

One-Variable Lattice-Valued Logics

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A General Question

What is the **one-variable fragment** of a first-order logic?

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What is the **S5 modal companion** of a first-order logic?

Some Particular Answers

- The one-variable fragments of first-order classical logic and first-order intuitionistic logic are S5 and MIPC, respectively.
- More generally, the one-variable fragments of first-order intermediate logics have been studied intensively by Bezhanishvili, Ono, and Suzuki, and, in the setting of linear frames, by Caicedo et al.
- One-variable fragments of certain first-order many-valued logics have also been studied in some depth; notably, the one-variable fragment of first-order Łukasiewicz logic was axiomatized by Rutledge in 1959.

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- One-variable fragments of certain first-order many-valued logics have also been studied in some depth; notably, the one-variable fragment of first-order Łukasiewicz logic was axiomatized by Rutledge in 1959.

A More Precise Question

How can we **axiomatize** the S5 modal companion of a first-order logic?

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How can we find **algebraic semantics** for a one-variable first-order logic?

Let \mathcal{L} be any algebraic language such that \mathcal{L}_2 contains \wedge and \vee , where \mathcal{L}_n denotes the set of operation symbols of \mathcal{L} of arity $n \in \mathbb{N}$.

An \mathcal{L} -**lattice** is an algebra $\mathbf{A} = \langle A, \{\star^{\mathbf{A}} \mid n \in \mathbb{N}, \star \in \mathcal{L}_n\} \rangle$ such that

- (i) $\langle A, \wedge^{\mathbf{A}}, \vee^{\mathbf{A}} \rangle$ is a lattice with order $x \leq^{\mathbf{A}} y : \Longleftrightarrow x \wedge^{\mathbf{A}} y = x$;
- (ii) $\star^{\mathbf{A}}$ is an n -ary operation on A for each $n \in \mathbb{N}$ and $\star \in \mathcal{L}_n$.

We call this algebra **complete** if its lattice reduct $\langle A, \wedge, \vee \rangle$ is complete.

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Formulas and Equations

Let $\text{Fm}_{\forall}^1(\mathcal{L})$ be the set of **one-variable \mathcal{L} -formulas** φ, ψ, \dots built as usual using unary predicates $\{P_i\}_{i \in \mathbb{N}}$, a variable x , connectives in \mathcal{L} , and \forall, \exists .

An $\text{Fm}_{\forall}^1(\mathcal{L})$ -**equation** is an expression $\varphi \approx \psi$, where $\varphi, \psi \in \text{Fm}_{\forall}^1(\mathcal{L})$.

E.g., if \mathcal{L} is the language of lattices, $\text{Fm}_{\forall}^1(\mathcal{L})$ -equations include

$$(\forall x)(P_1x \wedge P_2x) \approx (\forall x)(P_1x) \wedge (\forall x)(P_2x), \quad (\exists x)(P_1x) \approx (\forall x)(P_1x), \quad \dots$$

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Structures and Validity

An **A-structure** for a complete \mathcal{L} -lattice **A** is an ordered pair $\mathfrak{S} = \langle S, \mathcal{I} \rangle$ such that S is a non-empty set and $\mathcal{I}(P_i)$ is a map from S to A for $i \in \mathbb{N}$.

For $u \in S$, we define a map $\|\cdot\|_u^{\mathfrak{S}} : \text{Fm}_{\forall}^1(\mathcal{L}) \rightarrow A$ inductively by

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Consequence

Given any class of complete \mathcal{L} -lattices \mathcal{K} and set of $\text{Fm}_{\forall}^1(\mathcal{L})$ -equations $T \cup \{\varphi \approx \psi\}$, we define

$$T \models_{\mathcal{K}}^{\forall} \varphi \approx \psi : \Longleftrightarrow \text{for any } \mathbf{A} \in \mathcal{K} \text{ and } \mathbf{A}\text{-structure } \mathfrak{G}, \\ \mathfrak{G} \models \varphi' \approx \psi' \text{ for all } \varphi' \approx \psi' \in T \implies \mathfrak{G} \models \varphi \approx \psi.$$

Example

If \mathcal{K} consists of all complete Boolean algebras or Heyting algebras, $\models_{\mathcal{K}}^{\forall}$ is consequence in the one-variable fragment of first-order classical logic or intuitionistic logic, respectively. If \mathcal{K} consists of all *totally ordered* complete Heyting algebras or MV-algebras, $\models_{\mathcal{K}}^{\forall}$ is consequence in the one-variable fragment of first-order Gödel logic or Łukasiewicz logic, respectively.

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The Challenge

Can we axiomatize $\models_{\mathcal{K}}^{\forall}$?

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Can we axiomatize $\models_{\mathcal{K}}^{\forall}$ when \mathcal{K} satisfies certain conditions?

The Modal Perspective

Let $\text{Fm}_\square(\mathcal{L})$ be the set of propositional formulas α, β, \dots built using propositional atoms $\{p_i\}_{i \in \mathbb{N}}$, connectives in \mathcal{L} , and unary connectives \square, \diamond .

The standard translation from $\text{Fm}_\forall^1(\mathcal{L})$ to $\text{Fm}_\square(\mathcal{L})$ is defined by

$$\begin{aligned}(P_i x)^* &= p_i \\ (\star(\varphi_1, \dots, \varphi_n))^* &= \star(\varphi_1^*, \dots, \varphi_n^*) \\ ((\forall x)\varphi)^* &= \square\varphi^* \\ ((\exists x)\varphi)^* &= \diamond\varphi^*.\end{aligned}$$

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This translation extends in the obvious way to (sets of) equations.

An **m-lattice** is an algebra $\langle L, \wedge, \vee, \Box, \Diamond \rangle$ such that $\langle L, \wedge, \vee \rangle$ is a lattice and the following equations are satisfied:

$$(L1_{\Box}) \quad \Box x \wedge x \approx \Box x$$

$$(L2_{\Box}) \quad \Box(x \wedge y) \approx \Box x \wedge \Box y$$

$$(L3_{\Box}) \quad \Box \Diamond x \approx \Diamond x$$

$$(L1_{\Diamond}) \quad \Diamond x \vee x \approx \Diamond x$$

$$(L2_{\Diamond}) \quad \Diamond(x \vee y) \approx \Diamond x \vee \Diamond y$$

$$(L3_{\Diamond}) \quad \Diamond \Box x \approx \Box x.$$

It follows also that every m-lattice satisfies

$$(L4_{\Box}) \quad \Box \Box x \approx \Box x$$

$$(L5_{\Box}) \quad x \leq y \implies \Box x \leq \Box y$$

$$(L4_{\Diamond}) \quad \Diamond \Diamond x \approx \Diamond x$$

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An **m-lattice** is an algebra $\langle L, \wedge, \vee, \Box, \Diamond \rangle$ such that $\langle L, \wedge, \vee \rangle$ is a lattice and the following equations are satisfied:

$$(L1_{\Box}) \quad \Box x \wedge x \approx \Box x$$

$$(L2_{\Box}) \quad \Box(x \wedge y) \approx \Box x \wedge \Box y$$

$$(L3_{\Box}) \quad \Box \Diamond x \approx \Diamond x$$

$$(L1_{\Diamond}) \quad \Diamond x \vee x \approx \Diamond x$$

$$(L2_{\Diamond}) \quad \Diamond(x \vee y) \approx \Diamond x \vee \Diamond y$$

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It follows also that every m-lattice satisfies

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An **m- \mathcal{L} -lattice** is an algebra $\langle \mathbf{A}, \Box, \Diamond \rangle$ such that \mathbf{A} is an \mathcal{L} -lattice and $\langle \mathbf{A}, \wedge, \vee, \Box, \Diamond \rangle$ is an m-lattice, satisfying for each $\star \in \mathcal{L}_n$ ($n \in \mathbb{N}$):

$$(\star\Box) \quad \Box(\star(\Box x_1, \dots, \Box x_n)) \approx \star(\Box x_1, \dots, \Box x_n).$$

Given any variety \mathcal{V} of \mathcal{L} -lattices, let $m\mathcal{V}$ denote the variety consisting of all m- \mathcal{L} -lattices $\langle \mathbf{A}, \Box, \Diamond \rangle$ such that $\mathbf{A} \in \mathcal{V}$.

Example

If \mathcal{V} is the variety of Boolean algebras or Heyting algebras, $m\mathcal{V}$ consists of all monadic Boolean algebras or monadic Heyting algebras, respectively.

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Representing m- \mathcal{L} -Lattices

For any m- \mathcal{L} -lattice $\langle \mathbf{A}, \Box, \Diamond \rangle$, we obtain a subalgebra $\Box \mathbf{A}$ of \mathbf{A} , with

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For any complete \mathcal{L} -lattice \mathbf{A} and set W , define for $f \in A^W$ and $u \in W$,

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Then $\langle \mathbf{A}^W, \Box, \Diamond \rangle$ is an m- \mathcal{L} -lattice.

We call an m- \mathcal{L} -lattice **functional** if it embeds into one of these algebras.

Functional m- \mathcal{L} -Lattices

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A Correspondence Result

For any variety \mathcal{V} of \mathcal{L} -lattices, let $\overline{\mathcal{V}}$ denote the class of complete members of \mathcal{V} , and $m\mathcal{V}^F$ the class of functional members of $m\mathcal{V}$.

Proposition

For any variety \mathcal{V} of \mathcal{L} -lattices and set of $\text{Fm}_{\overline{\mathcal{V}}}^1(\mathcal{L})$ -equations $T \cup \{\varphi \approx \psi\}$,

$$T \models_{\overline{\mathcal{V}}}^{\forall} \varphi \approx \psi \iff T^* \models_{m\mathcal{V}^F} \varphi^* \approx \psi^*,$$

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A Functional Representation Theorem

Theorem (Bezhanishvili and Harding 2002)

Each monadic Heyting algebra is functional; that is, if \mathcal{V} is the variety of Heyting algebras, then each member of $m\mathcal{V}$ is functional.

A Functional Representation Theorem

Theorem

Let \mathcal{V} be a variety of \mathcal{L} -lattices that admits regular completions and has the super-amalgamation property. Then each member of $m\mathcal{V}$ is functional.

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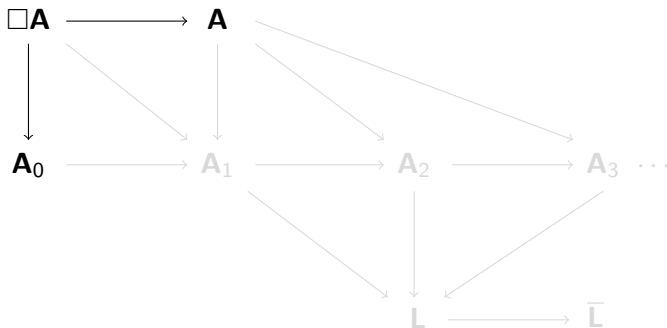
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Proof Idea

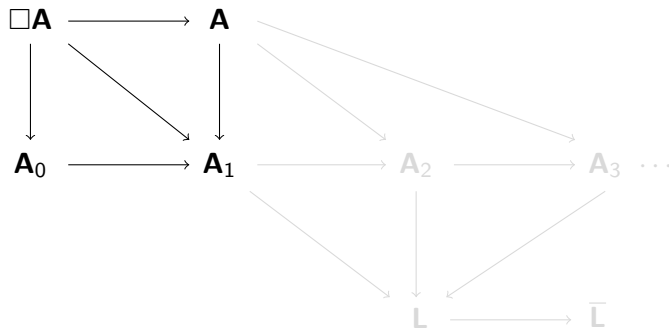
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We then obtain an embedding of $\langle \mathbf{A}, \square, \diamond \rangle$ into $\langle \bar{\mathbf{L}}^W, \square, \diamond \rangle$ by mapping each $x \in A$ to $f_x: W \rightarrow \bar{\mathbf{L}}$, where $f_x(i)$ is the image of x in $\bar{\mathbf{L}}$ via \mathbf{A}_i .

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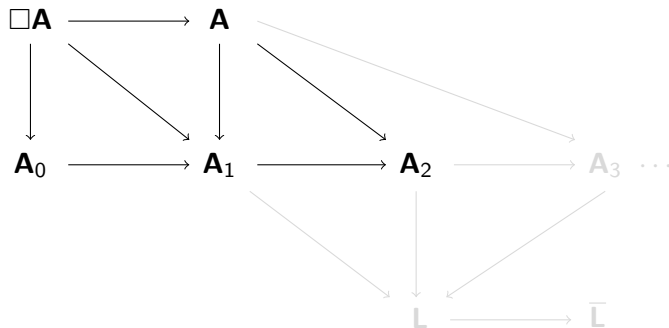
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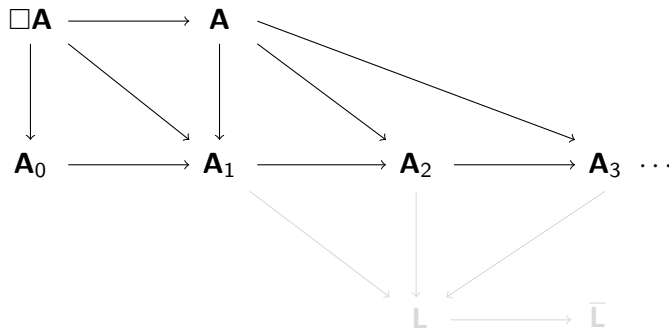
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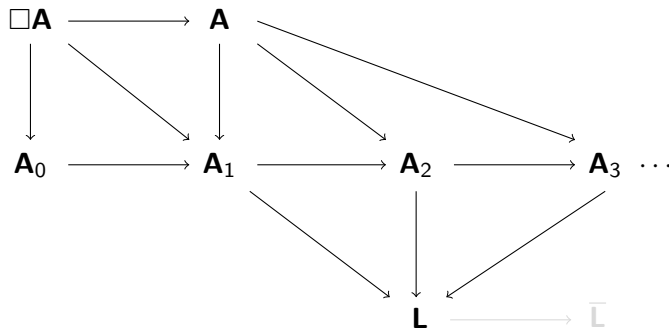
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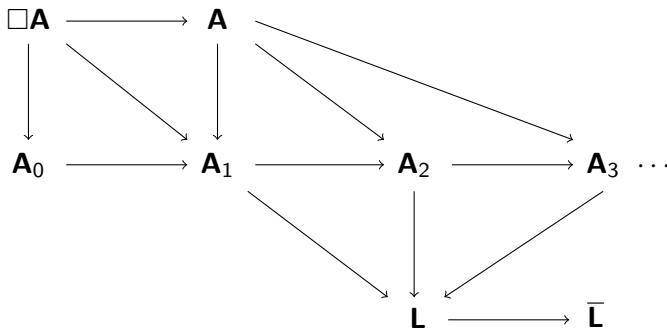
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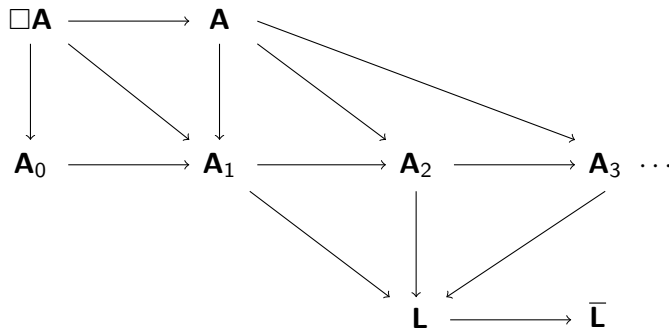
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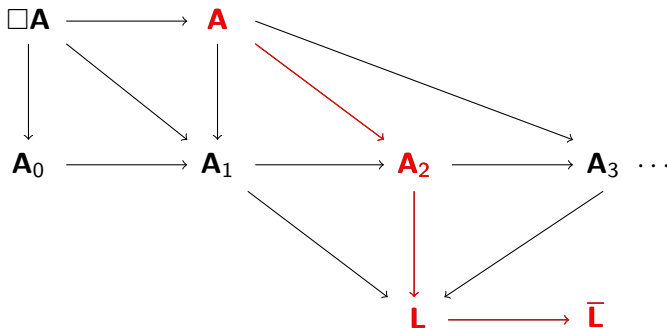
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A Completeness Theorem

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Our theorem applies to...

- Boolean algebras and Heyting algebras, yielding the well-known completeness results for S5 and MIPC;
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Concluding Remarks

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- In fact, regular completions are not needed for this result (so, e.g., it applies to any variety of Heyting algebras with super-amalgamation), but the functional representation theorem needs to be modified.
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Concluding Remarks

- We have presented a uniform axiomatization for the one-variable fragment of any first-order logic based on a variety of \mathcal{L} -lattices that admits regular completions and has the super-amalgamation property.
- In fact, regular completions are not needed for this result (so, e.g., it applies to any variety of Heyting algebras with super-amalgamation), but the functional representation theorem needs to be modified.
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