Strictly join irreducible varieties of residuated lattices

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We show how both these notions can be generalized for non-integral, non-commutative subvarieties of RL, characterizing join irreducibility in a large class of residuated lattices, that include for instance all normal varieties, representable varieties, and ℓ-groups.

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We show how both these notions can be generalized for non-integral, non-commutative subvarieties of RL, characterizing join irreducibility in a large class of residuated lattices, that include for instance all normal varieties, representable varieties, and ℓ-groups.

A key role is played by results implicit in Galatos (2003) concerning the axiomatization of the join of varieties of residuated lattices.

Residuated lattices

A **residuated lattice** is an algebra $\mathbf{A} = \langle A, \vee, \wedge, \cdot, /, \backslash, 1\rangle$ where

1. $\langle A, \vee, \wedge\rangle$ is a lattice;
2. $\langle A, \cdot, 1\rangle$ is a monoid;
3. $/$ and $\backslash$ are the right and left divisions w.r.t. $\cdot$, i.e. $x \cdot y \leq z$ iff $y \leq x\backslash z$ iff $x \leq z/y$.

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Residuated lattices form a variety RL; for an axiomatization, together with the many equations holding in these very rich structures, see Blount-Tsinakis (2003).

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A residuated lattice $\mathbf{A}$ is **integral** if it satisfies the equation $x \leq 1$; it is **commutative** if $\cdot$ is commutative; we will denote by CRL the variety of commutative residuated lattices, and by CIRL the variety of commutative and integral residuated lattices.

Halldén completeness and well-connectedness

The concept of well-connected algebra was introduced by L. Maksimova (1986) to characterize the disjunction property in intermediate logics, i.e. those logics $\mathcal{L}$ for which the corresponding algebraic semantics is a variety of Heyting algebras.

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Classical logic is Halldén complete but does not have the disjunction property, thus differentiating the two concepts.

Kihara and Ono's results

Theorem

Kihara-Ono (2008) For a variety V *of commutative and integral residuated lattices the following are equivalent:*

1. $\mathcal{L}_{\mathsf{V}}$ *is Halldén complete;*
2. V *is join irreducible;*
3. $\mathsf{V} = \mathsf{V}(\mathbf{A})$ *for some well-connected algebra* $\mathbf{A}$.

Theorem

Kihara-Ono (2008) If V *is a variety of commutative and integral residuated lattices and there is a subdirectly irreducible algebra* $\mathbf{A}$ *with* $\mathsf{V} = \boldsymbol{V}(\mathbf{A})$, *then* V *is join irreducible.*

One can extend the definition of well-connectedness to the nonintegral case is a natural way: a residuated lattice **A** is **well-connected** if 1 is **join prime** in **A**, i.e. $a \vee b \geq 1$ implies $a \geq 1$ or $b \geq 1$.

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If V is a variety of commutative residuated lattices then:

- every subdirectly irreducible algebra in V is well-connected;
- if W, Z are subvarieties of V axiomatized (relative to V) by $p \geq 1$ and $q \geq 1$ (and we make sure that p and q have no variables in common), then $\mathsf{W} \vee \mathsf{Z}$ is axiomatized relative to V by $p \vee q \geq 1$.

Both statements are false if we remove commutativity; for the first it is easy to find a finite and integral residuated lattice that is simple but not well-connected, while the second fails fore more general reasons discussed at length in Galatos (2004).

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To find a solution we must delve deeper into the structure of residuated lattices.

Congruence filters

A **filter** F of a residuated lattice $\mathbf{A}$ a subset of A that is a lattice filter containing 1 and is closed under multiplication; let $A^+ = \{a : a \geq 1\}$ and for any $\theta \in \mathrm{Con}(\mathbf{A})$

$$A^+/\theta = \bigcup \{a/\theta : a \geq 1\}$$

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Congruence filters form an algebraic lattice and the two mappings

$$\theta \longmapsto A^+/\theta \qquad\qquad F \longmapsto \theta_F = \{(a,b) : a/b, b/a \in F\}$$

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A residuated lattice is **normal** if every filter is a congruence filter and a variety of residuated lattices is **normal** if each of its members is normal.

Iterated conjugates

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An **iterated conjugate** in $\mathbf{A}$ is a unary term $\gamma_{a_1}(\gamma_{a_2}(\ldots\gamma_{a_n}(x)))$ where $a_1, \ldots, a_n \in A$ and $\gamma_{a_i} \in \{l_{a_i}, r_{a_i}\}$ for $i = 1 \ldots n$, $n \in \mathbb{N}$.

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Lemma

Let $\mathbf{A}$ be a residuated lattice and let $X \subseteq A$; then the congruence filter generated by X in $\mathbf{A}$ is the set

$$\mathrm{F}(X) = \{b \in A : \gamma_1(a_1)\dots\gamma_n(a_n) \leq b,\ n \in \mathbb{N},\ a_1, \dots, a_n \in X \cup \{1\}, \\ \gamma_i \in \Gamma^k(\mathbf{A}) \text{ for some } k \in \mathbb{N}, i = 1 \dots n\}.$$

The trick

We define a set $B^n(x, y)$ of equations in two variables x, y for all $n \in \mathbb{N}$ in the following way; let Γ^n be the set of iterated conjugates of *length* n (i.e. a composition of n left and right conjugates) over the appropriate language, with $\Gamma^0 = \{l_1\}$.

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For all $n \in \mathbb{N}$

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Let **A** be a residuated lattice and $a, b \in A$; we say that **A satisfies** $B^n(a, b)$, in symbols $\mathbf{A} \models B^n(a, b)$ if $\mathbf{A}, a, b \models B^n(x, y)$. i.e. $\gamma_1(a) \vee \gamma_2(b) = 1$ for all $\gamma_1, \gamma_2 \in \Gamma^n(\mathbf{A})$.

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We say that **A satisfies** $(G_{n,k})$ if for all $a, b \in A$, if $\mathbf{A} \models B^n(a, b)$, then $\mathbf{A} \models B^k(a, b)$.

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Lemma

Let V *be a variety of residuated lattices and let* $p(x_1, \ldots, x_n) \geq 1$, $q(y_1, \ldots, y_m) \geq 1$ *be two inequalities not holding in* V. *If* W *and* Z *are the subvarieties axiomatized by* $p \wedge 1 \approx 1$ *and* $q \wedge 1 \approx 1$ *respectively, then* $\mathsf{W} \vee \mathsf{Z}$ *is axiomatized by the set* $B(p,q) = \bigcup_{n \in \mathbb{N}} B^n(p,q)$. *Moreover if* V *satisfies* $(G_{l,l+1})$ *for some* $l \in \mathbb{N}$, *then* $\mathsf{W} \vee \mathsf{Z}$ *is axiomatized by the finite set* $B^l(p,q)$.

Γ-connectedness and Γ-completeness

A residuated lattice $\mathbf{A}$ is Γ^n-**connected** if for all $a, b \in A$, if $\gamma_1(a) \vee \gamma_2(b) = 1$ for all $\gamma_1, \gamma_2 \in \Gamma_n(\mathbf{A})$, then either $a \geq 1$ or $b \geq 1$.

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Lemma

Let V *be a variety of residuated lattices that satisfies* $(G_{n,n+1})$*. Then every subdirectly irreducible algebra in* V *is* Γ_n*-connected.*

Finally we complete the connections with logics: a logic $\mathcal{L}$ is Γ^n-**complete** if for all formulas ϕ and ψ which have no variables in common, if $\mathcal{L} \vdash B^n_{\mathcal{L}}(\phi, \psi)$ then either $\mathcal{L} \vdash \phi$ or $\mathcal{L} \vdash \psi$.

The main results

Theorem

Let V be a variety of residuated lattices satisfying $(G_{n,n+1})$ for some $n \in \mathbb{N}$; then the following are equivalent.

1. $\mathcal{L}_{\mathsf{V}}$ *is* Γ^n*-complete;*
2. V *is join irreducible;*
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Theorem

Let V *be a variety of residuated lattices that satisfies* $(G_{n,n+1})$ *for some* $n \in \mathbb{N}$*. Then* V *is join irreducible if and only if there is a subdirectly irreducible algebra* $\mathbf{A} \in \mathsf{V}$ *such that* $\boldsymbol{V}(\mathbf{A}) = \mathsf{V}$*.*

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THANK YOU!