Some proof-theoretical aspects of non-associative, non-commutative multi-modal linear logic

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Motivation – Categorial Grammars

- [Lambek'58] presented a logic describing natural language syntax.
- Lambek calculus contains the connectives $\otimes, \rightarrow, \leftarrow$. Contexts are lists.

PROPOSITIONAL RULES

$$\frac{\Gamma, F, G, \Pi \Rightarrow H}{\Gamma, F \otimes G, \Pi \Rightarrow H} \otimes L \qquad \frac{\Gamma \Rightarrow F \quad \Delta \Rightarrow G}{\Gamma, \Delta \Rightarrow F \otimes G} \otimes R$$

$$\frac{\Delta \Rightarrow F \quad \Gamma, G, \Pi \Rightarrow H}{\Gamma, \Delta, F \rightarrow G, \Pi \Rightarrow H} \rightarrow L \qquad \frac{F, \Gamma \Rightarrow G}{\Gamma \Rightarrow F \rightarrow G} \rightarrow R$$

$$\frac{\Delta \Rightarrow F \quad \Gamma, G, \Pi \Rightarrow H}{\Gamma, G \leftarrow F, \Delta, \Pi \Rightarrow H} \leftarrow L \qquad \frac{\Gamma, F \Rightarrow G}{\Gamma \Rightarrow G \leftarrow F} \leftarrow R$$

INITIAL AND CUT RULES

$$\frac{}{F \Rightarrow F} \text{ init} \qquad \frac{\Delta \Rightarrow F \quad \Gamma, F, \Pi \Rightarrow G}{\Gamma, \Delta, \Pi \Rightarrow G} \text{ cut}$$

Motivation – Natural Language

- Words are given types that correspond to syntactic categories, and parts of speech correspond to types.
- Sentence (s), noun (n), and noun phrase (np) are primitive.
- An intransitive verb is $np \rightarrow s$.
- An adjective is $n \leftarrow n$.
- One or more types are assigned to each word in the lexicon.

Motivation – Natural Language

Words	Types
the	$np \leftarrow n$
Hulk	n
is	$(np \rightarrow s) \leftarrow (n \leftarrow n), (np \rightarrow s) \leftarrow np$
green	$n \leftarrow n, np$
incredible	$n \leftarrow n$

(Moot, Retoré 2012)

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(Moot, Retoré 2012) The grammaticality of

“The Hulk is incredible.”

is attested by the proof

$$\dfrac{\dfrac{\dfrac{\dfrac{}{n \Rightarrow n}\,\text{init} \quad \dfrac{}{n \Rightarrow n}\,\text{init}}{n \leftarrow n, n \Rightarrow n}\,\leftarrow L}{n \leftarrow n \Rightarrow n \leftarrow n}\,\leftarrow R \qquad \dfrac{\dfrac{}{n \Rightarrow n}\,\text{init} \quad \dfrac{\dfrac{}{np \Rightarrow np}\,\text{init} \quad \dfrac{}{s \Rightarrow s}\,\text{init}}{np, np \rightarrow s \Rightarrow s}\,\rightarrow L}{np \leftarrow n, n, np \rightarrow s \Rightarrow s}\,\leftarrow L}{np \leftarrow n, n, (np \rightarrow s) \leftarrow (n \leftarrow n), n \leftarrow n \Rightarrow s}\,\leftarrow L$$

Linguistic Problems with Associativity

- Lambek Calculus over-generates: ungrammatical sentences like

  "The Hulk is green incredible."

- Indeed:

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is provable. So associativity is not a good feature here!

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- On the other hand, the following requires associativity:

  "the girl whom John loves:"

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How to solve that??

Capturing Grammaticality

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- Nested structures rather than lists.

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Non-associatively,

"The Hulk is incredible"

is still marked as grammatical, while

"The Hulk is green incredible"

is not.

Non-associative Lambek Calculus

Here $\Gamma\{\Delta\}$ means that Δ is a subtree of Γ.

PROPOSITIONAL RULES

$$\frac{\Gamma\{(F,G)\} \Rightarrow H}{\Gamma\{F \otimes G\} \Rightarrow H} \otimes L \qquad \frac{\Gamma_1 \Rightarrow F \quad \Gamma_2 \Rightarrow G}{(\Gamma_1,\Gamma_2) \Rightarrow F \otimes G} \otimes R$$

$$\frac{\Delta \Rightarrow F \quad \Gamma\{G\} \Rightarrow H}{\Gamma\{(\Delta, F \to G)\} \Rightarrow H} \to L \qquad \frac{(F,\Gamma) \Rightarrow G}{\Gamma \Rightarrow F \to G} \to R$$

$$\frac{\Delta \Rightarrow F \quad \Gamma\{G\} \Rightarrow H}{\Gamma\{(G \leftarrow F, \Delta)\} \Rightarrow H} \leftarrow L \qquad \frac{(\Gamma,F) \Rightarrow G}{\Gamma \Rightarrow G \leftarrow F} \leftarrow R$$

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Simulating Associativity

One can recapture the original associative system by introducing the following structural rules.

$$\frac{\Gamma\{((\Delta_1, \Delta_2), \Delta_3)\} \Rightarrow G}{\Gamma\{(\Delta_1, (\Delta_2, \Delta_3))\} \Rightarrow G}\ \text{A1} \qquad \frac{\Gamma\{(\Delta_1, (\Delta_2, \Delta_3))\} \Rightarrow G}{\Gamma\{((\Delta_1, \Delta_2), \Delta_3)\} \Rightarrow G}\ \text{A2}$$

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Our approach: Subexponentials!!!

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Our approach: Subexponentials!!!

Why: They allow more fine-grained control over usage of structural rules.

Outline

1. Linear logic with sub-exponentials (SELL)
2. Non-commutativity/associativity SELL
3. Undecidability
4. Concluding remarks

Outline

Intuitionistic linear logic in a nutshell

- Linear conjunctions: $\&$ (additive) and $\otimes$ (multiplicative)
  Linear disjunction: $\oplus$ (additive)
  Unities: $1, \bot$
  Linear implication: $\multimap$
  Exponential: $!$

Intuitionistic linear logic in a nutshell

- Linear conjunctions: & (additive) and $\otimes$ (multiplicative)
  Linear disjunction: $\oplus$ (additive)
  Unities: $1, \bot$
  Linear implication: $\multimap$
  Exponential: !
- By composing a proof of $F \multimap G$ and a proof of F we consume them to get a proof of G.
- Linear logic formulas behave like resources.
- Exponentials recover the full expressive power of intuitionistic and classical logic: in $!F$ and $?F$ we are allowed to use contraction and weakening.

Subexponentials [Danos,Joinet,Schellinx'93]

Exponentials in ILL:

$$\frac{\Gamma, F \Rightarrow G}{\Gamma, !\, F \Rightarrow G}\ !_L \qquad \frac{!\, F_1, \ldots, !\, F_n \Rightarrow F}{!\, F_1, \ldots, !\, F_n \Rightarrow !\, F}\ !_R$$

Subexponentials [Danos,Joinet,Schellinx'93]

Sub-exponentials in ILL:

$$\frac{\Gamma, F \Rightarrow G}{\Gamma, !^a F \Rightarrow G} \; !^a{}_L \qquad \frac{!^{a_1} F_1, \ldots, !^{a_n} F_n \Rightarrow F}{!^{a_1} F_1, \ldots, !^{a_n} F_n \Rightarrow !^a F} \; !^a{}_R, \text{ provided } a \preceq a_i$$

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Then:

$!^a F \not\equiv !^b F$ for any $a \neq b$.

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- lack of commutativity $\rightsquigarrow$ lists

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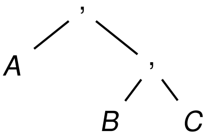
- lack of associativity $\rightsquigarrow$ trees – $(A, (B, C))$



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- exchange

$$\frac{\Gamma\{(\Delta_2, !^{e}\Delta_1)\} \Rightarrow G}{\Gamma\{(!^{e}\Delta_1, \Delta_2)\} \Rightarrow G}\ \mathrm{E1} \qquad \frac{\Gamma\{(!^{e}\Delta_2, \Delta_1)\} \Rightarrow G}{\Gamma\{(\Delta_1, !^{e}\Delta_2)\} \Rightarrow G}\ \mathrm{E2}$$

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About exponential rules...

$$\frac{\Gamma^{\uparrow i} \Rightarrow F}{\Gamma \Rightarrow !^i F}\ !^i R \qquad \frac{\Gamma\{F\} \Rightarrow G}{\Gamma\left\{!^i F\right\} \Rightarrow G}\ \mathrm{der}$$

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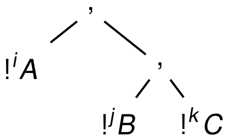
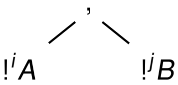
Example. Suppose that $i \preceq j$ but $i \not\preceq k$, and $\mathsf{W} \in f(k)$.

$$\Gamma = (!^i A, (!^j B, !^k C)) \qquad \Gamma^{\uparrow i} = (!^i A, !^j B)$$





System acLL_Σ

Subexponential signature $\Sigma = (I, \preceq, f)$ with:

1. A set of labels I.
2. A set of structural rules $f(i)$ licensed by each i.
3. A preorder $(I, \preceq)$ such that if $i \preceq j$ then $!^j F \Rightarrow !^i F$.
4. Upward-closure: if $i \preceq j$, then $f(i) \subseteq f(j)$.

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Theorem

If the sequent $\Gamma \Rightarrow F$ is provable in acLL_Σ, then it has a proof with no instances of the rule mcut*:*

$$\frac{\Delta \Rightarrow F \quad \Gamma\left\{{}^1F\right\}\ldots\left\{{}^nF\right\} \Rightarrow G}{\Gamma\left\{{}^1\Delta\right\}\ldots\left\{{}^n\Delta\right\} \Rightarrow G}\ \text{mcut}$$

Linguistic Examples

- The necessity of this more fine-grained control of associativity (instead of global associativity) is seen via a combination of these two examples.

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- Phrases like *"The superhero whom Hawkeye killed was incredible"* and *"... was green"* are analysed using $!^a$:

$$(np \leftarrow n, (n, ((n \rightarrow n) \leftarrow (s \leftarrow !^a np), (np, (np \rightarrow s) \leftarrow np)))),$$
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- On the other hand, global non-associativity prevents from deriving incorrect phrases like *"The superhero whom Hawkeye killed was green incredible."*

Outline

Decidability and undecidability Results

Acronym	System	Decidable?
L	Lambek calculus	✓
LL	(propositional) linear logic	✗
ILL	intuitionistic LL	✗
MALL	multiplicative-additive LL	✓
iMALL	intuitionistic MALL	✓
FL	full (multiplicative-additive) L	✓
cLL	non-commutative iMALL	✓
acLL_Σ	non-commutative, non-associative ILL with subexponentials	–
NL	non-associative L	✓
FNL	full (multiplicative-additive) NL	✓
MELL	multiplicative-exponential LL	unknown
SDML	simply dependent multimodal linear logics	–
SMALC_Σ	FL with subexponentials	–

Our undecidability Results

Theorem

If there exists such $s \in I$ that $f(s) \supseteq \{\mathsf{C}, \mathsf{W}\}$, then the derivability problem in acLL_Σ is undecidable. Moreover, this holds for the fragment with only $\otimes$, $\rightarrow$, $\oplus$, $!^s$.

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This result follows from undecidability of derivability from hypotheses (consequence relation) for the multiplicative-additive Lambek calculus [Chvalovský'15]. This result is a refinement of a result by [Tanaka'19].

Undecidability Results

Theorem

If there are $a, c \in I$ such that $f(a) = \{\mathrm{A1}, \mathrm{A2}\}$ and $f(c) \supseteq \{\mathrm{C}, \mathrm{W}, \mathrm{A1}, \mathrm{A2}\}$ then the derivability problem in acLL_Σ is undecidable, in the fragment with only $\rightarrow$, $!^a$, $!^c$.

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If there are $a, c \in I$ *such that* $f(a) = \{\mathsf{A1}, \mathsf{A2}\}$ *and* $f(c) \supseteq \{\mathsf{C}, \mathsf{W}, \mathsf{A1}, \mathsf{A2}\}$ *then the derivability problem in* acLL_Σ *is undecidable, in the fragment with only* $\rightarrow, !^a, !^c$.

This result is a purely multiplicative one, and it is based on the corresponding result for the associative system [Kanovich et al.'19].

Outline

We are working on:

- Decidability results on the intersection of the two undecidable fragments.
- Focusing.
- extensions to other normal modalities.

Thanks!!!

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