On proof equivalence for Modal Logics

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Mosaic Kickoff Meeting 2022 10/09/22

- What is a proof?
- When two proofs are the same?
- . . . and why should we care about?
- What is a Combinatorial Proof?
- When two proofs are the same?
- What about modal logics?
- . . . and their proof identity?

What is a proof?

A proof is...

- A sequence of instructions

A proof is...

- A sequence of instructions
- A strategy to win an argumentation

A proof is...

- A sequence of instructions
- A strategy to win an argumentation
- The sound relations between the components of a statement

When two proofs are the same?

Pythagorean theorem

There are many different proofs of the Pythagorean theorem

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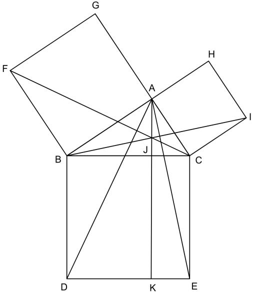

Pythagorean theorem

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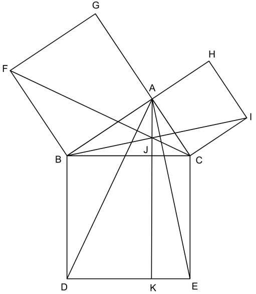

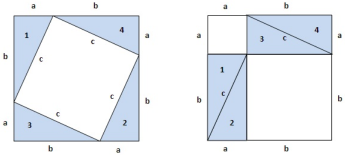

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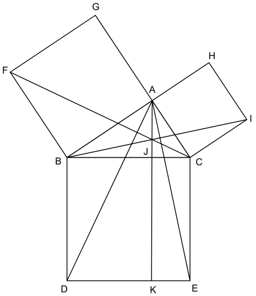

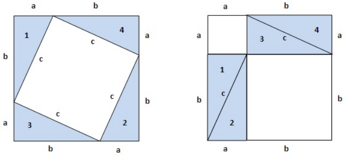

More proofs (122) available at

`http://www.cut-the-knot.org/pythagoras/index.shtml`

Pythagorean theorem

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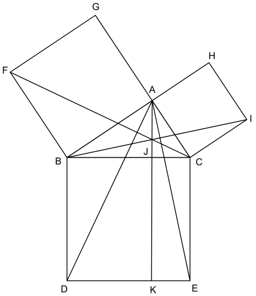
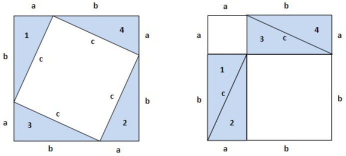


$\stackrel{?}{\cong}$



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Why should we care about?

Definition (Proof Theory)

Proof theory is the branch of mathematical logic that studies proofs as formal mathematical objects.

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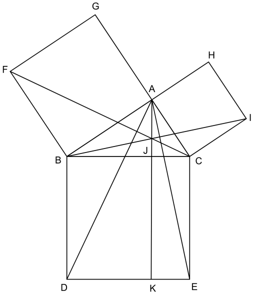
Every definition of a mathematical object comes with a notion of identity



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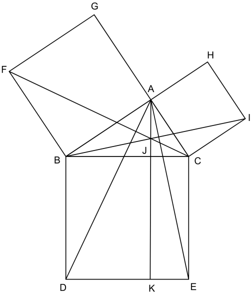

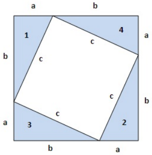

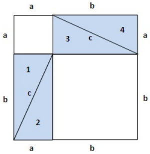

PROBLEM: no agreement on the meaning of “the same”

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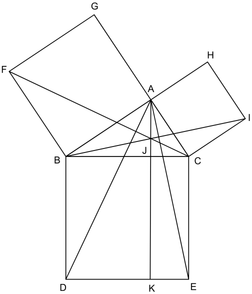

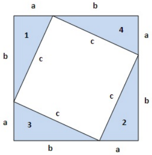

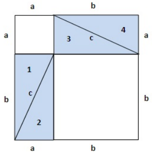

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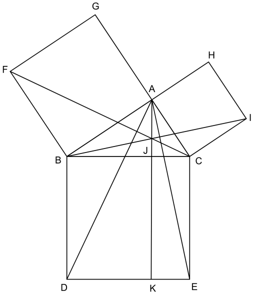
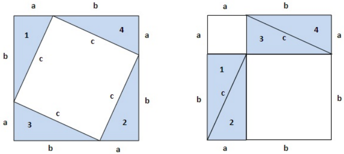


$\overset{?}{\simeq}$



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The 24th Hilbert problem[1]:

Criteria of simplicity, or proof of the greatest simplicity of certain proofs. [. . .]

[1] Found on notes discovered by Thiele in 2000

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Under a given set of conditions there can be but one simplest proof. [...]

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The 24th Hilbert problem[1]:

Criteria of simplicity, or proof of the greatest simplicity of certain proofs. [. . .] Under a given set of conditions there can be but one simplest proof. [. . .] Quite generally, if there are two proofs for a theorem, you must keep going until you have derived each from the other, or until it becomes quite evident what variant conditions (and aids) have been used in the two proofs. [. . .]

[1] Found on notes discovered by Thiele in 2000

Why computer scientists should we care about proof equivalence?

"[God] caused a tumult among them, by producing in them diverse languages, and causing that, through the multitude of those languages, they should not be able to understand one another."
(Flavius Josephus, Antiquities of the Jews, c. 94 CE)

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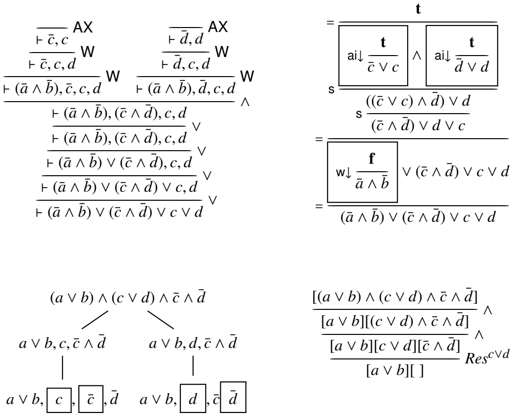

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Coq $\leftrightsquigarrow$ Lean

Equivalence via rule permutations

$$\dfrac{\Gamma_1, \Delta_1 \qquad \dfrac{\Gamma_2, \Delta_2, \Delta_3 \qquad \Gamma_3, \Delta_4}{\Gamma_2, \Gamma_3, \Delta_2, \Sigma_2}\,\rho_2}{\Gamma_1, \Gamma_2, \Gamma_3, \Sigma_1, \Sigma_2}\,\rho_1 \quad \equiv \quad \dfrac{\dfrac{\Gamma_1, \Delta_1 \qquad \Gamma_1, \Delta_2, \Delta_3}{\Gamma_1, \Gamma_2, \Sigma_1, \Delta_2,}\,\rho_1 \qquad \Gamma_3, \Delta_4}{\Gamma_1, \Gamma_2, \Gamma_3, \Sigma_1, \Sigma_2}\,\rho_2$$

$$\dfrac{\dfrac{\Gamma, \Delta_1, \Delta_2}{\Gamma, \Sigma_1, \Delta_2}\,\rho_1}{\Gamma, \Sigma_1, \Sigma_2}\,\rho_2 \quad \equiv \quad \dfrac{\dfrac{\Gamma, \Delta_1, \Delta_2}{\Gamma, \Delta_1, \Sigma_2}\,\rho_2}{\Gamma, \Sigma_1, \Sigma_2}\,\rho_1$$

$$\dfrac{\dfrac{\Gamma, \Delta_1, \Delta_2 \qquad \Gamma_2, \Delta_3}{\Gamma_1, \Gamma_2, \Delta_1, \Sigma_2}\,\rho_2}{\Gamma_1, \Gamma_2, \Sigma_1, \Sigma_2}\,\rho_1 \quad \equiv \quad \dfrac{\dfrac{\Gamma, \Delta_1, \Delta_2}{\Gamma, \Sigma_1, \Delta_2}\,\rho_1 \qquad \Gamma_2, \Delta_3}{\Gamma_1, \Gamma_2, \Sigma_1, \Sigma_2}\,\rho_2$$

We consider some derivations to be the same proof:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{a, \bar{a}}\,\mathsf{AX} \quad \dfrac{}{\bar{b}, b}\,\mathsf{AX}}{a, \bar{a} \otimes \bar{b}, b}\,\otimes}{a \parr (\bar{a} \otimes \bar{b}), b}\,\parr \qquad \dfrac{\dfrac{}{c, \bar{c}}\,\mathsf{AX} \quad \dfrac{}{\bar{d}, d}\,\mathsf{AX}}{c, \bar{c} \otimes \bar{d}, d}\,\otimes}{\dfrac{a \parr (\bar{a} \otimes \bar{b}), b \otimes c, d, \bar{c} \otimes \bar{d}}{a \parr (\bar{a} \otimes \bar{b}), (b \otimes c) \parr d, \bar{c} \otimes \bar{d}}\,\parr}\,\otimes$$

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$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{a,\bar{a}}\,\mathsf{AX} \qquad \dfrac{}{\bar{b}, b}\,\mathsf{AX}}{a, \bar{a}\otimes \bar{b}, b}\,\otimes \qquad \dfrac{\dfrac{}{c,\bar{c}}\,\mathsf{AX}\qquad \dfrac{}{\bar{d}, d}\,\mathsf{AX}}{c,\bar{c}\otimes\bar{d}, d}\,\otimes}{a,(\bar{a}\otimes\bar{b}), b\otimes c, d, \bar{c}\otimes \bar{d}}\,\otimes}{a \parr (\bar{a}\otimes \bar{b}), b\otimes c, d,\bar{c}\otimes\bar{d}}\,\parr}{a\parr(\bar{a}\otimes\bar{b}),(b\otimes c)\parr d,\bar{c}\otimes\bar{d}}\,\parr$$

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$$
\dfrac{
  \dfrac{}{a, \bar{a}}\,\mathsf{AX}
  \qquad
  \dfrac{
    \dfrac{
      \dfrac{}{\bar{b}, b}\,\mathsf{AX}
      \quad
      \dfrac{}{c, \bar{c}}\,\mathsf{AX}
    }{\bar{b}, b \otimes c, \bar{c}}\,\otimes
    \qquad
    \dfrac{}{\bar{d}, d}\,\mathsf{AX}
  }{\bar{b}, b \otimes c, \bar{c} \otimes \bar{d}, d}\,\otimes
}{
  \dfrac{
    \dfrac{a, \bar{a} \otimes \bar{b}, b \otimes c, d, \bar{c} \otimes \bar{d}}{a \parr (\bar{a} \otimes \bar{b}), b \otimes c, d, \bar{c} \otimes \bar{d}}\,\parr
  }{a \parr (\bar{a} \otimes \bar{b}), (b \otimes c) \parr d, \bar{c} \otimes \bar{d}}\,\parr
}\,\otimes
$$

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$$\dfrac{\dfrac{\dfrac{}{a,\bar{a}}\,\mathsf{AX} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\bar{b},b}\,\mathsf{AX} \quad \dfrac{}{c,\bar{c}}\,\mathsf{AX}}{\bar{b},b\otimes c,\bar{c}}\,\otimes \qquad \dfrac{}{\bar{d},d}\,\mathsf{AX}}{\bar{b},b\otimes c,\bar{c}\otimes\bar{d},d}\,\otimes}{(b\otimes c)\parr d,\bar{c}\otimes\bar{d}}\,\parr}{a,\bar{a}\otimes\bar{b},(b\otimes c)\parr d,\bar{c}\otimes\bar{d}}\,\otimes}{a\parr(\bar{a}\otimes\bar{b}),(b\otimes c)\parr d,\bar{c}\otimes\bar{d}}\,\parr$$

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$$\dfrac{\dfrac{\dfrac{\dfrac{}{a,\bar{a}}\,\mathsf{AX} \quad \dfrac{}{\bar{b},b}\,\mathsf{AX}}{a,\bar{a}\otimes\bar{b},b}\,\otimes}{a \parr (\bar{a}\otimes\bar{b}),b}\,\parr \qquad \dfrac{\dfrac{}{c,\bar{c}}\,\mathsf{AX} \quad \dfrac{}{\bar{d},d}\,\mathsf{AX}}{c,\bar{c}\otimes\bar{d},d}\,\otimes}{\dfrac{a \parr (\bar{a}\otimes\bar{b}),b\otimes c,d,\bar{c}\otimes\bar{d}}{a \parr (\bar{a}\otimes\bar{b}),(b\otimes c)\parr d,\bar{c}\otimes\bar{d}}\,\parr}\,\otimes \quad \simeq \quad \dfrac{\dfrac{}{a,\bar{a}}\,\mathsf{AX} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\bar{b},b}\,\mathsf{AX} \quad \dfrac{}{c,\bar{c}}\,\mathsf{AX}}{\bar{b},b\otimes c,\bar{c}}\,\otimes \quad \dfrac{}{\bar{d},d}\,\mathsf{AX}}{\bar{b},b\otimes c,\bar{c}\otimes\bar{d},d}\,\otimes}{(b\otimes c)\parr d,\bar{c}\otimes\bar{d}}\,\parr}{\dfrac{a,\bar{a}\otimes\bar{b},(b\otimes c)\parr d,\bar{c}\otimes\bar{d}}{a \parr (\bar{a}\otimes\bar{b}),(b\otimes c)\parr d,\bar{c}\otimes\bar{d}}\,\parr}\,\otimes$$

Sequences are... sequential (no space for parallelism)

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{}{\vdash a, \bar{a}}\,\mathsf{AX} \quad \dfrac{}{\vdash b, \bar{b}}\,\mathsf{AX}
    }{\vdash a, \bar{a} \wedge \bar{b}, b}\,\wedge
  }{\vdash a \vee (\bar{a} \wedge \bar{b}), b}\,\vee
  \quad
  \dfrac{
    \dfrac{}{\vdash c, \bar{c}}\,\mathsf{AX} \quad \dfrac{}{\vdash d, \bar{d}}\,\mathsf{AX}
  }{\vdash c, \bar{c} \wedge \bar{d}, d}\,\wedge
}{
  \dfrac{\vdash a \vee (\bar{a} \wedge \bar{b}), b \wedge c, d, \bar{c} \wedge \bar{d}}{\vdash a \vee (\bar{a} \wedge \bar{b}), (b \wedge c) \vee d, \bar{c} \wedge \bar{d}}\,\vee
}\,\wedge
$$

$$\simeq$$

$$
\dfrac{
  \dfrac{
    \dfrac{}{\vdash a, \bar{a}}\,\mathsf{AX}
    \quad
    \dfrac{
      \dfrac{
        \dfrac{
          \dfrac{}{\vdash b, \bar{b}}\,\mathsf{AX} \quad \dfrac{}{\vdash c, \bar{c}}\,\mathsf{AX}
        }{\vdash \bar{b}, b \wedge c, \bar{c}}\,\wedge
        \quad
        \dfrac{}{\vdash d, \bar{d}}\,\mathsf{AX}
      }{\vdash \bar{b}, b \wedge c, d, \bar{c} \wedge \bar{d}}\,\wedge
    }{\vdash (b \wedge c) \vee d, \bar{c} \wedge \bar{d}}\,\vee
  }{\vdash a, (\bar{a} \wedge \bar{b}), (b \wedge c) \vee d, \bar{c} \wedge \bar{d}}\,\wedge
}{\vdash a \vee (\bar{a} \wedge \bar{b}), (b \wedge c) \vee d, \bar{c} \wedge \bar{d}}\,\vee
$$

Tracking relations instead of permuting rules

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{AX}
\UnaryInfC{$a, \bar{a}$}
\AxiomC{}
\RightLabel{AX}
\UnaryInfC{$\bar{b}, b$}
\RightLabel{$\otimes$}
\BinaryInfC{$a, \bar{a}\otimes\bar{b}, b$}
\RightLabel{$\parr$}
\UnaryInfC{$a \parr (\bar{a} \otimes \bar{b}), b$}
\AxiomC{}
\RightLabel{AX}
\UnaryInfC{$c, \bar{c}$}
\AxiomC{}
\RightLabel{AX}
\UnaryInfC{$\bar{d}, d$}
\RightLabel{$\otimes$}
\BinaryInfC{$c, \bar{c}\otimes\bar{d}, d$}
\RightLabel{$\otimes$}
\BinaryInfC{$a \parr (\bar{a} \otimes \bar{b}), b\otimes c, d, \bar{c} \otimes \bar{d}$}
\RightLabel{$\parr$}
\UnaryInfC{$a \parr (\bar{a} \otimes \bar{b}), (b \otimes c) \parr d, \bar{c} \otimes \bar{d}$}
\end{prooftree}
$$

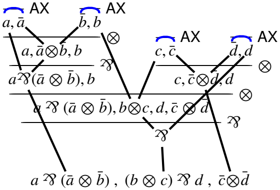
$$\simeq$$

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{AX}
\UnaryInfC{$a, \bar{a}$}
\AxiomC{}
\RightLabel{AX}
\UnaryInfC{$\bar{b}, b$}
\AxiomC{}
\RightLabel{AX}
\UnaryInfC{$c, \bar{c}$}
\RightLabel{$\otimes$}
\BinaryInfC{$\bar{b}, b\otimes c, \bar{c}$}
\AxiomC{}
\RightLabel{AX}
\UnaryInfC{$\bar{d}, d$}
\RightLabel{$\otimes$}
\BinaryInfC{$\bar{b}, b \otimes c, \bar{c}\otimes\bar{d}, d$}
\RightLabel{$\parr$}
\UnaryInfC{$(b \otimes c) \parr d, \bar{c} \otimes \bar{d}$}
\RightLabel{$\otimes$}
\BinaryInfC{$a, \bar{a}\otimes\bar{b}, (b \otimes c) \parr d, \bar{c} \otimes \bar{d}$}
\RightLabel{$\parr$}
\UnaryInfC{$a \parr (\bar{a} \otimes \bar{b}), (b \otimes c) \parr d, \bar{c} \otimes \bar{d}$}
\end{prooftree}
$$

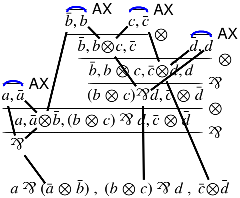
$$
\dfrac{\dfrac{\dfrac{\dfrac{\overset{\frown}{a,\bar{a}}\ \mathsf{AX} \qquad \overset{\frown}{\bar{b},b}\ \mathsf{AX}}{a,\bar{a}\otimes\bar{b},b}\otimes}{a \parr (\bar{a}\otimes\bar{b}),b}\parr \qquad \dfrac{\overset{\frown}{c,\bar{c}}\ \mathsf{AX} \qquad \overset{\frown}{\bar{d},d}\ \mathsf{AX}}{c,\bar{c}\otimes\bar{d},d}\otimes}{a \parr (\bar{a}\otimes\bar{b}),b\otimes c,d,\bar{c}\otimes\bar{d}}\otimes}{a \parr (\bar{a}\otimes\bar{b}),(b\otimes c)\parr d,\bar{c}\otimes\bar{d}}\parr
$$

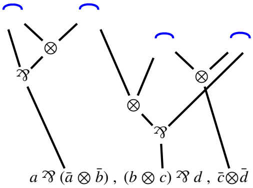
$$\simeq$$

$$
\dfrac{\dfrac{\overset{\frown}{a,\bar{a}}\ \mathsf{AX} \qquad \dfrac{\dfrac{\dfrac{\overset{\frown}{\bar{b},b}\ \mathsf{AX} \qquad \overset{\frown}{c,\bar{c}}\ \mathsf{AX}}{\bar{b},b\otimes c,\bar{c}}\otimes \qquad \overset{\frown}{\bar{d},d}\ \mathsf{AX}}{\bar{b},b\otimes c,\bar{c}\otimes\bar{d},d}\otimes}{(b\otimes c)\parr d,\bar{c}\otimes\bar{d}}\parr}{a,\bar{a}\otimes\bar{b},(b\otimes c)\parr d,\bar{c}\otimes\bar{d}}\otimes}{a \parr (\bar{a}\otimes\bar{b}),(b\otimes c)\parr d,\bar{c}\otimes\bar{d}}\parr
$$

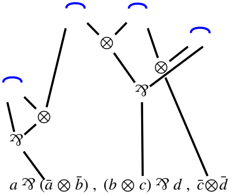

AX
a, ā
AX
b̄, b
a, ā⊗b̄, b
a ⅋ (ā ⊗ b̄), b
AX
c, c̄
AX
d̄, d
c, c̄⊗d̄, d
a ⅋ (ā ⊗ b̄), b⊗c, d, c̄ ⊗ d̄
a ⅋ (ā ⊗ b̄) , (b ⊗ c) ⅋ d , c̄⊗d̄

≃


AX
b̄, b
AX
c, c̄
b̄, b⊗c, c̄
AX
d̄, d
b̄, b ⊗ c, c̄⊗d̄, d
(b ⊗ c) ⅋ d, c̄ ⊗ d̄
AX
a, ā
a, ā⊗b̄, (b ⊗ c) ⅋ d, c̄ ⊗ d̄
a ⅋ (ā ⊗ b̄) , (b ⊗ c) ⅋ d , c̄⊗d̄



$\simeq$



This is an MLL-proof net [Girard ’87]

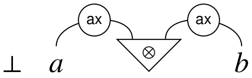
I have a bad news and a good news

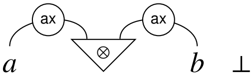
Problem: no proof nets* for extensions of MLL

$$
\frac{\dfrac{\dfrac{}{a, \bar{a}}\,\mathsf{ax}}{a, \bar{a}, \bot}\,\bot \qquad \dfrac{}{b, \bar{b}}\,\mathsf{ax}}{a, \bar{a} \otimes \bar{b}, b, \bot}\,\otimes
\quad \equiv \quad
\frac{\dfrac{\dfrac{}{a, \bar{a}}\,\mathsf{ax} \qquad \dfrac{}{a, \bar{a}}\,\mathsf{ax}}{a, \bar{a} \otimes \bar{b}, a}\,\otimes}{a, \bar{a} \otimes \bar{b}, b, \bot}\,\bot
\quad \equiv \quad
\frac{\dfrac{}{a, \bar{a}}\,\mathsf{ax} \qquad \dfrac{\dfrac{}{b, \bar{b}}\,\mathsf{ax}}{b, \bar{b}, \bot}\,\bot}{a, \bar{a} \otimes \bar{b}, b, \bot}\,\otimes
$$

Problem: no proof nets* for extensions of MLL

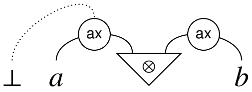
$$\dfrac{\dfrac{\dfrac{}{a,\bar a}\,\mathsf{ax}}{a,\bar a,\bot}\,\bot \qquad \dfrac{}{b,\bar b}\,\mathsf{ax}}{a,\bar a\otimes\bar b, b,\bot}\,\otimes \quad \equiv \quad \dfrac{\dfrac{\dfrac{}{a,\bar a}\,\mathsf{ax}\qquad\dfrac{}{a,\bar a}\,\mathsf{ax}}{a,\bar a\otimes\bar b,a}\,\otimes}{a,\bar a\otimes\bar b,b,\bot}\,\bot \quad \equiv \quad \dfrac{\dfrac{}{a,\bar a}\,\mathsf{ax}\qquad \dfrac{\dfrac{}{b,\bar b}\,\mathsf{ax}}{b,\bar b,\bot}\,\bot}{a,\bar a\otimes\bar b,b,\bot}\,\otimes$$

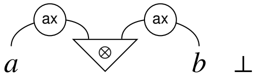




Problem: no proof nets* for extensions of MLL

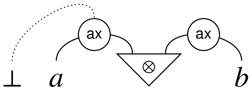
$$\dfrac{\dfrac{\dfrac{}{a, \bar{a}}\,\mathsf{ax}}{a, \bar{a}, \bot}\,\bot \qquad \dfrac{}{b, \bar{b}}\,\mathsf{ax}}{a, \bar{a} \otimes \bar{b}, b, \bot}\,\otimes \quad \equiv \quad \dfrac{\dfrac{\dfrac{}{a, \bar{a}}\,\mathsf{ax} \qquad \dfrac{}{a, \bar{a}}\,\mathsf{ax}}{a, \bar{a} \otimes \bar{b}, a}\,\otimes}{a, \bar{a} \otimes \bar{b}, b, \bot}\,\bot \quad \equiv \quad \dfrac{\dfrac{}{a, \bar{a}}\,\mathsf{ax} \qquad \dfrac{\dfrac{}{b, \bar{b}}\,\mathsf{ax}}{b, \bar{b}, \bot}\,\bot}{a, \bar{a} \otimes \bar{b}, b, \bot}\,\otimes$$

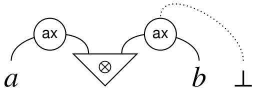




Problem: no proof nets* for extensions of MLL

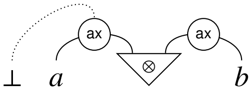
$$
\dfrac{\dfrac{\dfrac{}{a, \bar{a}}\,\mathsf{ax}}{a, \bar{a}, \bot}\,\bot \qquad \dfrac{}{b, \bar{b}}\,\mathsf{ax}}{a, \bar{a} \otimes \bar{b}, b, \bot}\,\otimes
\quad \equiv \quad
\dfrac{\dfrac{\dfrac{}{a, \bar{a}}\,\mathsf{ax} \qquad \dfrac{}{a, \bar{a}}\,\mathsf{ax}}{a, \bar{a} \otimes \bar{b}, a}\,\otimes}{a, \bar{a} \otimes \bar{b}, b, \bot}\,\bot
\quad \equiv \quad
\dfrac{\dfrac{}{a, \bar{a}}\,\mathsf{ax} \qquad \dfrac{\dfrac{}{b, \bar{b}}\,\mathsf{ax}}{b, \bar{b}, \bot}\,\bot}{a, \bar{a} \otimes \bar{b}, b, \bot}\,\otimes
$$

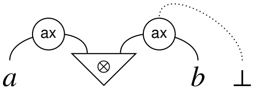




Problem: no proof nets* for extensions of MLL

$$\dfrac{\dfrac{\dfrac{}{a,\bar{a}}\,\text{ax}}{a,\bar{a},\bot}\,\bot \qquad \dfrac{}{b,\bar{b}}\,\text{ax}}{a,\bar{a}\otimes\bar{b},b,\bot}\,\otimes \quad\equiv\quad \dfrac{\dfrac{\dfrac{}{a,\bar{a}}\,\text{ax} \qquad \dfrac{}{a,\bar{a}}\,\text{ax}}{a,\bar{a}\otimes\bar{b},a}\,\otimes}{a,\bar{a}\otimes\bar{b},b,\bot}\,\bot \quad\equiv\quad \dfrac{\dfrac{}{a,\bar{a}}\,\text{ax} \qquad \dfrac{\dfrac{}{b,\bar{b}}\,\text{ax}}{b,\bar{b},\bot}\,\bot}{a,\bar{a}\otimes\bar{b},b,\bot}\,\otimes$$

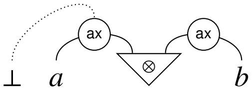


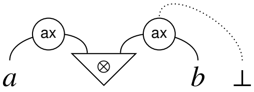


* :proof equivalence is P-space

Problem: no proof nets* for extensions of MLL

$$\dfrac{\dfrac{\dfrac{}{a,\bar{a}}\,\mathsf{ax}}{a,\bar{a},\bot}\,\bot \qquad \dfrac{}{b,\bar{b}}\,\mathsf{ax}}{a,\bar{a}\otimes\bar{b},b,\bot}\,\otimes \quad\equiv\quad \dfrac{\dfrac{\dfrac{}{a,\bar{a}}\,\mathsf{ax} \qquad \dfrac{}{a,\bar{a}}\,\mathsf{ax}}{a,\bar{a}\otimes\bar{b},a}\,\otimes}{a,\bar{a}\otimes\bar{b},b,\bot}\,\bot \quad\equiv\quad \dfrac{\dfrac{}{a,\bar{a}}\,\mathsf{ax} \qquad \dfrac{\dfrac{}{b,\bar{b}}\,\mathsf{ax}}{b,\bar{b},\bot}\,\bot}{a,\bar{a}\otimes\bar{b},b,\bot}\,\otimes$$



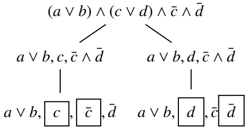


* :proof equivalence is P-space BUT translation and check are P-time

We have combinatorial proofs [Hughes 2006]

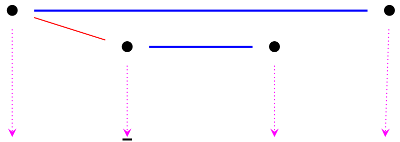
$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \bar{c}, c}\,\text{AX}}{\vdash \bar{c}, c, d}\,\text{W}}{\vdash (\bar{a} \wedge \bar{b}), \bar{c}, c, d}\,\text{W} \qquad \dfrac{\dfrac{\dfrac{}{\vdash \bar{d}, d}\,\text{AX}}{\vdash \bar{d}, c, d}\,\text{W}}{\vdash (\bar{a} \wedge \bar{b}), \bar{d}, c, d}\,\text{W}}{\vdash (\bar{a} \wedge \bar{b}), (\bar{c} \wedge \bar{d}), c, d}\,\wedge}{\vdash (\bar{a} \wedge \bar{b}), (\bar{c} \wedge \bar{d}), c, d}\,\vee}{\vdash (\bar{a} \wedge \bar{b}) \vee (\bar{c} \wedge \bar{d}), c, d}\,\vee}{\vdash (\bar{a} \wedge \bar{b}) \vee (\bar{c} \wedge \bar{d}) \vee c, d}\,\vee}{\vdash (\bar{a} \wedge \bar{b}) \vee (\bar{c} \wedge \bar{d}) \vee c \vee d}\,\vee
$$

$$
{=}\,\dfrac{\mathbf{t}}{\text{s}\,\dfrac{\boxed{\text{ai}\!\downarrow \dfrac{\mathbf{t}}{\bar{c} \vee c}} \wedge \boxed{\text{ai}\!\downarrow \dfrac{\mathbf{t}}{\bar{d} \vee d}}}{\text{s}\,\dfrac{((\bar{c} \vee c) \wedge \bar{d}) \vee d}{{=}\,\dfrac{(\bar{c} \wedge \bar{d}) \vee d \vee c}{{=}\,\dfrac{\boxed{\text{w}\!\downarrow \dfrac{\mathbf{f}}{\bar{a} \wedge \bar{b}}} \vee (\bar{c} \wedge \bar{d}) \vee c \vee d}{(\bar{a} \wedge \bar{b}) \vee (\bar{c} \wedge \bar{d}) \vee c \vee d}}}}}
$$



$$
\dfrac{\dfrac{\dfrac{[(a \vee b) \wedge (c \vee d) \wedge \bar{c} \wedge \bar{d}]}{[a \vee b][(c \vee d) \wedge \bar{c} \wedge \bar{d}]}\,\wedge}{[a \vee b][c \vee d][\bar{c} \wedge \bar{d}]}\,\wedge}{[a \vee b][\;]}\,Res^{c \vee d}
$$

We have combinatorial proofs [Hughes 2006]



$$(\bar{a} \wedge \bar{b}) \vee (\bar{c} \wedge \bar{d}) \vee (d \vee c)$$

- Canonical representation of proofs
- Proof system (Cook-Reckhow)
- Classical, Intuitionistic, Relavant, Modal, MELL, Costructive Modal

Combinatorial Proofs for Classical Logic

Formulas

$$A, B \coloneqq a \mid \bar{a} \mid A \wedge B \mid A \vee B$$

Sequent Calculus LK

$$\text{ax}\,\frac{}{a, \bar{a}} \quad \vee\,\frac{\Gamma, A, B}{\Gamma, A \vee B} \quad \wedge\,\frac{\Gamma, A \quad B, \Delta}{\Gamma, A \wedge B, \Delta} \quad \text{w}\,\frac{\Gamma}{\Gamma, A} \quad \text{c}\,\frac{\Gamma, A, A}{\Gamma, A}$$

Theorem

LK *is a sound and complete proof system for classical logic.*

Formulas

$$A, B \coloneqq a \mid \bar{a} \mid A \wedge B \mid A \vee B$$

Sequent Calculus LK

$$\text{ax}\,\frac{}{a, \bar{a}} \quad \vee\,\frac{\Gamma, A, B}{\Gamma, A \vee B} \quad \wedge\,\frac{\Gamma, A \quad B, \Delta}{\Gamma, A \wedge B, \Delta} \quad \text{w}\,\frac{\Gamma}{\Gamma, A} \quad \text{c}\,\frac{\Gamma, A, A}{\Gamma, A} \quad \Bigg| \quad \frac{\Gamma, A \quad \bar{A}, \Delta}{\Gamma, \Delta}\,\text{cut}$$

Theorem

LK *is a sound and complete proof system for classical logic.*

Theorem

Cut elimination holds in LK.

Definition

A combinatorial proof of a formula F is an axiom-preserving skew fibration

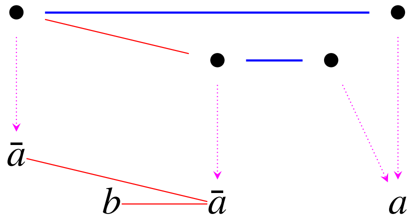
$$f : \mathcal{G} \to [\![F]\!]$$

from a **RB**-cograph $\mathcal{G}$ to the cograph of F.

Ideas:

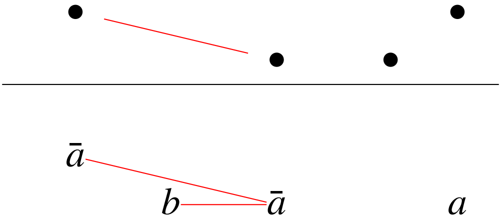
- cograph = graph enconding a formula
- **RB**-cograph = MLL proof nets
- skew fibration = $\{W_\downarrow, C_\downarrow\}$-derivations

Cographs[2]



[2] Duffin 1965

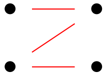
Cographs[2]



[2] Duffin 1965

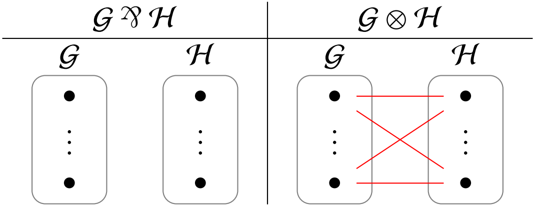
Definition

A **cograph** is a graph containing no four vertices such that

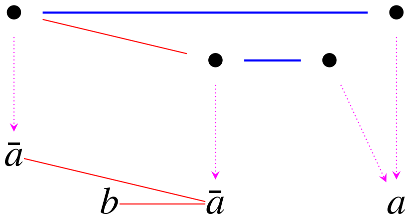


Theorem

A graph is a cograph iff constructed from single-vertices graphs using the graph operations

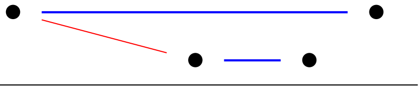


RB-cographs[3]



[3] Retoré 1993

RB-cographs[3]



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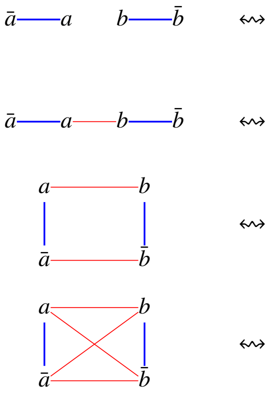


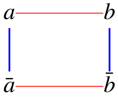
Theorem

*A **RB**-cograph is the encoding of an* MLL*-derivation iff it is æ-connected and æ-acyclic*

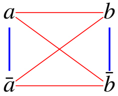
$\rightsquigarrow$

$$\text{mix}\,\frac{\text{AX}\,\dfrac{}{a,\bar{a}} \qquad \text{AX}\,\dfrac{}{b,\bar{b}}(Noderivation)}{\bar{a},a,b,\bar{b}}$$

$\rightsquigarrow$



$\rightsquigarrow$



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Theorem

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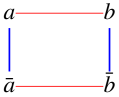
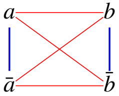
$\rightsquigarrow$ $$\dfrac{\dfrac{}{a, \bar{a}}\,\text{AX} \qquad \dfrac{}{b, \bar{b}}\,\text{AX}\,(Noderivation)}{\bar{a}, a, b, \bar{b}}\,\text{mix}$$

$\rightsquigarrow$ $$\dfrac{\dfrac{}{a, \bar{a}}\,\text{AX} \qquad \dfrac{}{b, \bar{b}}\,\text{AX}}{\bar{a}, a \otimes b, \bar{b}}\,\otimes$$



$\rightsquigarrow$



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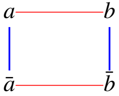
RB-cographs encoding proofs

$\leftrightsquigarrow$

$$\text{mix}\,\dfrac{\text{AX}\,\dfrac{}{a,\bar{a}} \qquad \text{AX}\,\dfrac{}{b,\bar{b}}(Noderivation)}{\bar{a},a,b,\bar{b}}$$

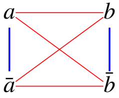
$\leftrightsquigarrow$

$$\otimes\,\dfrac{\text{AX}\,\dfrac{}{a,\bar{a}} \qquad \text{AX}\,\dfrac{}{b,\bar{b}}}{\bar{a},a\otimes b,\bar{b}}$$



$\leftrightsquigarrow$

No derivation



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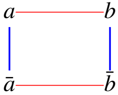
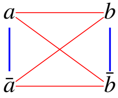


$\leftrightsquigarrow$ No derivation



$\leftrightsquigarrow$

$$\otimes\ \dfrac{⅋\ \dfrac{\text{AX}\ \dfrac{}{a,\bar{a}}}{a ⅋ \bar{a}} \qquad ⅋\ \dfrac{\text{AX}\ \dfrac{}{b,\bar{b}}}{b ⅋ \bar{b}}}{(a ⅋ \bar{a})\otimes(b ⅋ \bar{b})}$$

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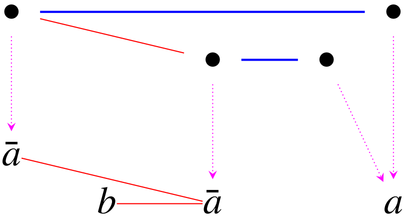
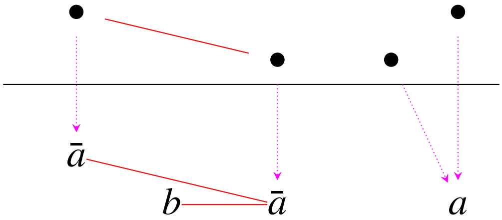
Skew Fibrations[4]



[4]Hughes 2005; Straßburger RTA2007

Skew Fibrations[4]



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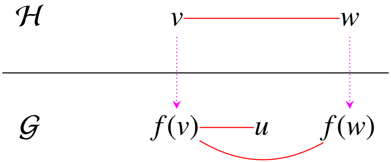
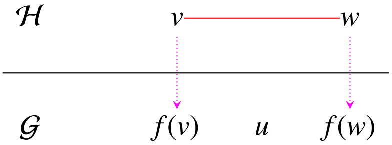


Definition

- A graph **homomorphism** $f : \mathcal{H} \to \mathcal{G}$ between two graphs is a map $f : V_{\mathcal{H}} \to V_{\mathcal{G}}$ preserving $\frown$-edges;



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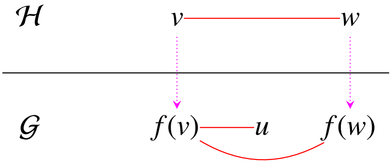

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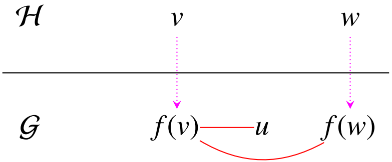
$$f(v) \overset{\mathcal{G}}{\frown} f(w) \Rightarrow v \overset{\mathcal{H}}{\frown} w$$



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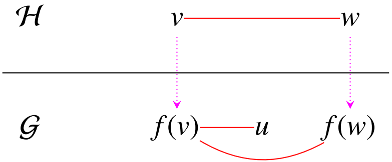

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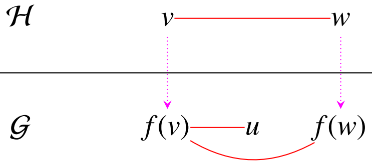
Skew Fibration



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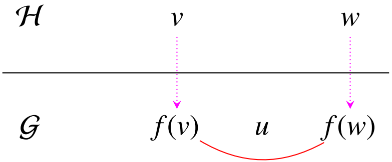

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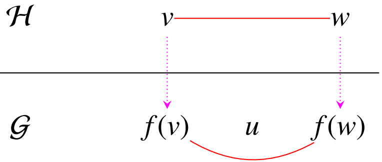

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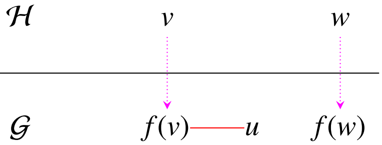

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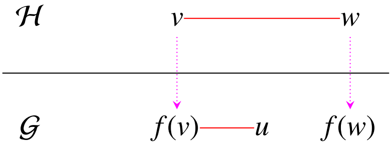
- A **skew fibration** is an homomorphism $f : \mathcal{H} \to \mathcal{G}$ such that

$$f(v) \overset{\mathcal{G}}{\frown} u \Rightarrow v \overset{\mathcal{H}}{\frown} w \text{ for a } w \text{ such that } f(w) \overset{\mathcal{G}}{\not\frown} u$$



Definition

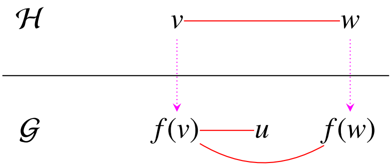
- A graph **homomorphism** $f : \mathcal{H} \to \mathcal{G}$ between two graphs is a map $f : V_{\mathcal{H}} \to V_{\mathcal{G}}$ preserving $\frown$-edges;
- A **fibration** is an homomorphism $f : \mathcal{H} \to \mathcal{G}$ such that

$$f(v) \overset{\mathcal{G}}{\frown} f(w) \Rightarrow v \overset{\mathcal{H}}{\frown} w$$

- A **skew fibration** is an homomorphism $f : \mathcal{H} \to \mathcal{G}$ such that

$$f(v) \overset{\mathcal{G}}{\frown} u \Rightarrow v \overset{\mathcal{H}}{\frown} w \text{ for a } w \text{ such that } f(w) \not\overset{\mathcal{G}}{\frown} u$$

Skew Fibration



Definition

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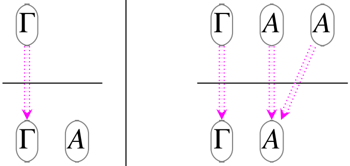
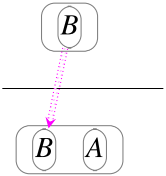
$$f(v) \overset{\mathcal{G}}{\frown} u \Rightarrow v \overset{\mathcal{H}}{\frown} w \text{ for a } w \text{ such that } f(w) \overset{\mathcal{G}}{\not\frown} u$$

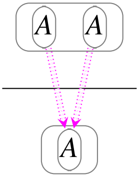
$\mathsf{W}\,\dfrac{\Gamma}{\Gamma, A}$	$\mathsf{C}\,\dfrac{\Gamma, A, A}{\Gamma, A}$



fibration

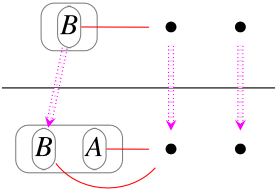
$$\mathsf{w}_{\downarrow}\ \frac{C\,[B]}{C\,[B \vee A]} \qquad \mathsf{c}_{\downarrow}\ \frac{C\,[A \vee A]}{C\,[A]}$$

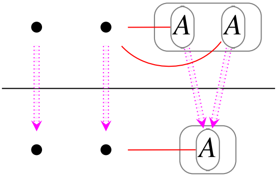




fibration

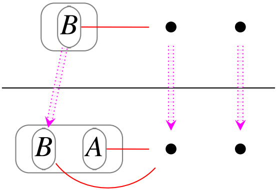
$$\mathsf{w}_{\downarrow}\ \frac{C\,[B]}{C\,[B \vee A]} \qquad\qquad \mathsf{c}_{\downarrow}\ \frac{C\,[A \vee A]}{C\,[A]}$$





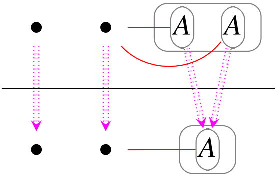
fibration

$$\mathsf{w}{\downarrow}\ \frac{C\,[B]}{C\,[B \vee A]}$$



skew

$$\mathsf{c}{\downarrow}\ \frac{C\,[A \vee A]}{C\,[A]}$$



fibration

Skew Fibrations[5]

Theorem (Decomposition)

$$F' \overset{\{\mathsf{w}_\downarrow, \mathsf{c}_\downarrow\}}{\vdash\!\!\!-} F \Longrightarrow \textit{there is a skew fibration } f : [\![F']\!] \to [\![F]\!]$$

$$\begin{array}{c} F' \\ \| \{\mathsf{w}_\downarrow, \mathsf{c}_\downarrow\} \\ F \end{array} \quad \Longrightarrow \quad \begin{array}{c} F' \\ \| \mathsf{m}_\downarrow \\ G' \\ \| \mathsf{ac}_\downarrow \\ G \\ \| \mathsf{w}_\downarrow \\ F \end{array} \quad \Longrightarrow \quad \begin{array}{c} [\![F']\!] \\ \downarrow \text{bijective} \\ [\![G']\!] \\ \downarrow \text{surjective} \\ [\![G]\!] \\ \downarrow \text{injective} \\ [\![F]\!] \end{array}$$

$$\mathsf{m}_\downarrow \frac{C\,[(A \wedge C) \vee (B \wedge D)]}{C\,[(A \vee B) \wedge (C \vee D)]}$$

$$\mathsf{ac}_\downarrow \frac{C\,[a \vee a]}{C\,[a]}$$

$$\mathsf{w}_\downarrow \frac{C\,[B]}{C\,[A \vee B]}$$

[5] Hughes 2005 ; Straßburger RTA2007

Reassembling the pieces

Theorem (Decomposition)

$$\overset{\mathsf{LK}}{\vdash} F \Longrightarrow \overset{\mathsf{MLL}}{\vdash} F' \overset{\{\mathsf{W}_\downarrow,\mathsf{C}_\downarrow\}}{\vdash} F$$

$$\begin{array}{c} \Vert \,\mathsf{LK} \\ F \end{array}$$

Theorem

Every LK *derivation can be represented by a combinatorial proof*

Theorem (Decomposition)

$$\overset{\mathsf{LK}}{\vdash} F \Longrightarrow \overset{\mathsf{MLL}}{\vdash} F' \overset{\{\mathsf{W}_{\downarrow},\mathsf{C}_{\downarrow}\}}{\vdash} F$$

$$\begin{array}{c} \Vert \mathsf{LK} \\ F \end{array} \quad \Longrightarrow \quad \begin{array}{c} \mathfrak{D}' \Vert \mathsf{MLL} \\ F' \\ \mathfrak{D} \Vert \{\mathsf{W}_{\downarrow},\mathsf{C}_{\downarrow}\} \\ F \end{array}$$

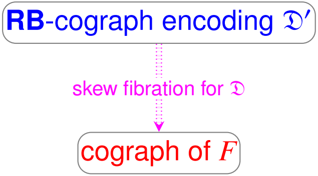
Theorem

Every LK *derivation can be represented by a combinatorial proof*

Theorem (Decomposition)

$$\overset{\text{LK}}{\vdash} F \Longrightarrow \overset{\text{MLL}}{\vdash} F' \overset{\{\text{W}_\downarrow, \text{C}_\downarrow\}}{\vdash} F$$

$$\begin{array}{c} \| \text{LK} \\ F \end{array} \quad \Longrightarrow \quad \begin{array}{c} \mathfrak{D}' \| \text{MLL} \\ F' \\ \mathfrak{D} \| \{\text{W}_\downarrow, \text{C}_\downarrow\} \\ F \end{array} \quad \Longrightarrow$$



Theorem

Every LK *derivation can be represented by a combinatorial proof*

Fact (Cook-Reckhow)

Check whether a syntactic object represents a valid proof can be done by means of a polynomial time algorithm.

- Check if a graph is a cograph
- Check if a **RB**-cograph is æ-connected and æ-acyclic
- Check if a map $f : \mathcal{H} \to \mathcal{G}$ between cograph is a skew fibration
- Check if f is axiom-preserving

Theorem

Combinatorial Proofs form a proof system for classical logic.

Proof equivalence in Classical Logic

Independent rules	$\dfrac{\Gamma_1,\Delta_1 \quad \dfrac{\Gamma_2,\Delta_2,\Delta_3 \quad \Gamma_3,\Delta_4}{\Gamma_2,\Gamma_3,\Delta_2,\Sigma_2}\rho_2}{\Gamma_1,\Gamma_2,\Gamma_3,\Sigma_1,\Sigma_2}\rho_1 = \dfrac{\dfrac{\Gamma_1,\Delta_1 \quad \Gamma_1,\Delta_2,\Delta_3}{\Gamma_1,\Gamma_2,\Sigma_1,\Delta_2,}\rho_1 \quad \Gamma_3,\Delta_4}{\Gamma_1,\Gamma_2,\Gamma_3,\Sigma_1,\Sigma_2}\rho_2$ $\dfrac{\dfrac{\Gamma,\Delta_1,\Delta_2}{\Gamma,\Sigma_1,\Delta_2}\rho_1}{\Gamma,\Sigma_1,\Sigma_2}\rho_2 = \dfrac{\dfrac{\Gamma,\Delta_1,\Delta_2}{\Gamma,\Delta_1,\Sigma_2}\rho_2}{\Gamma,\Sigma_1,\Sigma_2}\rho_1$ $\quad \dfrac{\dfrac{\Gamma,\Delta_1,\Delta_2 \quad \Gamma_2,\Delta_3}{\Gamma_1,\Gamma_2,\Delta_1,\Sigma_2}\rho_2}{\Gamma_1,\Gamma_2,\Sigma_1,\Sigma_2}\rho_1 = \dfrac{\dfrac{\Gamma,\Delta_1,\Delta_2}{\Gamma,\Sigma_1,\Delta_2}\rho_1 \quad \Gamma_2,\Delta_3}{\Gamma_1,\Gamma_2,\Sigma_1,\Sigma_2}\rho_2$
WC-comonad	$\dfrac{\dfrac{\Gamma,A,A,B,B}{\Gamma,A,B}2\times\mathsf{C}}{\Gamma,A\vee B}\vee \equiv_{\mathsf{c}} \dfrac{\dfrac{\Gamma,A,A,B,B\vdash C}{\Gamma,A\vee B,A\vee B}2\times\vee}{\Gamma,A\vee B}\mathsf{C}$ $\quad \dfrac{\dfrac{\Gamma}{\Gamma,A,B}2\times\mathsf{W}}{\Gamma,A\vee B}\vee \equiv_{\mathsf{c}} \dfrac{\Gamma}{\Gamma,A\vee B}\mathsf{W}$ $\dfrac{\dfrac{\Gamma,A,A}{\Gamma,A}\mathsf{C}}{\Gamma,A,A}\mathsf{W} \equiv_{\mathsf{c}} \Gamma,A,A$ $\quad \dfrac{\dfrac{\Gamma,A}{\Gamma,A,A}\mathsf{W}}{\Gamma,A}\mathsf{C} \equiv_{\mathsf{c}} \Gamma,A\vdash B$
Excising and Unfolding	$\dfrac{\Gamma,A \quad \dfrac{\Delta}{B,\Delta}\mathsf{W}}{\Gamma,\Delta,A\supset B}\wedge \equiv_{\mathsf{e}} \dfrac{\Delta}{\Gamma,\Delta,A\wedge B}\mathsf{W}$ $\quad \dfrac{\Gamma,A \quad \dfrac{\Delta,B,B}{\Delta,B}\mathsf{C}}{\Gamma,A\wedge B}\wedge \equiv_{\mathsf{u}} \dfrac{\dfrac{\Gamma,A \quad \dfrac{\Gamma,A \quad \Delta,B,B}{\Gamma,\Delta,A\wedge B,B}\supset^{\mathsf{L}}}{\Gamma,\Gamma,\Delta,A\wedge B,A\wedge B}\supset^{\mathsf{L}}}{\Gamma,\Delta,A\wedge B}\mathsf{C}$

Combinatorial Proofs for Modal Logic

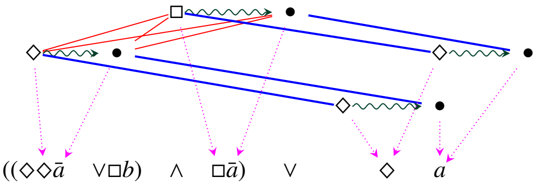
Modal Logic S4 [A. & Straßburger 2019]

Modal Formulas

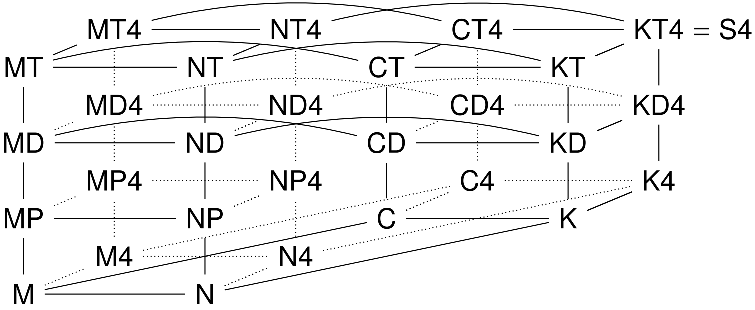
$$A, B := a \mid \bar{a} \mid A \wedge B \mid A \vee B \mid \Box A \mid \Diamond A$$

Sequent Calculus Rules

$$\mathsf{LK} \cup \left\{ \mathsf{K}\,\frac{A, \Gamma}{\Box A, \Diamond \Gamma} \;,\; \mathsf{D}\,\frac{A, \Gamma}{\Diamond A, \Diamond \Gamma} \;,\; \mathsf{T}_{\downarrow}\,\frac{C\,[A]}{C\,[\Diamond A]} \;,\; 4_{\downarrow}\,\frac{C\,[\Diamond\Diamond A]}{C\,[\Diamond A]} \right\}$$



- Encode modalities using special vertices and additional edges
- Encode K and D as classes of modal vertices
- Encode $\mathsf{T}_{\downarrow}$ and $4_{\downarrow}$ as graphs homomorphisms



$$\frac{A, \Gamma}{\Box A, \Diamond \Gamma}\,\mathsf{K} \quad \frac{A, \Gamma}{\Diamond A, \Diamond \Gamma}\,\mathsf{D} \quad \frac{A, B}{\Box A, \Diamond B}\,\mathsf{M} \quad \frac{A}{\Box A}\,\mathsf{N} \quad \frac{A}{\Diamond A}\,\mathsf{P} \quad \frac{A, \Sigma}{\Box A, \Diamond \Sigma}\,\mathsf{C}\ (\text{where } |\Sigma| > 0)$$

Proof equivalence in Classical Modal Logic

Independent rules	$\dfrac{\Gamma_1,\Delta_1 \quad \dfrac{\Gamma_2,\Delta_2,\Delta_3 \quad \Gamma_3,\Delta_4}{\Gamma_2,\Gamma_3,\Delta_2,\Sigma_2}\rho_2}{\Gamma_1,\Gamma_2,\Gamma_3,\Sigma_1,\Sigma_2}\rho_1 = \dfrac{\dfrac{\Gamma_1,\Delta_1 \quad \Gamma_1,\Delta_2,\Delta_3}{\Gamma_1,\Gamma_2,\Sigma_1,\Delta_2,}\rho_1 \quad \Gamma_3,\Delta_4}{\Gamma_1,\Gamma_2,\Gamma_3,\Sigma_1,\Sigma_2}\rho_2$ $\dfrac{\dfrac{\Gamma,\Delta_1,\Delta_2}{\Gamma,\Sigma_1,\Delta_2}\rho_1}{\Gamma,\Sigma_1,\Sigma_2}\rho_2 = \dfrac{\dfrac{\Gamma,\Delta_1,\Delta_2}{\Gamma,\Delta_1,\Sigma_2}\rho_2}{\Gamma,\Sigma_1,\Sigma_2}\rho_1$ $\quad$ $\dfrac{\dfrac{\Gamma,\Delta_1,\Delta_2 \quad \Gamma_2,\Delta_3}{\Gamma_1,\Gamma_2,\Delta_1,\Sigma_2}\rho_2}{\Gamma_1,\Gamma_2,\Sigma_1,\Sigma_2}\rho_1 = \dfrac{\dfrac{\Gamma,\Delta_1,\Delta_2}{\Gamma,\Sigma_1,\Delta_2}\rho_1 \quad \Gamma_2,\Delta_3}{\Gamma_1,\Gamma_2,\Sigma_1,\Sigma_2}\rho_2$
WC-comonad	$\dfrac{\dfrac{\Gamma,A,A,B,B}{\Gamma,A,B}2\times \mathsf{C}}{\Gamma,A\vee B}\vee \equiv_{\mathsf{c}} \dfrac{\dfrac{\Gamma,A,A,B,B\vdash C}{\Gamma,A\vee B,A\vee B}2\times\vee}{\Gamma,A\vee B}\mathsf{C}$ $\quad$ $\dfrac{\dfrac{\Gamma}{\Gamma,A,B}2\times \mathsf{W}}{\Gamma,A\vee B}\vee \equiv_{\mathsf{c}} \dfrac{\Gamma}{\Gamma,A\vee B}\mathsf{W}$ $\dfrac{\dfrac{\Gamma,A,A}{\Gamma,A}\mathsf{C}}{\Gamma,A,A}\mathsf{W} \equiv_{\mathsf{c}} \Gamma,A,A$ $\quad$ $\dfrac{\dfrac{\Gamma,A}{\Gamma,A,A}\mathsf{W}}{\Gamma,A}\mathsf{C} \equiv_{\mathsf{c}} \Gamma,A\vdash B$
Excising and Unfolding	$\dfrac{\Gamma,A \quad \dfrac{\Delta}{B,\Delta}\mathsf{W}}{\Gamma,\Delta,A\supset B}\wedge \equiv_{\mathsf{e}} \dfrac{\overline{\Delta}}{\Gamma,\Delta,A\wedge B}\mathsf{W}$ $\quad$ $\dfrac{\Gamma,A \quad \dfrac{\Delta,B,B}{\Delta,B}\mathsf{C}}{\Gamma,A\wedge B}\wedge \equiv_{\mathsf{u}} \dfrac{\dfrac{\Gamma,A \quad \dfrac{\Gamma,A \quad \Delta,B,B}{\Gamma,\Delta,A\wedge B,B}\supset^{\mathsf{L}}}{\Gamma,\Gamma,\Delta,A\wedge B,A\wedge B}\supset^{\mathsf{L}}}{\Gamma,\Delta,A\wedge B}\mathsf{C}$
Structural vs K	$\dfrac{\dfrac{\Gamma,A}{\Gamma,B,A}\mathsf{W}}{\Diamond\Gamma,\Diamond B,\Box A}\mathsf{K} \equiv_{\Box\mathsf{c}} \dfrac{\dfrac{\Gamma,A}{\Diamond\Gamma,\Box A}\mathsf{K}}{\Diamond\Gamma,\Diamond B,\Box A}\mathsf{W}$ $\quad$ $\dfrac{\dfrac{\Gamma,B,B,A}{\Gamma,B,A}\mathsf{C}}{\Diamond\Gamma,\Diamond B,\Box A}\mathsf{K} \equiv_{\Box\mathsf{c}} \dfrac{\dfrac{\Gamma,B,B,A}{\Diamond\Gamma,\Diamond B,\Diamond B,\Box A}\mathsf{K}^{\Box}}{\Diamond\Gamma,\Diamond B,\Box A}\mathsf{C}$

Constructive Modal Logic

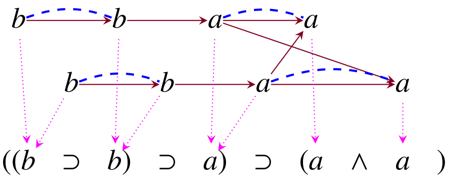
Formulas

$$A, B := a \mid A \wedge B \mid A \supset B$$

Sequent Calculus Rules

$$\frac{}{a \vdash a}\,\mathsf{ax} \quad \frac{\Gamma, B \vdash A}{\Gamma \vdash B \supset A}\supset^{\mathsf{R}} \quad \frac{\Gamma, B, C \vdash A}{\Gamma, B \wedge C \vdash A}\wedge^{\mathsf{L}} \quad \frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \wedge B}\wedge^{\mathsf{R}} \quad \frac{\Gamma \vdash A \quad \Delta, B \vdash C}{\Gamma, \Delta, A \supset B \vdash C}\supset^{\mathsf{L}}$$

$$\frac{}{\vdash 1}\,\mathsf{1} \quad \frac{\Gamma, B, B \vdash A}{\Gamma, B \vdash A}\,\mathsf{C} \quad \frac{\Gamma \vdash A}{\Gamma, B \vdash A}\,\mathsf{W}$$

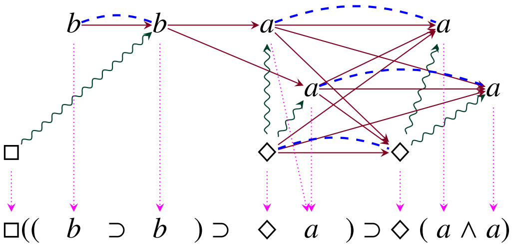


Modal Formulas

$$A, B := a \mid A \wedge B \mid A \supset B \mid \Box A \mid \Diamond A \mid 1$$

Additional Sequent Calculus Rules

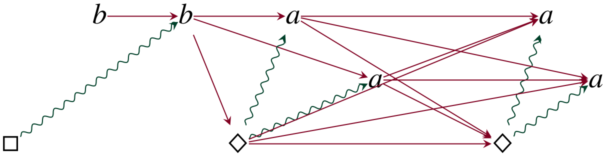
$$\frac{\Gamma \vdash A}{\Box\Gamma \vdash \Box A}\ \mathsf{K}_\Box \qquad \frac{B, \Gamma \vdash A}{\Diamond B, \Box\Gamma \vdash \Diamond A}\ \mathsf{K}_\Diamond \qquad \frac{B, \Gamma \vdash A}{\Box\Gamma \vdash \Diamond A}\ \mathsf{D}$$



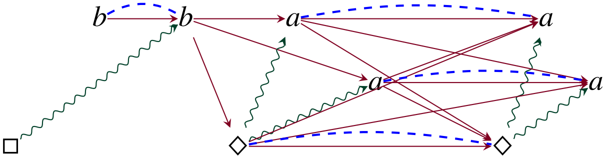
- Encode modalities using special vertices and additional edges
- Encode K and D as by links on modal vertices
- Encode $\mathsf{T}_\downarrow$ and $4_\downarrow$ as graphs homomorphisms

$\Longrightarrow$ Game Semantics for CK and CD [A., Catta & Straßburger 2021]

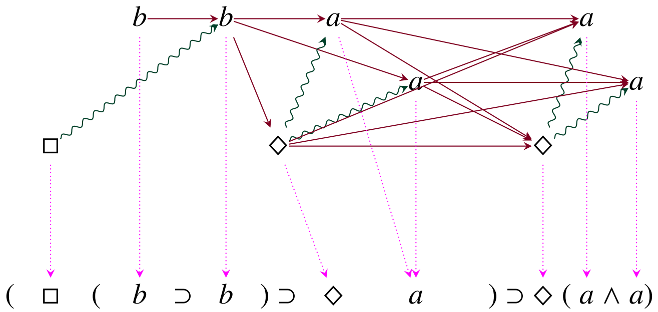
- Arenas for modal formulas



- Arenas for modal formulas
- Linear proofs = arenas + specific vertices partitions



- Arenas for modal formulas
- Linear proofs = arenas + specific vertices partitions
- Deep-WC derivations = Specific morphisms



- Arenas for modal formulas
- Linear proofs = arenas + specific vertices partitions
- Deep-WC derivations = Specific morphisms
- We can factorize CK proofs

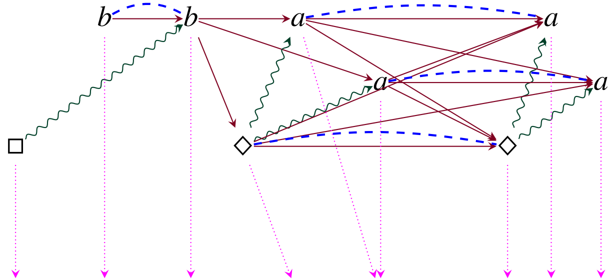
$$\Vert \mathsf{IMLL\text{-}X}^{\bullet}$$

$$\Box \; ((\; b \; \supset \; b \;) \supset \; \Diamond \; (a \wedge a) \;) \supset \Diamond \, (\, a \wedge a)$$

$$\Vert \mathsf{LI}^{\bullet}_{\downarrow}$$

$$(\; \Box \; (\; b \; \supset \; b \;) \supset \; \Diamond \; a \;) \supset \Diamond \, (\, a \wedge a)$$

- Arenas for modal formulas
- Linear proofs = arenas + specific vertices partitions
- Deep-WC derivations = Specific morphisms
- We can factorize CK proofs
- We have combinatorial proofs for CK!



$$(\Box(b \supset b) \supset \Diamond a) \supset \Diamond(a \wedge a)$$

As advertised:

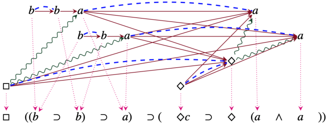
- Polynomial Correcntess Criterion
- Sound and Complete w.r.t. sequent calculus
- Rule-free representation of proofs

Proof equivalence in Constructive Modal Logic

Independent rules	$\equiv$
WC-comonad	$\dfrac{\dfrac{\Gamma,A,A,B,B\vdash C}{\Gamma,A,B\vdash C}\,2\times\mathsf{C}}{\Gamma,A\wedge B\vdash C}\,\wedge^{\mathsf{L}} \equiv_{\mathsf{c}} \dfrac{\dfrac{\Gamma,A,A,B,B\vdash C}{\Gamma,A\wedge B,A\wedge B\vdash C}\,2\times\wedge^{\mathsf{L}}}{\Gamma,A\vdash B}\,\mathsf{C}$ $\quad \dfrac{\dfrac{\Gamma\vdash C}{\Gamma,A,B\vdash C}\,2\times\mathsf{W}}{\Gamma,A\wedge B\vdash C}\,\wedge^{\mathsf{L}} \equiv_{\mathsf{c}} \dfrac{\Gamma\vdash C}{\Gamma,A\wedge B\vdash C}\,\mathsf{W}$ $\dfrac{\dfrac{\Gamma,A,A\vdash B}{\Gamma,A\vdash B}\,\mathsf{C}}{\Gamma,A,A\vdash B}\,\mathsf{W} \equiv_{\mathsf{c}} \Gamma,A,A\vdash B$ $\quad \dfrac{\dfrac{\Gamma,A\vdash B}{\Gamma,A,A\vdash B}\,\mathsf{W}}{\Gamma,A\vdash B}\,\mathsf{C} \equiv_{\mathsf{c}} \Gamma,A\vdash B$
Excising and Unfolding	$\dfrac{\Gamma\vdash A \quad \dfrac{\Delta\vdash C}{B,\Delta\vdash C}\,\mathsf{W}}{\Gamma,\Delta,A\supset B\vdash C}\,\supset^{\mathsf{L}} \equiv_{\mathsf{e}} \dfrac{\Delta\vdash C}{\Gamma,\Delta,A\supset B\vdash C}\,\mathsf{W}$ $\Big\Vert$ $\dfrac{\Gamma\vdash A \quad \dfrac{\Delta,B,B\vdash C}{\Delta,B\vdash C}\,\mathsf{C}}{\Gamma,A\supset B\vdash C}\,\supset^{\mathsf{L}} \equiv_{\mathsf{u}} \dfrac{\dfrac{\Gamma\vdash A \quad \dfrac{\Gamma\vdash A \quad \Delta,B,B\vdash C}{\Gamma,\Delta,A\supset B,B\vdash C}\,\supset^{\mathsf{L}}}{\Gamma,\Gamma,\Delta,A\supset B,A\supset B\vdash C}\,\supset^{\mathsf{L}}}{\Gamma,\Delta,A\supset B\vdash C}\,\mathsf{C}$
Structural vs K	$\dfrac{\dfrac{\Gamma\vdash A}{\Gamma,B\vdash A}\,\mathsf{W}}{\Box\Gamma,\Box B\vdash\Box A}\,\mathsf{K}_\Box \equiv_{\Box\mathsf{c}} \dfrac{\dfrac{\Gamma\vdash A}{\Box\Gamma\vdash\Box A}\,\mathsf{K}_\Box}{\Box\Gamma,\Box B\vdash\Box A}\,\mathsf{W}$ $\quad \dfrac{\dfrac{\Gamma,B,B\vdash A}{\Gamma,B\vdash A}\,\mathsf{C}}{\Box\Gamma,\Box B\vdash\Box A}\,\mathsf{K}_\Box \equiv_{\Box\mathsf{c}} \dfrac{\dfrac{\Gamma,B,B\vdash A}{\Box\Gamma,\Box B,\Box B\vdash\Box A}\,\mathsf{K}_\Box}{\Box\Gamma,\Box B\vdash\Box A}\,\mathsf{C}$ $\dfrac{\dfrac{\Gamma,B\vdash A}{\Gamma,B,C\vdash A}\,\mathsf{W}}{\Box\Gamma,\Diamond B,\Box C,\vdash\Box A}\,\mathsf{K}_\Diamond \equiv_{\Box\mathsf{c}} \dfrac{\dfrac{\Gamma,B\vdash A}{\Box\Gamma,\Diamond B\vdash\Diamond A}\,\mathsf{K}_\Diamond}{\Box\Gamma,\Diamond B,\Box C\vdash\Box A}\,\mathsf{W}$ $\quad \dfrac{\dfrac{\Gamma,B,C,C\vdash A}{\Gamma,B,C\vdash A}\,\mathsf{C}}{\Box\Gamma,\Diamond B,\Box C\vdash\Diamond A}\,\mathsf{K}_\Box \equiv_{\Box\mathsf{c}} \dfrac{\dfrac{\Gamma,B,C,C\vdash A}{\Box\Gamma,\Diamond B,\Box C,\Box C\Box\vdash\Diamond A}\,\mathsf{K}_\Box}{\Box\Gamma,\Diamond B,\Box C\vdash\Diamond A}\,\mathsf{C}$
Jumps	$\dfrac{\dfrac{\dfrac{\Gamma\vdash A}{\Gamma,B\vdash A}\,\mathsf{W}}{\Box\Gamma,\Diamond B\vdash\Diamond A}\,\mathsf{K}_\Diamond}{\Box\Gamma,\Diamond B,\Diamond C\vdash\Diamond A}\,\mathsf{W} \equiv_{\Diamond\mathsf{w}} \dfrac{\dfrac{\dfrac{\Gamma\vdash A}{\Gamma,C\vdash A}\,\mathsf{W}}{\Box\Gamma,\Diamond C\vdash\Diamond A}\,\mathsf{K}_\Diamond}{\Box\Gamma,\Diamond B,\Diamond C\vdash\Diamond A}\,\mathsf{W}$

$\equiv_{\mathsf{CP}} := (\equiv\cup\equiv_{\mathsf{e}}\cup\equiv_{\mathsf{c}})\qquad \equiv_{\lambda} := (\equiv_{\mathsf{CP}}\cup\equiv_{\mathsf{u}})\qquad \equiv_{\mathsf{WIS}} := (\equiv_{\lambda}\cup\equiv_{\Box\mathsf{c}})\qquad \equiv_{\Diamond\mathsf{w}} := (\equiv_{\mathsf{WIS}}\cup\equiv_{\Box\mathsf{c}})$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{b \vdash b}\,\mathsf{AX}}{\vdash b \supset b}\,\supset^{\mathsf{R}} \qquad \dfrac{}{a \vdash a}\,\mathsf{AX}}{(b \supset b) \supset a \vdash a}\,\supset^{\mathsf{L}} \qquad \dfrac{\dfrac{\dfrac{}{b \vdash b}\,\mathsf{AX}}{\vdash b \supset b}\,\supset^{\mathsf{R}} \qquad \dfrac{}{a \vdash a}\,\mathsf{AX}}{(b \supset b) \supset a \vdash a}\,\supset^{\mathsf{L}}}{(b \supset b) \supset a, (b \supset b) \supset a \vdash a \wedge a}\,\wedge^{\mathsf{R}}}{(b \supset b) \supset a \vdash a \wedge a}\,\mathsf{C}}{c, (b \supset b) \supset a \vdash a \wedge a}\,\mathsf{W}}{\Diamond c, \Box((b \supset b) \supset a) \vdash \Diamond(a \wedge a)}\,\mathsf{K}^{\Diamond}}{\Box(b \supset b) \supset a) \supset (\Diamond c \supset \Diamond(a \wedge a))}\,\supset^{\mathsf{R}}
$$



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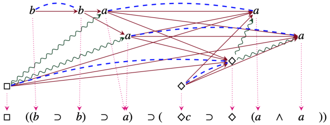

Sum up:

- We have combinatorial Proofs for
  - Classical (Normal) Modal Logics in the S4 hyper-cube
  - Constructive Modal Logics CK and CD
- . . . which are proof systems [Cook-Reckhow]
- . . . providing a (resource-sensitive) proof equivalence
- it is possible to define stronger proof equivalences

Related works/Future works/Works in Progress:

- Game semantics for CK and CD [A., Catta & Straßburger 2021]
- Combinatorial Proofs and Game Semantics for CS4 [WIP]
- Combinatorial Proofs as proof certificates (with modules)

Thanks

Thanks

Questions?