One-Variable Lattice-Valued Logics

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A General Question

What is the **one-variable fragment** of a first-order logic ?

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What is the **S5 modal companion** of a first-order logic ?

Some Particular Answers

- The one-variable fragments of first-order classical logic and first-order intuitionistic logic are S5 and MIPC, respectively.
- More generally, the one-variable fragments of first-order intermediate logics have been studied intensively by Bezhanishvili, Ono, and Suzuki, and, in the setting of linear frames, by Caicedo et al.
- One-variable fragments of certain first-order many-valued logics have also been studied in some depth; notably, the one-variable fragment of first-order Łukasiewicz logic was axiomatized by Rutledge in 1959.

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A More Precise Question

How can we **axiomatize** the S5 modal companion of a first-order logic ?

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How can we find **algebraic semantics** for a one-variable first-order logic ?

$\mathcal{L}$-Lattices

Let $\mathcal{L}$ be any algebraic language such that $\mathcal{L}_2$ contains $\wedge$ and $\vee$, where $\mathcal{L}_n$ denotes the set of operation symbols of $\mathcal{L}$ of arity $n \in \mathbb{N}$.

An $\mathcal{L}$-**lattice** is an algebra $\mathbf{A} = \langle A, \{\star^{\mathbf{A}} \mid n \in \mathbb{N}, \star \in \mathcal{L}_n\}\rangle$ such that

(i) $\langle A, \wedge^{\mathbf{A}}, \vee^{\mathbf{A}}\rangle$ is a lattice with order $x \leq^{\mathbf{A}} y :\Longleftrightarrow x \wedge^{\mathbf{A}} y = x$;

(ii) $\star^{\mathbf{A}}$ is an n-ary operation on A for each $n \in \mathbb{N}$ and $\star \in \mathcal{L}_n$.

We call this algebra **complete** if its lattice reduct $\langle A, \wedge, \vee\rangle$ is complete.

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Formulas and Equations

Let $\mathrm{Fm}^1_\forall(\mathcal{L})$ be the set of **one-variable $\mathcal{L}$-formulas** $\varphi, \psi, \ldots$ built as usual using unary predicates $\{P_i\}_{i\in\mathbb{N}}$, a variable x, connectives in $\mathcal{L}$, and $\forall, \exists$.

An $\mathrm{Fm}^1_\forall(\mathcal{L})$-**equation** is an expression $\varphi \approx \psi$, where $\varphi, \psi \in \mathrm{Fm}^1_\forall(\mathcal{L})$.

E.g., if $\mathcal{L}$ is the language of lattices, $\mathrm{Fm}^1_\forall(\mathcal{L})$-equations include

$$(\forall x)(P_1x \wedge P_2x) \approx (\forall x)(P_1x) \wedge (\forall x)(P_2x), \quad (\exists x)(P_1x) \approx (\forall x)(P_1x), \quad \ldots$$

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Structures and Validity

An **A-structure** for a complete $\mathcal{L}$-lattice **A** is an ordered pair $\mathfrak{S} = \langle S, \mathcal{I} \rangle$ such that S is a non-empty set and $\mathcal{I}(P_i)$ is a map from S to A for $i \in \mathbb{N}$.

For $u \in S$, we define a map $\|\cdot\|_u^{\mathfrak{S}} : \mathrm{Fm}_\forall^1(\mathcal{L}) \to A$ inductively by

$$\begin{aligned} \|P_i x\|_u^{\mathfrak{S}} &= \mathcal{I}(P_i)(u) \\ \|\star(\varphi_1, \ldots, \varphi_n)\|_u^{\mathfrak{S}} &= \star^{\mathbf{A}}(\|\varphi_1\|_u^{\mathfrak{S}}, \ldots, \|\varphi_n\|_u^{\mathfrak{S}}) \\ \|(\forall x)\varphi\|_u^{\mathfrak{S}} &= \bigwedge\{\|\varphi\|_v^{\mathfrak{S}} \mid v \in S\} \\ \|(\exists x)\varphi\|_u^{\mathfrak{S}} &= \bigvee\{\|\varphi\|_v^{\mathfrak{S}} \mid v \in S\}. \end{aligned}$$

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Consequence

Given any class of complete $\mathcal{L}$-lattices $\mathcal{K}$ and set of $\mathrm{Fm}^1_\forall(\mathcal{L})$-equations $T \cup \{\varphi \approx \psi\}$, we define

$$T \models^\forall_\mathcal{K} \varphi \approx \psi \;:\Longleftrightarrow\; \text{for any } \mathbf{A} \in \mathcal{K} \text{ and } \mathbf{A}\text{-structure } \mathfrak{S},$$
$$\mathfrak{S} \models \varphi' \approx \psi' \text{ for all } \varphi' \approx \psi' \in T \implies \mathfrak{S} \models \varphi \approx \psi.$$

Example

If $\mathcal{K}$ consists of all complete Boolean algebras or Heyting algebras, $\models^\forall_\mathcal{K}$ is consequence in the one-variable fragment of first-order classical logic or intuitionistic logic, respectively. If $\mathcal{K}$ consists of all *totally ordered* complete Heyting algebras or MV-algebras, $\models^\forall_\mathcal{K}$ is consequence in the one-variable fragment of first-order Gödel logic or Łukasiewicz logic, respectively.

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The Challenge

Can we axiomatize $\models^{\forall}_{\mathcal{K}}$?

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Can we axiomatize $\models^{\forall}_{\mathcal{K}}$ when $\mathcal{K}$ satisfies certain conditions ?

The Modal Perspective

Let $\mathrm{Fm}_{\Box}(\mathcal{L})$ be the set of propositional formulas $\alpha, \beta, \ldots$ built using propositional atoms $\{p_i\}_{i \in \mathbb{N}}$, connectives in $\mathcal{L}$, and unary connectives $\Box, \Diamond$.

The standard translation from $\mathrm{Fm}^1_{\forall}(\mathcal{L})$ to $\mathrm{Fm}_{\Box}(\mathcal{L})$ is defined by

$$\begin{aligned}
(P_i x)^* &= p_i \\
(\star(\varphi_1, \ldots, \varphi_n))^* &= \star(\varphi_1^*, \ldots, \varphi_n^*) \\
((\forall x)\varphi)^* &= \Box \varphi^* \\
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This translation extends in the obvious way to (sets of) equations.

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m-Lattices

An **m-lattice** is an algebra $\langle L, \wedge, \vee, \Box, \Diamond \rangle$ such that $\langle L, \wedge, \vee \rangle$ is a lattice and the following equations are satisfied:

$$
\begin{array}{llll}
(\mathrm{L1}_\Box) & \Box x \wedge x \approx \Box x & (\mathrm{L1}_\Diamond) & \Diamond x \vee x \approx \Diamond x \\
(\mathrm{L2}_\Box) & \Box(x \wedge y) \approx \Box x \wedge \Box y & (\mathrm{L2}_\Diamond) & \Diamond(x \vee y) \approx \Diamond x \vee \Diamond y \\
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m-Lattices

An **m-lattice** is an algebra $\langle L, \wedge, \vee, \Box, \Diamond \rangle$ such that $\langle L, \wedge, \vee \rangle$ is a lattice and the following equations are satisfied:

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m-$\mathcal{L}$-Lattices

An **m-$\mathcal{L}$-lattice** is an algebra $\langle \mathbf{A}, \Box, \Diamond \rangle$ such that $\mathbf{A}$ is an $\mathcal{L}$-lattice and $\langle A, \wedge, \vee, \Box, \Diamond \rangle$ is an m-lattice, satisfying for each $\star \in \mathcal{L}_n$ ($n \in \mathbb{N}$):

$$(\star_\Box) \qquad \Box(\star(\Box x_1, \ldots, \Box x_n)) \approx \star(\Box x_1, \ldots, \Box x_n).$$

Given any variety $\mathcal{V}$ of $\mathcal{L}$-lattices, let $m\mathcal{V}$ denote the variety consisting of all m-$\mathcal{L}$-lattices $\langle \mathbf{A}, \Box, \Diamond \rangle$ such that $\mathbf{A} \in \mathcal{V}$.

Example

If $\mathcal{V}$ is the variety of Boolean algebras or Heyting algebras, $m\mathcal{V}$ consists of all monadic Boolean algebras or monadic Heyting algebras, respectively.

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Representing m-$\mathcal{L}$-Lattices

For any m-$\mathcal{L}$-lattice $\langle \mathbf{A}, \Box, \Diamond \rangle$, we obtain a subalgebra $\Box\mathbf{A}$ of $\mathbf{A}$, with

$$\Box A := \{\Box x \mid x \in A\} = \{\Diamond x \mid x \in A\},$$

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Functional m-$\mathcal{L}$-Lattices

For any complete $\mathcal{L}$-lattice $\mathbf{A}$ and set W, define for $f \in A^W$ and $u \in W$,

$$\Box f(u) := \bigwedge_{v \in W} f(v) \quad \text{and} \quad \Diamond f(u) := \bigvee_{v \in W} f(v).$$

Then $\langle \mathbf{A}^W, \Box, \Diamond \rangle$ is an m-$\mathcal{L}$-lattice.

We call an m-$\mathcal{L}$-lattice **functional** if it embeds into one of these algebras.

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A Correspondence Result

For any variety $\mathcal{V}$ of $\mathcal{L}$-lattices, let $\overline{\mathcal{V}}$ denote the class of complete members of $\mathcal{V}$, and $m\mathcal{V}^F$ the class of functional members of $m\mathcal{V}$.

Proposition

For any variety $\mathcal{V}$ of $\mathcal{L}$-lattices and set of $\mathrm{Fm}^1_\forall(\mathcal{L})$-equations $T \cup \{\varphi \approx \psi\}$,

$$T \models^{\forall}_{\overline{\mathcal{V}}} \varphi \approx \psi \quad \Longleftrightarrow \quad T^* \models_{m\mathcal{V}^F} \varphi^* \approx \psi^*,$$

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Question. When can we replace $m\mathcal{V}^F$ with $m\mathcal{V}$ in this equivalence?

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A Functional Representation Theorem

Theorem (Bezhanishvili and Harding 2002)

Each monadic Heyting algebra is functional; that is, if $\mathcal{V}$ is the variety of Heyting algebras, then each member of $m\mathcal{V}$ is functional.

A Functional Representation Theorem

Theorem

Let $\mathcal{V}$ be a variety of $\mathcal{L}$-lattices that admits regular completions and has the super-amalgamation property. Then each member of $m\mathcal{V}$ is functional.

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Proof Idea

Given any $\langle \mathbf{A}, \Box, \Diamond \rangle \in m\mathcal{V}$, let $W := \mathbb{N}^{>0}$, and $\mathbf{A}_0 := \mathbf{A}$. For each $i \in W$, there exists a super-amalgam $\mathbf{A}_i \in \mathcal{V}$ of $\langle \Box\mathbf{A}, \mathbf{A}_{i-1}, \mathbf{A} \rangle$. Then the direct limit $\mathbf{L} \in \mathcal{V}$ of the $\mathbf{A}_i$'s embeds regularly into a complete $\mathcal{L}$-lattice $\overline{\mathbf{L}} \in \mathcal{V}$.

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We then obtain an embedding of $\langle \mathbf{A}, \Box, \Diamond \rangle$ into $\langle \overline{\mathbf{L}}^W, \Box, \Diamond \rangle$ by mapping each $x \in A$ to $f_x \colon W \to \overline{L}$, where $f_x(i)$ is the image of x in $\overline{L}$ via $\mathbf{A}_i$.

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A Completeness Theorem

Theorem

Let $\mathcal{V}$ be a variety of $\mathcal{L}$-lattices that admits regular completions and has the super-amalgamation property.

Then for any set of $\mathrm{Fm}^1_\forall(\mathcal{L})$-equations $T \cup \{\varphi \approx \psi\}$,

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Case Studies

Our theorem applies to...

- Boolean algebras and Heyting algebras, yielding the well-known completeness results for S5 and MIPC;
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Concluding Remarks

- We have presented a uniform axiomatization for the one-variable fragment of any first-order logic based on a variety of $\mathcal{L}$-lattices that admits regular completions and has the super-amalgamation property.
- In fact, regular completions are not needed for this result (so, e.g., it applies to any variety of Heyting algebras with super-amalgamation), but the functional representation theorem needs to be modified.
- If a variety does not have the super-amalgamation property, we might try a proof-theoretic approach or use the (weaker) super generalized amalgamation property.
- We would also like to extend our methods beyond varieties . . .

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