# Structural completeness and lattice of extensions in many-valued logics with rational constants

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# Łukasiewicz, Gödel and Product logics

- $Ł_\infty$ Infinite valued Łukasiewicz logic, finite valued $Ł_n$ (1920)
- **G** Gödel-Dummett Logic (1959), finite valued $G_n$ (1932)
- **P** Product logic (1996)

Common frame:

- Fuzzy logics
- Algebraizable logics

# Axiomatic Extensions of Basic fuzzy logic

- $Ł_\infty = BL + \neg\neg\varphi \to \varphi$
- $G = BL + \varphi \to \varphi * \varphi$
- $P = BL + \neg(\varphi \wedge \neg\varphi) + \neg\varphi \vee ((\varphi \to \varphi * \psi) \to \psi)$

- $Ł_\infty$, $G$ and $P$ have some type of strong standard completeness theorem

- $L_\infty$ is algebraizable and the class $MVA$ of all MV-algebras is its equivalent (quasi)variety semantics.
- $G$ is algebraizable and the class $GA$ of all Gödel algebras is its equivalent (quasi)variety semantics.
- $P$ is algebraizable and the class $PA$ of all Product algebras is its equivalent (quasi)variety semantics.

# Algebraic logic

$$\text{Deductive Systems} \longleftrightarrow \text{Quasivarieties}$$

$$L \longleftrightarrow \mathbb{K}$$

$$\langle Prop(X), \vdash_L \rangle \longleftrightarrow \langle Eq(X), \models_{\mathbb{K}} \rangle$$

$$\text{Axiomatic Extensions} \longleftrightarrow \text{(Relative) Varieties}$$

$$L_{AE}(L) \cong L_{(R)\mathcal{V}}(\mathbb{K})$$

$$\text{Finitary Extensions of } L \longleftrightarrow \text{Quasivarieties of } \mathbb{K}$$

$$L_{FE}(L) \cong L_{\mathcal{Q}}(\mathbb{K})$$

# Admissibility Theory

A rule $\psi_0, \ldots, \psi_{n-1}/\varphi$ is $L$-**admissible** in $L$ iff for each substitution $\sigma$, $\vdash_L \sigma(\psi_i)$ for every $i < n$ implies $\vdash_L \sigma(\varphi)$.

A logic $L$ is **structurally complete** iff every admissible rule is a derivable rule.

A logic is **hereditarily** structurally complete iff every extension is structurally complete.

*Every logic $L$ admits a unique structural completion $L^+$*

$$L \text{ is structurally complete} \quad \longleftrightarrow \quad \mathbb{K} = Q(\mathbf{F}_{\mathbb{K}}(\omega))$$

$$L \text{ is hereditarily structurally complete} \quad \longleftrightarrow \quad L_{(R)V}(\mathbb{K}) = L_Q(\mathbb{K})$$

# Łukasiewicz logic

- $MVA = V([0,1]_L) = V([0,1]_L \cap \mathbb{Q}) = Q([0,1]_L) = Q([0,1]_L \cap \mathbb{Q})$.
  Every finite valued logic $Ł_n$ is an axiomatic extension of $Ł_\infty$.
  Proper consistent axiomatic extension of $L_\infty$ are given by

  $$\{\mathrm{L}_m \mid m \in I\} \cup \{\mathrm{L}_n^\omega \mid n \in J\}$$

  where $I$ and $J$ are finite subsets of integers $\geq 1$, not both empty.
  $L_V(MVA)$ is a countable infinite Pseudo-Boolean algebra.

- $Ł_\infty$ is not structurally complete.
  Structurally complete axiomatic extensions of $Ł_\infty$ are $Ł_2$ and $Ł_2^\omega$.
  $L_Q(MVA)$ is as complicated as it can be.
  $MVA$ is $Q$-universal. i.e. For every quasivariety $\mathbb{K}$ of finite language,
  $L_Q(\mathbb{K}) \in HS(L_Q(MVA))$

# Gödel logic

- $GA = V([0,1]_G) = V([0,1]_G \cap \mathbb{Q}) = Q([0,1]_G) = Q([0,1]_G \cap \mathbb{Q})$.
  Proper axiomatic extensions of G are exactly finite valued Gödel logics $G_n$
  $L_V(GA)$ is isomorphic to the chain $\omega + 1$.

- G is hereditarily structurally complete.
  $L_Q(GA) = L_V(GA)$ is isomorphic to the chain $\omega + 1$.

# Product logic

- $PA = V([0,1]_P) = V([0,1]_P \cap \mathbb{Q}) = Q([0,1]_P) = Q([0,1]_P \cap \mathbb{Q})$.
  P has a unique consistent proper axiomatic extension, namely CPC.
  $L_V(PA)$ is a 3-element chain $\{TA, BA, PA\}$.

- P is hereditarily structurally complete.
  $L_Q(PA) = L_V(PA)$ is a 3-element chain $\{TA, BA, PA\}$.

# Rational expansions

Let $C = \{c_q : q \in [0,1] \cap \mathbb{Q}\}$ be a set of new propositional constants.

Rational Łukasiewicz logic **RŁ** (Rational Gödel logic **RG**, Rational Product logic **RP**) is the expansion of $Ł_\infty$ (G, P) with the constants in $C$ and the following bookkepping axioms:

$(c_q \cdot c_r) \leftrightarrow c_{q*r}$  $(c_q \to c_r) \leftrightarrow c_{q\Rightarrow r}$  $c_0 \leftrightarrow \bot$  $c_1 \leftrightarrow \top$

for every $q, r \in [0,1] \cap \mathbb{Q}$

$R$Ł is a conservative expansion of $Ł_\infty$
$RG$ is a conservative expansion of $G$
$RP$ is a conservative expansion of $P$

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# Canonical Completeness

$$\mathbf{R}_L = \langle [\mathbf{0},\mathbf{1}]_{\mathbf{L}}, \{q : q \in [0,1] \cap Q\} \rangle$$

$$\mathbf{R}_G = \langle [\mathbf{0},\mathbf{1}]_{\mathbf{G}}, \{q : q \in [0,1] \cap Q\} \rangle \qquad \text{i.e. } c_q^{\mathbf{R}_*} = q$$

$$\mathbf{R}_P = \langle [\mathbf{0},\mathbf{1}]_{\mathbf{P}}, \{q : q \in [0,1] \cap Q\} \rangle$$

*Let* $\varphi \in Prop_C(X)$

- $\vdash_{RL} \varphi$ *iff* $\models_{\mathbf{R}_L} \varphi$
- $\vdash_{RG} \varphi$ *iff* $\models_{\mathbf{R}_G} \varphi$
- $\vdash_{RP} \varphi$ *iff* $\models_{\mathbf{R}_P} \varphi$

$\mathbf{Q_L}$, $\mathbf{Q_G}$, $\mathbf{Q_P}$ denote the rational subalgebras of $\mathbf{R_L}$, $\mathbf{R_G}$ and $\mathbf{R_P}$.

# Algebraic rational expansions

- $R Ł$ is algebraizable and the class $RMV$ of all rational MV-algebras is its equivalent variety semantics. Moreover, $RMV = V(\mathbf{R}_L)$.
- $RG$ is algebraizable and the class $RGA$ of all rational Gödel algebras is its equivalent variety semantics. Moreover, $RGA = V(\mathbf{R}_G)$.
- $RP$ is algebraizable and the class $RPA$ of all rational Product algebras is its equivalent variety semantics. Moreover, $RPA = V(\mathbf{R}_P)$.

$\mathbf{A} = \langle \mathcal{A}, \{a_q : q \in [0,1] \cap \mathbb{Q}\}\rangle$ (i.e. $c_q^{\mathbf{A}} = a_q$) is a **rational MV-algebra** (**rational Gödel algebra**, **rational Product algebra**) iff $\mathcal{A} \in MVA$ (GA, PA) and following bookkepping equations hold:

$c_q \cdot c_r \approx c_{q*r} \qquad c_q \to c_r \approx c_{q \Rightarrow r} \qquad c_0 \approx 0 \qquad c_1 \approx 1$

for every $q, r \in [0,1] \cap \mathbb{Q}$

# Goal

To algebraically study axiomatic and finitary extensions of RŁ, RG and RP.

To obtain a description of $L_V(RMV)$, $L_V(RGA)$ and $L_V(RPA)$.

To obtain a description of $L_Q(RMV)$, $L_Q(RGA)$ and $L_Q(RPA)$.

To obtain a base of admissible rules for each extension.

# Rational Łukasiewicz logic

## Canonical standard algebra

Theorem

$\mathbf{R}_L$ *is the only standard rational MV-algebra.*

## Rational definability

*Every rational element in* $[0,1]$ *is implicitly definable in* $[\mathbf{0},\mathbf{1}]_{\mathbf{L}}$

## Unique constant interpretation

$C(\mathbf{A})$ denote the constant subalgebra of $\mathbf{A}$

Corollary

*If* $\mathbf{A} \in RMV$ *is non trivial, then* $C(\mathbf{A}) \cong \mathbf{Q}_L$

# Varieties and quasivarieties of RMV

**Theorem**

*RMV has non trivial subvarieties.*
*$L_V(RMV)$ is a 2-element chain $\{TA, RMV\}$.*

$$\begin{array}{ccc} RMV & = & V(\mathbf{R_L}) = V(\mathbf{Q_L}) = Q(\mathbf{R_L}) = Q(\mathbf{Q_L}) \\ \cup & & \\ TA & & \end{array}$$

**Theorem**

*RŁ is structurally complete.*

**Corollary**

*$L_Q(RMV)$ is a 2-element chain $\{TA, RMV\}$.*

# Rational Gödel Logic

**Non canonical standard rational Gödel algebras:**

Let $r \in (0,1]$,

$$\mathbf{R}_{[0,r)} = \langle [\mathbf{0},\mathbf{1}]_{\mathbf{G}}, \{c_q^{\mathbf{R}_{[0,r)}} : q \in [0,1] \cap \mathbb{Q}\} \rangle$$

$c_q^{\mathbf{R}_{[0,r)}} = q$ for each $q \in [0,r) \cap \mathbb{Q}$ and $c_q^{\mathbf{R}_{[0,r)}} = 1$ for each $q \in [r,1] \cap \mathbb{Q}$.

$$\mathbf{R}_{[0,1)} = \mathbf{R}_G$$

Let $p \in [0,1) \cap \mathbb{Q}$,

$$\mathbf{R}_{[0,p]} = \langle [\mathbf{0},\mathbf{1}]_{\mathbf{G}}, \{c_q^{\mathbf{R}_{[0,p]}} : q \in [0,1] \cap \mathbb{Q}\} \rangle$$

$c_q^{\mathbf{R}_{[0,p]}} = q$ for each $q \in [0,p] \cap \mathbb{Q}$ and $c_q^{\mathbf{R}_{[0,p]}} = 1$ for each $q \in (p,1] \cap \mathbb{Q}$.

$$\mathbf{R}_{[0,0]} = [\mathbf{0},\mathbf{1}]_{\mathbf{G}}$$

# Rational Gödel Logic

$$\mathbf{Q}_r := C(\mathbf{R}_{[0,r)}), \qquad r \in (0,1]$$

$$\mathbf{Q}_p^0 := C(\mathbf{R}_{[0,p]}) \qquad p \in [0,1) \cap \mathbb{Q}$$

*If $\mathbf{A} \in RGA$ is non trivial, then either $C(\mathbf{A}) \cong \mathbf{Q}_r$ for some $r \in (0,1]$ or $C(\mathbf{A}) \cong \mathbf{Q}_p^0$ for some $p \in [0,1) \cap \mathbb{Q}$.*

$$\mathbf{Q}_p^\gamma := \mathbf{Q}_p^0 \oplus \gamma + 1 \qquad \gamma \in \omega + 1$$

# Varieties of RGA

**Theorem**

- *Every nontrivial variety $V$ of rational Gödel algebras is of the form $V(\boldsymbol{Q}_r)$ for some $r \in (0,1]$ or $V(\boldsymbol{Q}_p^{\gamma})$ for some $\gamma \in \omega + 1$ and $p \in [0,1) \cap \mathbb{Q}$ . Moreover $V(\boldsymbol{Q}_r) = V(\boldsymbol{R}_{[0,r)})$ and $V(\boldsymbol{Q}_p^{\omega}) = V(\boldsymbol{R}_{[0,p]})$*
- *$L_V(RGA)$ is an uncountable chain isomorphic to the poset obtained adding a new bottom element to the Dedekind–MacNeille completion of $([0,1) \cap \mathbb{Q}) \times_{lex} (\omega + 1)$.*

# Structural completeness

### Theorem

- *For* $r \in (0,1]$*,* $V(\mathbf{Q}_r)$ *is not structurally complete and* $Q(\mathbf{Q}_r)$ *is its structural completion.*
- *For* $p \in (0,1) \cap \mathbb{Q}$*,* $V(\mathbf{Q}_p^{\gamma})$ *is not structurally complete and* $Q(\mathbf{Q}_p^{\gamma})$ *is its structural completion.*

### Theorem

- *A base for the admissible rules of* $RG_r$ *is given by the following rules:*

$$c_q \vee \varphi / \varphi \qquad q \in (0,r) \cap \mathbb{Q}$$

- *A base for the admissible rules of* $RG_p^{\gamma}$ *is given by the following rule:*

$$c_p \vee \varphi / \varphi$$

# Quasivarieties of RGA

- $L_V(RGA)$ is a uncountable chain in $L_Q(RGA)$
- $\{Q(\mathbf{Q}_r) : r \in (0,1]\}$ is an uncountable antichain in $L_Q(RGA)$
- Is RGA Q-universal?

# Rational Product Logic

**Non canonical standard rational Product algebra:**

$\mathbf{SR_P} = \langle [\mathbf{0}, \mathbf{1}]_\mathbf{P}, \{c_q^{\mathbf{SR_P}} : q \in [0,1] \cap \mathbb{Q}\} \rangle$
where $c_q^{\mathbf{SR_P}} = 1$ for every $q \in (0,1] \cap \mathbb{Q}$ and $c_0^{\mathbf{SR_P}} = 0$.

$C(\mathbf{SR_P}) \cong \mathbf{B_2}$ and $\mathbf{SR_P} = [\mathbf{0}, \mathbf{1}]_\mathbf{P}$

*If* $\mathbf{A} \in RPA$ *is non trivial, then either*

- $C(\mathbf{A}) \cong \mathbf{Q_P}$
- $C(\mathbf{A}) \cong \mathbf{B_2}$

# Varieties of RPA

**Theorem**

*$L_V(RPA)$ is a four-element chain.*

$$
\begin{array}{ccl}
RPA & = & V(\mathbf{R_P}) = V(\mathbf{Q_P}) = Q(\{\mathbf{SR_P}, \mathbf{R_P}\}) \\
\cup & & \\
PA & = & V(\mathbf{SR_P}) \\
\cup & & \\
BA & = & V(\mathbf{B_2}) \\
\cup & & \\
TA & &
\end{array}
$$

# Structural completeness

**Theorem**

*RP is not structurally complete*

**Theorem**

$Q(\mathbf{Q_P})$ *is the structural completion of RPA.*

**Theorem**

*A base for the admissible rules of RP is given by the following rules:*
*For every* $q \in (0,1) \cap \mathbb{Q}$,

$$c_q \vee \varphi / \varphi$$

*For every* $r \in (0,1) \cap \mathbb{Q}$, $n \in \omega$ *such that* $\sqrt[n]{r}$ *is irrational,*

$$(c_r \leftrightarrow \alpha^n) \vee \varphi / \varphi$$

# Quasivarieties of RPA

**Theorem**

*RPA is Q-universal.*

Proof:
$S(\mathcal{P}_{<\omega}(\text{Prime}))$ is the lattice of subalgebras of $\langle \mathcal{P}_{<\omega}(\text{Prime}), \cup, \emptyset \rangle$

[Adams-Dziobiak] If $\mathbb{K}$ is a quasivariety and $\mathbb{M} \in L_Q(\mathbb{K})$ and $h\colon L_Q(\mathbb{M}) \to S(\mathcal{P}_{<\omega}(\text{Prime}))$ is surjective, then $\mathbb{K}$ is Q-universal.

$X = \{p_1, \ldots, p_n\} \in \mathcal{P}_{<\omega}(\text{Prime})$,
$\mathbf{A}_X$ be the subalgebra of $(\mathbf{R_P})^n$ generated by $(\frac{1}{\sqrt{p_1}}, \ldots, \frac{1}{\sqrt{p_n}})$
$\mathbb{M} = Q(\{\mathbf{A}_X : X \in \mathcal{P}_{<\omega}(\text{Prime})\})$
$h\colon L_Q(\mathbb{M}) \to S(\mathcal{P}_{<\omega}(\text{Prime}))$, $h(\mathbb{H}) = \{X : \mathbf{A}_X \in \mathbb{H}\}$

# Resume and concluding remarks

- *R*Ł is hereditarily structurally complete and has no proper consistent extensions.
- *RG* is not structurally complete. It has an uncountable chain of axiomatic extensions, and there is an uncountable antichain of finitary extensions. We do not know if RGA is Q-universal.
- *RP* is not structurally complete. Axiomatic extensions form a 4-element chain. RPA is Q-universal.
- Adding constants may seem innocuous, however it has an impact in structural completeness and the subvariety and subquasivariety lattices.

**Thank you**

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# Quasivarieties of RGA

$Q(\mathbf{Q}_G) = Q(\mathbf{R}_G) \subsetneq RGA$
$c_{\frac{1}{2}} \approx 1 \Rightarrow 0 \approx 1$ is valid in $\mathbf{Q}_G$ but does not hold in $\mathbf{Q}_{\frac{1}{2}} \in RGA$

$r \in (0,1]$, $Q(\mathbf{Q}_r) = Q(\mathbf{R}_{[0,r)}) \subsetneq V(\mathbf{Q}_r)$
$q \in (0,r) \cap \mathbb{Q}$,
$c_q \approx 1 \Rightarrow 0 \approx 1$ is valid in $\mathbf{Q}_r$ but does not hold in $\mathbf{Q}_q \in V(\mathbf{Q}_r)$.

$p \in (0,1) \cap \mathbb{Q}$ and $\gamma \in \omega + 1$, $Q(\mathbf{Q}_p^\gamma) \subsetneq V(\mathbf{Q}_r^\gamma)$
$c_p \approx 1 \Rightarrow 0 \approx 1$ is valid in $\mathbf{Q}_p^\gamma$ but does not hold in $\mathbf{Q}_p \in V(\mathbf{Q}_p^\gamma)$.

$Q(\mathbf{Q}_0^\omega) = V([\mathbf{0},\mathbf{1}]_\mathbf{G}) = GA$ and $Q(\mathbf{Q}_0^n) = V(\mathbf{G}_n) = GA_n$

# Quasivarieties of RPA

### Proposition

$Q(\mathbf{Q_P}) \subsetneq Q(\mathbf{R_P}) \subsetneq Q(\{\mathbf{SR_P}, \mathbf{R_P}\}) = RPA$

Proof:

$c_{\frac{1}{2}} \approx 1 \Rightarrow 0 \approx 1$ is valid in $\mathbf{R_P}$ and does not hold in $\mathbf{SR_P}$.

$x^2 \approx c_{\frac{1}{2}} \Rightarrow 0 \approx 1$ is valid in $\mathbf{Q_P}$ and does not hold in $\mathbf{R_P}$. □

### Corollary

*RP is not structurally complete*