W)$;
- ▶ $\textcolor{red}{R_{\nu^c}} \subseteq X \times Y \quad w R_{\nu^c} Z \text{ iff } Z \notin \nu(w)$;
- ▶ $\textcolor{red}{R_\nexists} \subseteq Y \times X \quad Z R_\nexists w \text{ iff } w \notin Z$ for all $x \in X$ and $Z \in Y$.

$$\nabla \varphi := [\nu^c] \langle \nexists \rangle \varphi$$

$$\mathbb{N}, w \Vdash \nabla \varphi$$

$$\text{iff } \forall Z (Z \notin \nu(w) \Rightarrow \varphi^\mathbb{N} \not\subseteq Z)$$

$$\text{iff } \forall Z (w R_{\nu^c} Z \Rightarrow \exists z (z \notin Z \& z \in \varphi^\mathbb{N}))$$

$$\text{iff } \forall Z (w R_{\nu^c} Z \Rightarrow \exists z (Z R_\nexists z \& z \in \varphi^\mathbb{N}))$$

$$\text{iff } \forall Z (w R_{\nu^c} Z \Rightarrow Z \Vdash \langle \nexists \rangle \varphi)$$

$$\text{iff } \mathbb{K}, w \Vdash [\nu^c] \langle \nexists \rangle \varphi$$

Monotone modal logic as a 2-sorted frame

A monotone neighbourhood frame [Chellas 80], [Herzig⁺ 96], [Hansen 03]

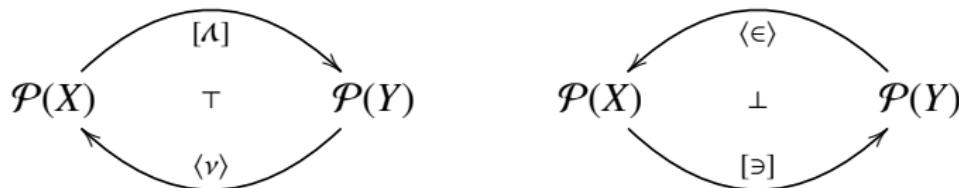
$$\mathbb{N} := (W, \nu : W \rightarrow \mathcal{P}\mathcal{P}(W))$$

can be represented as a **2-sorted n-frame**:

$$\mathbb{K} := (X, Y, R_\nu, R_\exists, R_{\nu^c}, R_{\nexists})$$

and as a **heterogeneous m-algebra**:

$$\mathbb{H} := (\mathcal{P}(X), \mathcal{P}(Y), \langle \nu \rangle, [\exists], [\nu^c], \langle \nexists \rangle)$$



- ▶ $\langle \nu \rangle$ and $[\exists]$ (resp. $[\nu^c]$ and $\langle \nexists \rangle$) **multi-type normal operators**.

Monotone modal logic algebraically

Let $\mathbb{A}_1, \mathbb{A}_2$ be complete lattices and $\nabla : \mathbb{A}_1 \rightarrow \mathbb{A}_2$ be a monotone map. We define maps:

- ▶ $[\exists], \langle \emptyset \rangle : \mathbb{A}_1 \rightarrow \mathcal{P}(\mathbb{A}_1);$
- ▶ $\langle v \rangle, [v^c] : \mathcal{P}(\mathbb{A}_1) \rightarrow \mathbb{A}_2;$

$$[\exists]a := \{b \in \mathbb{A} \mid b \leq a\} \quad \langle v \rangle B := \bigvee \{\nabla b \mid b \in B\}$$

$$[v^c]B := \bigwedge \{\nabla b \mid b \notin B\} \quad \langle \emptyset \rangle a := \{b \in \mathbb{A} \mid a \not\leq b\}.$$

Then $[\exists], \langle \emptyset \rangle, \langle v \rangle, [v^c]$ are **normal operators** and

$$\nabla a = \langle v \rangle [\exists]a = [v^c] \langle \emptyset \rangle a.$$

Positional translation

If \mathbb{F} is a monotone n-frame, $\varphi \Rightarrow \psi$ is an \mathcal{L}_∇ -sequent, \mathbb{F}^* its associated two-sorted n-frame, then

$$\mathbb{F} \Vdash \varphi \Rightarrow \psi \text{ iff } \mathbb{F}^* \Vdash \tau(\varphi \Rightarrow \psi).$$

$$\begin{array}{rcl} \tau(\varphi \Rightarrow \psi) & := & \tau_1(\varphi) \vdash \tau_2(\psi) \\ \hline \tau_1(p) & := & p & \tau_2(p) & := & p \\ \tau_1(\varphi \wedge \psi) & := & \tau_1(\varphi) \wedge \tau_1(\psi) & \tau_2(\varphi \wedge \psi) & := & \tau_2(\varphi) \wedge \tau_2(\psi) \\ \tau_1(\nabla \varphi) & := & \langle v \rangle [\exists] \tau_1(\varphi) & \tau_2(\nabla \varphi) & := & [v^c] \langle \emptyset \rangle \tau_2(\varphi) \end{array}$$

- ▶ Positional translation allows us to transform **more** sequents into analytic inductive sequents.

\mathcal{L} ukasiewicz logic and MV-algebras

An MV-algebra $\langle X, \oplus, \neg, 0 \rangle$ is a set X s.t.:

$$\text{MV1 } x \oplus (y \oplus z) = (x \oplus y) \oplus z$$

$$\text{MV2 } x \oplus y = y \oplus x$$

$$\text{MV3 } x \oplus 0 = x$$

$$\text{MV4 } \neg\neg x = x$$

$$\text{MV5 } x \oplus \neg 0 = \neg 0$$

$$\text{MV6 } \neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$$

A Hilbert-style axiomatization in the fragment $\{\rightarrow, \mathbf{0}\}$ is the following:

$$\mathcal{L}1 \quad A \rightarrow (B \rightarrow A)$$

$$\mathcal{L}2 \quad (A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))$$

$$\mathcal{L}3 \quad ((A \rightarrow B) \rightarrow B) \rightarrow ((B \rightarrow A) \rightarrow A) \quad \text{recall } (B \rightarrow A) \rightarrow A \equiv B \vee A$$

$$\mathcal{L}4 \quad ((A \rightarrow \mathbf{0}) \rightarrow (B \rightarrow \mathbf{0})) \rightarrow (B \rightarrow A)$$

Łukasiewicz operators: basic properties

normal binary diamond

normal binary box

$A \odot \mathbf{0} = \mathbf{0} = \mathbf{0} \odot A$ $(A \vee B) \odot C = (A \odot C) \vee (B \odot C)$ $C \odot (A \vee B) = (C \odot A) \vee (C \odot B)$	$A \oplus \mathbf{1} = \mathbf{1} = \mathbf{1} \oplus A$ $(A \wedge B) \oplus C = (A \oplus C) \wedge (B \oplus C)$ $C \oplus (A \wedge B) = (C \oplus A) \wedge (C \oplus B)$
$A \ominus \mathbf{1} = \mathbf{0} = \mathbf{0} \ominus A$ $(A \vee B) \ominus C = (A \ominus C) \vee (B \ominus C)$ $C \ominus (A \wedge B) = (C \ominus A) \vee (C \ominus B)$	$A \rightarrow \mathbf{1} = \mathbf{1} = \mathbf{0} \rightarrow A$ $(A \vee B) \rightarrow C = (A \rightarrow C) \wedge (B \rightarrow C)$ $C \rightarrow (A \wedge B) = (C \rightarrow A) \wedge (C \rightarrow B)$

residuation

$$\begin{array}{lll} A \odot B \leq C & \text{iff} & B \leq A \rightarrow C \\ C \leq B \oplus A & \text{iff} & C \ominus A \leq B \end{array}$$

ALBA rules for regular connectives (PSZ16)

- ▶ Adjunction rules (only for **unary** regular connectives):

$$\frac{f(\phi) \leq \psi}{f(\perp) \leq \psi \quad \phi \leq \blacksquare_f \psi}$$

- ▶ Approximation rules:

$$\frac{\mathbf{i} \leq f(\phi)}{[\mathbf{i} \leq f(\perp)] \text{ } \wp \text{ } [\mathbf{j} \leq \phi \quad \mathbf{i} \leq f(\mathbf{j})]}$$

$$\frac{\mathbf{i} \leq k(\bar{\phi}_{\epsilon_k^+}, \bar{\psi}_{\epsilon_k^-})}{\wp_{P \subseteq \epsilon_k^+, N \subseteq \epsilon_k^-} (\mathbf{i} \leq k(\bar{\mathbf{j}}_P, \bar{\perp}_{\epsilon_k^+ \setminus P}, \bar{\mathbf{m}}_N, \bar{\top}_{\epsilon_k^- \setminus N}) \text{ } \& \text{ } \bigwedge_{e \in P} (\mathbf{j}_e \leq \phi_e) \text{ } \& \text{ } \bigwedge_{e \in N} (\psi_e \leq \mathbf{m}_e))}$$

ALBA succeeds: But only when non-unary regular connectives appear exclusively in the skeleton.

Spelling out the approximation rule

We have:

$$i \leq \psi_1 \oplus \psi_2 \Leftrightarrow$$

- ▶ $[i \leq 0 \oplus 0]$ OR
- ▶ $[i \leq j_1 \oplus 0 \ \& \ j_1 \leq \psi_1]$ OR
- ▶ $[i \leq 0 \oplus j_2 \ \& \ j_2 \leq \psi_2]$ OR
- ▶ $[i \leq j_1 \oplus j_2 \ \& \ j_1 \leq \psi_1 \ \& \ j_2 \leq \psi_2].$

An example

$$\forall[(p \ominus q) \oplus q \leq p \vee q]$$

iff $\forall[i \leq (p \ominus q) \oplus q \ \& \ p \vee q \leq m \Rightarrow i \leq m]$

iff $\forall[i \leq (j_1 \ominus n) \oplus j_2 \ \& \ j_1 \leq m \ \& \ j_2 \leq n \ \& \ j_2 \leq m \Rightarrow i \leq m] \& [\dots]$

iff $\forall[j_1 \leq m \ \& \ j_2 \leq n \ \& \ j_2 \leq m \Rightarrow (j_1 \ominus n) \oplus j_2 \leq m] \& [\dots]$

Which yields the following **structural rule**:

$$\text{L3} \frac{X_1 \vdash Y_1 \quad X_2 \vdash Y_2 \quad X_2 \vdash Y_3}{(X_1 \hat{\ominus} Y_2) \hat{\oplus} X_2 \vdash Y_1 \check{\vee} Y_3}$$

The red bracket

- ▶ We have 3 cases:
 1. $i \leq 0 \oplus 0 \ \& \ p \vee q \leq m \Rightarrow i \leq m.$
 2. $i \leq j_2 \ \& \ j_2 \leq q \ \& \ p \vee q \leq m \Rightarrow i \leq m.$
 3. $i \leq j_1 \ \& \ j_1 \leq p \ominus q \ \& \ p \vee q \leq m \Rightarrow i \leq m.$
- ▶ All 3 cases are tautological statements.

Putting everything together

$$A ::= p \mid \top \mid \perp \mid \neg A \mid A \sqcap A \mid A \sqcup A$$
$$\alpha ::= [\exists]A \mid \langle \not\models \rangle A$$
$$\phi ::= \langle \nu \rangle \alpha \mid [\nu^c] \alpha \mid 1 \mid 0 \mid \sim \phi \mid \phi \wedge \phi \mid \phi \vee \phi \mid \phi \oplus \phi \mid \phi \ominus \phi$$

- A1. From $A \vdash B$ infer $\langle \nu \rangle [\exists]A \vdash [\nu^c] \langle \not\models \rangle B$;
- A2. $\langle \nu \rangle [\exists] (\neg A) \vdash \sim \langle \nu \rangle [\exists] A$ and $\sim [\nu^c] \langle \not\models \rangle A \vdash [\nu^c] \langle \not\models \rangle \neg A$;
- A3a. $(\langle \nu \rangle [\exists] A \ominus [\nu^c] \langle \not\models \rangle (A \sqcap B)) \oplus \langle \nu \rangle [\exists] B \vdash [\nu^c] \langle \not\models \rangle (A \sqcup B)$;
- A3b. $\langle \nu \rangle [\exists] (A \sqcup B) \vdash ([\nu^c] \langle \not\models \rangle A \ominus \langle \nu \rangle [\exists] (A \sqcap B)) \oplus [\nu^c] \langle \not\models \rangle B$;
- Nec. from $\top \vdash A$ infer $1 \vdash [\nu^c] \langle \not\models \rangle A$.

Structural rules

► A1.

$$\text{M } \frac{\langle \hat{\beta} \rangle \langle \hat{\epsilon} \rangle \Gamma \vdash \Delta}{\langle \hat{\alpha}^c \rangle \langle \hat{v} \rangle \Gamma \vdash \Delta}$$

► A3a.

$$\frac{\langle \hat{\beta} \rangle (\langle \hat{\epsilon} \rangle X \hat{\wedge} \langle \hat{\epsilon} \rangle Y) \vdash Z \quad \langle \hat{\beta} \rangle \langle \hat{\epsilon} \rangle X \vdash W \quad \langle \hat{\beta} \rangle \langle \hat{\epsilon} \rangle Y \vdash W}{\langle \hat{v} \rangle X \hat{\oplus} (\langle \hat{v} \rangle Y \hat{\ominus} [\check{v}^c]Z) \vdash [\check{v}^c]W}$$

► Nec.

$$\text{N } \frac{\langle \hat{\beta} \rangle \hat{T} \vdash \Gamma}{\hat{1} \vdash [\check{v}^c]\Gamma}$$

Conclusions

- ▶ Proof system for probabilistic logics
- ▶ Modular tools to tackle the problems
- ▶ What about cut elimination?

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Thanks!