What is the cost of cut?

Elaine Pimentel
UCL, UK

Joint work with T. Lang, C. Olarte and C. Fermüller

LATD&MOSAIC

07 September 2022

Motivation

- Game interpretation of bottom-up proof search in sequent systems.

Motivation

- Game interpretation of bottom-up proof search in sequent systems.
- Our view of game semantics: a playground for illuminating specific intuitions underlying certain proof systems.

Motivation

- Game interpretation of bottom-up proof search in sequent systems.
- Our view of game semantics: a playground for illuminating specific intuitions underlying certain proof systems.
- Resource consciousness: motivation usually remains metaphorical [Girard'87]

Motivation

- Game interpretation of bottom-up proof search in sequent systems.
- Our view of game semantics: a playground for illuminating specific intuitions underlying certain proof systems.
- Resource consciousness: motivation usually remains metaphorical [Girard'87]
- Gentzen's sequent calculus is a (the?) natural starting point for connecting inference and resource consciousness.

Motivation

- Game interpretation of bottom-up proof search in sequent systems.
- Our view of game semantics: a playground for illuminating specific intuitions underlying certain proof systems.
- Resource consciousness: motivation usually remains metaphorical [Girard'87]
- Gentzen's sequent calculus is a (the?) natural starting point for connecting inference and resource consciousness.
- To *breathe life into the resource metaphor*, we need dynamics

  $\Longrightarrow$ game semantics for substructural sequent calculi.

Motivation

- Game interpretation of bottom-up proof search in sequent systems.
- Our view of game semantics: a playground for illuminating specific intuitions underlying certain proof systems.
- Resource consciousness: motivation usually remains metaphorical [Girard'87]
- Gentzen's sequent calculus is a (the?) natural starting point for connecting inference and resource consciousness.
- To *breathe life into the resource metaphor*, we need dynamics

  $\Longrightarrow$ game semantics for substructural sequent calculi.
- Better understanding of resource conscious reasoning, which is often cited as a motivation for substructural logics.

Motivation

- Game interpretation of bottom-up proof search in sequent systems.
- Our view of game semantics: a playground for illuminating specific intuitions underlying certain proof systems.
- Resource consciousness: motivation usually remains metaphorical [Girard'87]
- Gentzen's sequent calculus is a (the?) natural starting point for connecting inference and resource consciousness.
- To *breathe life into the resource metaphor*, we need dynamics

  $\Longrightarrow$ game semantics for substructural sequent calculi.
- Better understanding of resource conscious reasoning, which is often cited as a motivation for substructural logics.
- Side effect: notion of cost of cuts!

Outline

Outline

Dialogues as foundations

Dialogues

A Proponent P tries to defend a logically complex statement against attacks by an Opponent O. The dialogue stepwise reduces complex assertions to their components.

Dialogues as foundations

Dialogues

A Proponent P tries to defend a logically complex statement against attacks by an Opponent O. The dialogue stepwise reduces complex assertions to their components.

X/Y stands for P/O or O/P

statement by X	attack by Y	defense by X
$A \wedge B$	l? or r? (Y chooses)	A or B, accordingly
$A \vee B$	?	A or B (X chooses)
$A \supset B$	A	B

Dialogues as foundations

Dialogues

A Proponent P tries to defend a logically complex statement against attacks by an Opponent O. The dialogue stepwise reduces complex assertions to their components.

X/Y stands for P/O or O/P

statement by X	attack by Y	defense by X
$A \wedge B$	l? or r? (Y chooses)	A or B, accordingly
$A \vee B$	?	A or B (X chooses)
$A \supset B$	A	B

Winning conditions for P:

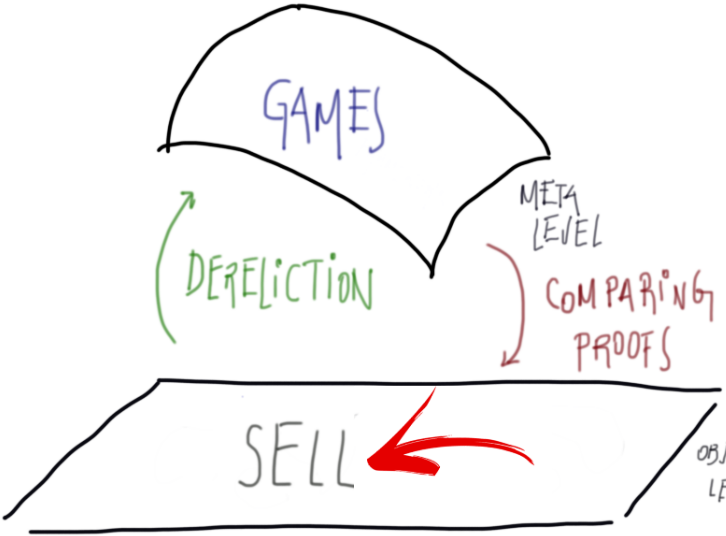
W: O has already granted P's active formula

W$\perp$: O has granted $\perp$

[Lorenzen'60] attempted to *justify constructive logic*. The completeness result w.r.t. LJ came much later [Felscher'85].

Outline

The object-level



Linear logic in a nutshell

- Linear conjunctions: $\&$ (additive) and $\otimes$ (multiplicative)
  Linear disjunctions: $\oplus$ (additive) and $\parr$ (multiplicative)
  Unities: $\top, 1, 0, \bot$
  Linear implication: $\multimap$
  Exponentials: $!, ?$

Linear logic in a nutshell

- Linear conjunctions: $\&$ (additive) and $\otimes$ (multiplicative)
  Linear disjunctions: $\oplus$ (additive) and $\invamp$ (multiplicative)
  Unities: $\top, 1, 0, \bot$
  Linear implication: $\multimap$
  Exponentials: $!, ?$
- By composing a proof of $A \multimap B$ and a proof of A we consume them to get a proof of B.
- Linear logic formulas behave like resources.
- Exponentials recover the full expressive power of intuitionistic and classical logic: in $!B$ and $?B$ we are allowed to use contraction and weakening.

Affine intuitionistic multiplicative additive LL ($\mathcal{C}$)

Sequent System for $\mathcal{C}$

$$\frac{\Delta_1 \longrightarrow A \quad \Delta_2 \longrightarrow B}{\Delta_1, \Delta_2 \longrightarrow A \otimes B} \otimes_R \qquad \frac{\Gamma \longrightarrow A \quad \Gamma \longrightarrow B}{\Gamma \longrightarrow A \,\&\, B} \&_R \qquad \frac{\Gamma, A \longrightarrow B}{\Gamma \longrightarrow A \multimap B} \multimap_R$$

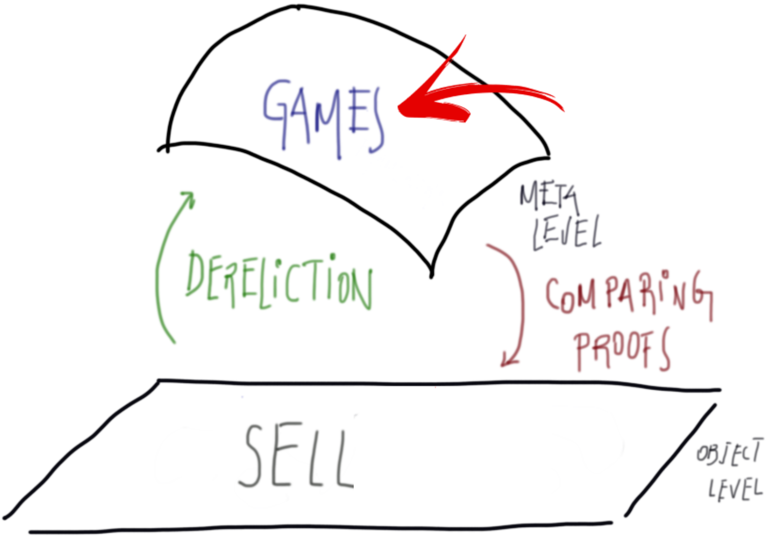
$$\frac{\Gamma, A, B \longrightarrow C}{\Gamma, A \otimes B \longrightarrow C} \otimes_L \qquad \frac{\Delta_1 \longrightarrow A \quad \Delta_2, B \longrightarrow C}{\Delta_1, \Delta_2, A \multimap B \longrightarrow C} \multimap_L \qquad \frac{\Gamma, A_i \longrightarrow B}{\Gamma, A_1 \,\&\, A_2 \longrightarrow B} \&_{L_i}$$

$$\frac{\Gamma, A \longrightarrow C \quad \Gamma, B \longrightarrow C}{\Gamma, A \oplus B \longrightarrow C} \oplus_L \qquad \frac{\Gamma \longrightarrow A_i}{\Gamma \longrightarrow A_1 \oplus A_2} \oplus_{R_i}$$

$$\frac{}{\Gamma, p \longrightarrow p} I \qquad \frac{}{\Gamma \longrightarrow 1} 1_R \qquad \frac{}{\Gamma, 0 \longrightarrow A} 0_L$$

Outline

The meta-level



The game for $\mathcal{C}$ [Fermüller,Lang17]

- Formulas are seen as resources that can be build from atomic propositions, units $0, 1$ and the constructors $\otimes, \&, \oplus, \multimap$
- States: multisets of sequents of the form $\Gamma \longrightarrow F$
- Two players: P and O. Player P starts the game and selects a sequent S from the current state.
- The game proceeds in rounds with two possible succ. states:

$$\begin{array}{llll} (1) & G \cup \{S\} & \rightsquigarrow & G \cup \{S'\} \\ (2) & G \cup \{S\} & \rightsquigarrow & G \cup \{S_1\} \cup \{S_2\} \end{array}$$

- P chooses a sequent S among the current game state, a principal formula in S and a matching rule instance r.
- P acts as the scheduler of the game.

Multiplicative vs Additive

Both are (right) branching rules:

$$\frac{\Gamma \longrightarrow A \quad \Gamma \longrightarrow B}{\Gamma \longrightarrow A \,\&\, B} \;\&_R \qquad \frac{\Gamma_1 \longrightarrow A \quad \Gamma_2 \longrightarrow B}{\Gamma_1, \Gamma_2 \longrightarrow A \otimes B} \;\otimes_R$$

However, the intended meaning is different:

- $A \,\&\, B$: P must be prepared to play either A or B (O choice) but only one game is actually played.
- $A \otimes B$: both subgames, A and B must be played and P must win both.

Multiplicative vs Additive

Both are (right) branching rules:

$$\frac{\Gamma \longrightarrow A \quad \Gamma \longrightarrow B}{\Gamma \longrightarrow A \,\&\, B} \,\&_R \qquad \frac{\Gamma_1 \longrightarrow A \quad \Gamma_2 \longrightarrow B}{\Gamma_1, \Gamma_2 \longrightarrow A \otimes B} \otimes_R$$

However, the intended meaning is different:

- $A \,\&\, B$: P must be prepared to play either A or B (O choice) but only one game is actually played.
- $A \otimes B$: both subgames, A and B must be played and P must win both.

Branching structure

Both definitions (a single or a parallel game) are equivalent: the existence of winning strategies for P remains the same.
However, semantically, they provide different viewpoints of the connectives.

The game for $\mathcal{C}$

$$\dfrac{}{p, q \oplus r \longrightarrow (p \otimes q) \oplus (p \otimes r)} \oplus_{R_1}$$

The game for $\mathcal{C}$

$$\dfrac{\dfrac{}{p, q \oplus r \longrightarrow (p \otimes q)}\ \otimes_R}{p, q \oplus r \longrightarrow (p \otimes q) \oplus (p \otimes r)}\ \oplus_{R_1}$$

The game for $\mathcal{C}$

$$\dfrac{\dfrac{p \overset{☺}{\longrightarrow} p \quad q \oplus r \longrightarrow q}{p, q \oplus r \longrightarrow (p \otimes q)} \otimes_R}{p, q \oplus r \longrightarrow (p \otimes q) \oplus (p \otimes r)} \oplus_{R_1}$$

The game for $\mathcal{C}$

$$\dfrac{\dfrac{p \xrightarrow{\ddot\smile} p \qquad \dfrac{q \xrightarrow{\ddot\smile} q}{q \oplus r \longrightarrow q}\,\oplus_L}{p, q \oplus r \longrightarrow (p \otimes q)}\,\otimes_R}{p, q \oplus r \longrightarrow (p \otimes q) \oplus (p \otimes r)}\,\oplus_{R_1}$$

The game for $\mathcal{C}$

$$
\dfrac{
  \dfrac{
    p \stackrel{\text{☺}}{\longrightarrow} p
    \qquad
    \dfrac{q \stackrel{\text{☺}}{\longrightarrow} q \quad r \stackrel{\text{☹}}{\longrightarrow} q}{q \oplus r \longrightarrow q}\,\oplus_L
  }{p, q \oplus r \longrightarrow (p \otimes q)}\,\otimes_R
}{p, q \oplus r \longrightarrow (p \otimes q) \oplus (p \otimes r)}\,\oplus_{R_1}
$$

The game for $\mathcal{C}$

$$\dfrac{\dfrac{\dfrac{p \overset{☺}{\longrightarrow} p \quad q \overset{☺}{\longrightarrow} q}{p, q \longrightarrow (p \otimes q)}\ \otimes_R}{p, q \longrightarrow (p \otimes q) \oplus (p \otimes r)}\ \oplus_{R_1} \qquad p, r \longrightarrow (p \otimes q) \oplus (p \otimes r)}{p, q \oplus r \longrightarrow (p \otimes q) \oplus (p \otimes r)}\ \oplus_L$$

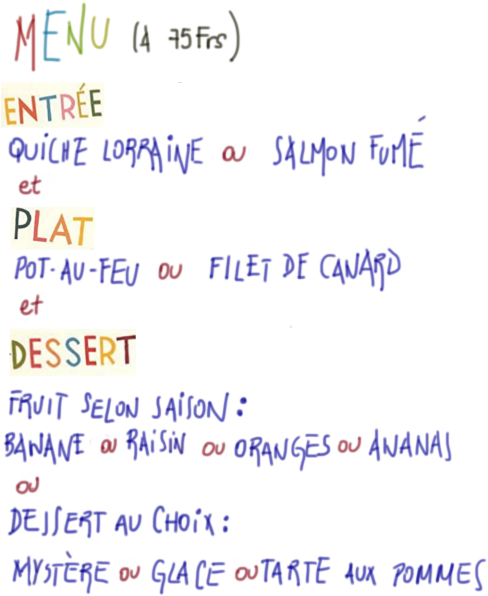
The game for $\mathcal{C}$

$$\dfrac{p, q \longrightarrow (p \otimes q) \oplus (p \otimes r) \qquad \dfrac{\dfrac{p \overset{☺}{\longrightarrow} p \qquad r \overset{☺}{\longrightarrow} r}{p, r \longrightarrow p \otimes r} \otimes_R}{p, r \longrightarrow (p \otimes q) \oplus (p \otimes r)} \oplus_{R_2}}{p, q \oplus r \longrightarrow (p \otimes q) \oplus (p \otimes r)} \oplus_L$$

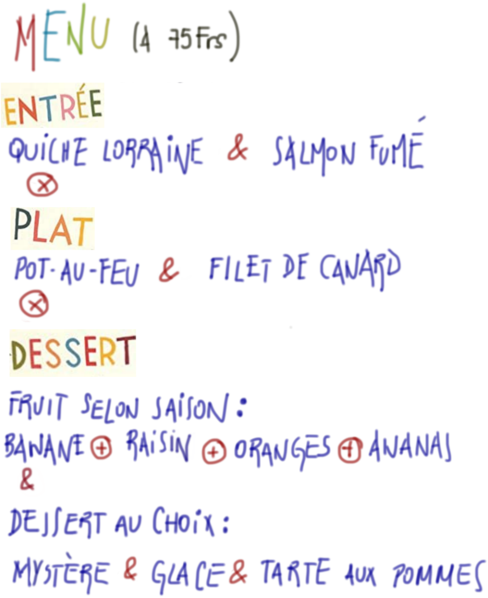
The game for $\mathcal{C}$

$$
\dfrac{
\dfrac{
\dfrac{p \overset{\smiley}{\longrightarrow} p \quad q \overset{\smiley}{\longrightarrow} q}{p, q \longrightarrow (p \otimes q)}\ \otimes_R
}{p, q \longrightarrow (p \otimes q) \oplus (p \otimes r)}\ \oplus_{R_1}
\qquad
\dfrac{
\dfrac{p \overset{\smiley}{\longrightarrow} p \quad r \overset{\smiley}{\longrightarrow} r}{p, r \longrightarrow p \otimes r}\ \otimes_R
}{p, r \longrightarrow (p \otimes q) \oplus (p \otimes r)}\ \oplus_{R_2}
}{p, q \oplus r \longrightarrow (p \otimes q) \oplus (p \otimes r)}\ \oplus_L
$$

Lafont's menu revisited

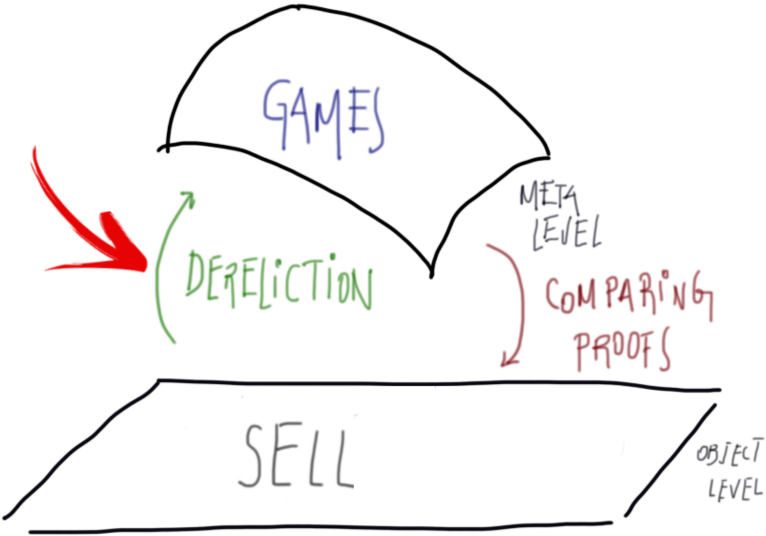


Lafont's menu revisited



Outline

Intended meaning



Subexponentials [Danos,Joinet,Schellinx'93]

Exponentials in ILL:

$$\frac{\Gamma, A \longrightarrow C}{\Gamma, !\,A \longrightarrow C}\ !_L \qquad \frac{!\,A_1, \ldots, !\,A_n \longrightarrow A}{!\,A_1, \ldots, !\,A_n \longrightarrow !\,A}\ !_R$$

Subexponentials [Danos,Joinet,Schellinx'93]

Sub-exponentials in ILL:

$$\frac{\Gamma, A \longrightarrow C}{\Gamma, !^a A \longrightarrow C}\ {!^a}_L \qquad \frac{!^{a_1} A_1, \ldots, !^{a_n} A_n \longrightarrow A}{!^{a_1} A_1, \ldots, !^{a_n} A_n \longrightarrow !^a A}\ {!^a}_R, \text{ provided } a \preceq a_i$$

Subexponentials [Danos,Joinet,Schellinx'93]

Sub-exponentials in ILL:

$$\frac{\Gamma, A \longrightarrow C}{\Gamma, !^a A \longrightarrow C}\ !^a{}_L \qquad \frac{!^{a_1}A_1, \ldots, !^{a_n}A_n \longrightarrow A}{!^{a_1}A_1, \ldots, !^{a_n}A_n \longrightarrow !^a A}\ !^a{}_R, \text{ provided } a \preceq a_i$$

Then:

$!^a A \not\equiv !^b A$ for any $a \neq b$.

Assumptions plus cost – system $\mathcal{C}(\mathbb{R}^+)$

Augment assumptions with costs, where assumptions are formulas occurring negatively on sequents.

$$\frac{\Gamma, !^a A, A \longrightarrow C}{\Gamma, !^a A \longrightarrow C} \; !^a{}_L, \; a \in \mathbb{R}^+$$

The game $\mathcal{G}_{\mathcal{C}}(\mathbb{R}^+)$ [Lang,Olarte,Pimentel,Fermüller'19]

- States: tuples (H, b), where H is a finite multiset of $\mathbb{R}^+$-valued sequents and $b \in \mathbb{R}$ is a budget.
- Rounds: the successor state is determined according to rules that fit one of the two following schemes:

$$
\begin{array}{llcl}
(1) & (G \cup \{S\}, b) & \rightsquigarrow & (G \cup \{S'\}, b') \\
(2) & (G \cup \{S\}, b) & \rightsquigarrow & (G \cup \{S^1\} \cup \{S^2\}, b)
\end{array}
$$

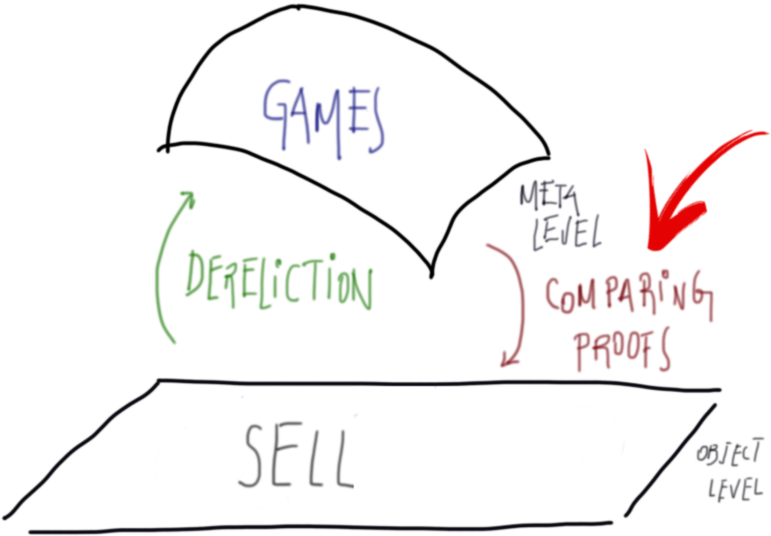
The game $\mathcal{G}_{\mathcal{C}}(\mathbb{R}^+)$ [Lang,Olarte,Pimentel,Fermüller'19]

- States: tuples (H, b), where H is a finite multiset of $\mathbb{R}^+$-valued sequents and $b \in \mathbb{R}$ is a budget.
- Rounds: the successor state is determined according to rules that fit one of the two following schemes:

$$\begin{array}{llll} (1) & (G \cup \{S\}, b) & \rightsquigarrow & (G \cup \{S'\}, b') \\ (2) & (G \cup \{S\}, b) & \rightsquigarrow & (G \cup \{S^1\} \cup \{S^2\}, b) \end{array}$$

- Depending on the r, the round proceeds as follows:
  1. If the rule r is not $!_L$, then the game proceeds as before, with budget b.
  2. Budget decrease: $!_L$ with premise S' and principal formula $!^a A$, then the game proceeds in the game state $(G \cup \{S'\}, b - a)$.
  3. To win the game: non negative final budget.

Properties


GAMES
META LEVEL
DERELICTION
COMPARING PROOFS
SELL
OBJECT LEVEL

Labelled system $\mathcal{C}^{\ell}(\mathbb{R}^+)$

- Weak adequacy: information about the budget b is lost in the proof theoretic representation.
- In other words, the game $\mathcal{G}_{\mathcal{C}}(\mathbb{R}^+)$ is more expressive than the calculus $\mathcal{C}(\mathbb{R}^+)$.

Labelled system $\mathcal{C}^\ell(\mathbb{R}^+)$

- Weak adequacy: information about the budget b is lost in the proof theoretic representation.
- In other words, the game $\mathcal{G}_\mathcal{C}(\mathbb{R}^+)$ is more expressive than the calculus $\mathcal{C}(\mathbb{R}^+)$.
- To overcome this mismatch: a labelled extension of $\mathcal{C}(\mathbb{R}^+)$.
- A $\mathcal{C}^\ell(\mathbb{R}^+)$-proof is build from labelled sequents

$$b : \Gamma \longrightarrow A$$

where $\Gamma \longrightarrow A$ is a $\mathcal{C}(\mathbb{R}^+)$ sequent and $b \in \mathbb{R}^+$.

Sequent rules for $\mathcal{C}^{\ell}(\mathbb{R}^{+})$

Labelled sequent system for $\mathcal{C}^{\ell}(\mathbb{R}^{+})$

$$\frac{a : !\Gamma, \Delta_1 \longrightarrow A \quad b : !\Gamma, \Delta_2 \longrightarrow B}{a + b : !\Gamma, \Delta_1, \Delta_2 \longrightarrow A \otimes B} \otimes_R \qquad \frac{a : \Gamma \longrightarrow A \quad b : \Gamma \longrightarrow B}{\max\{a, b\} : \Gamma \longrightarrow A \,\&\, B} \&_R$$

$$\frac{c : \Gamma, !^a A, A \longrightarrow C}{a + c : \Gamma, !^a A \longrightarrow C} \;!_L$$

$$\frac{}{b : \Gamma, p \longrightarrow p} \; I \; b \geq 0 \qquad \frac{}{b : \Gamma \longrightarrow 1} \; 1_R \; b \geq 0 \qquad \frac{}{b : \Gamma, 0 \longrightarrow A} \; 0_L \; b \geq 0$$

Example

You have white and black socks in a drawer in a completely dark room. How many socks do you have to take out blindly to be sure of having a matching pair?

Example

You have white and black socks in a drawer in a completely dark room. How many socks do you have to take out blindly to be sure of having a matching pair?

- Matching pair: $(w \otimes w) \oplus (b \otimes b)$;
- Act of drawing a random sock: $!^1(w \oplus b)$.

Example

You have white and black socks in a drawer in a completely dark room. How many socks do you have to take out blindly to be sure of having a matching pair?

- Matching pair: $(w \otimes w) \oplus (b \otimes b)$; 
- Act of drawing a random sock: $!^1(w \oplus b)$.

  What is the smallest n s.t. $n : !^1(w \oplus b) \longrightarrow (w \otimes w) \oplus (b \otimes b)$ is provable?

Example

You have white and black socks in a drawer in a completely dark room. How many socks do you have to take out blindly to be sure of having a matching pair?

- Matching pair: $(w \otimes w) \oplus (b \otimes b)$;
- Act of drawing a random sock: $!^1(w \oplus b)$.

  What is the smallest n s.t. $n : !^1(w \oplus b) \longrightarrow (w \otimes w) \oplus (b \otimes b)$ is provable?

The answer, of course, is 3:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{0 : !^1(w \oplus b), w, w, w \oplus b \longrightarrow w \otimes w}\ \otimes_R, I}{0 : !^1(w \oplus b), w, w, w \oplus b \longrightarrow F}\ \oplus_L \qquad \dfrac{\dfrac{\dfrac{}{0 : !^1(w \oplus b), w, b, w \longrightarrow (w \otimes w)}\ \otimes_R, I}{0 : !^1(w \oplus b), w, b, w \longrightarrow F}\ \oplus_R \quad \dfrac{\dfrac{}{0 : !^1(w \oplus b), w, b, b \longrightarrow b \otimes b}\ \otimes_R, I}{0 : !^1(w \oplus b), w, b, b \longrightarrow F}}{0 : !^1(w \oplus b), w, b, w \oplus b \longrightarrow F}\ \oplus_L}{0 : !^1(w \oplus b), w, w \oplus b, w \oplus b \longrightarrow F}\ \oplus_L\quad \Xi}{0 : !^1(w \oplus b), w \oplus b, w \oplus b, w \oplus b \longrightarrow F}\ \oplus_L}{3 : !^1(w \oplus b) \longrightarrow F}\ 3 \times !_L
$$

Game theoretically, P must be prepared for any of the choices of O when she decides to select $w \oplus b$ (on the left).

Results

Theorem

Given a $\mathcal{C}(\mathbb{R}^+)$-proof Ξ of a sequent S, there exists a smallest budget with $\mathtt{cost}(\Xi)$ that suffices to win the game $\mathcal{G}_{\mathcal{C}}(\mathbb{R}^+)$ on S when following the strategy corresponding to Ξ.

Spectrum

$\text{spec}(S) := \{\mathtt{cost}(\Xi) \mid \Xi \text{ is an } \mathcal{C}(\mathbb{R}^+)\text{-proof of } S\}.$

Theorem

If $\vdash_{\mathcal{C}(\mathbb{R}^+)} \Gamma \longrightarrow A$, then spec$(\Gamma \longrightarrow A)$ has a least element. In other words, there is a smallest b such that $\vdash_{\mathcal{C}^{\ell}(\mathbb{R}^+)} \Gamma \longrightarrow_b A$.

Outline

Cut-elimination

$\mathcal{C}(\mathbb{R}^+)$ inherits the admissibility of the following cut rule from SELL:

$$\frac{!\Gamma, \Delta_1 \longrightarrow A \quad !\Gamma, \Delta_2, A \longrightarrow C}{!\Gamma, \Delta_1, \Delta_2 \longrightarrow C}\ cut$$

Note: Remember that bangs occur negatively only.

Cut-elimination

$\mathcal{C}(\mathbb{R}^+)$ inherits the admissibility of the following cut rule from SELL:

$$\frac{!\Gamma, \Delta_1 \longrightarrow A \quad !\Gamma, \Delta_2, A \longrightarrow C}{!\Gamma, \Delta_1, \Delta_2 \longrightarrow C}\ cut$$

Note: Remember that bangs occur negatively only.

Theorem

For $f(a,b) = a + b$, the following cut rule is admissible in $\mathcal{C}^\ell(\mathbb{R}^+)$:

$$\frac{a : !\Gamma, \Delta_1 \longrightarrow A \quad b : !\Gamma, \Delta_2, A \longrightarrow C}{f(a,b) : !\Gamma, \Delta_1, \Delta_2 \longrightarrow C}\ cut_\ell$$

Moreover, whenever cut_ℓ is admissible w.r.t. $f : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$, then $a + b \le f(a,b)$.

What if we add exponentials to succedents?

$$\frac{b : \Gamma^{\preceq a} \longrightarrow A}{b : \Gamma \longrightarrow\ !^{a} A}$$

What if we add exponentials to succedents?

$$\frac{b : \Gamma^{\preceq a} \longrightarrow A}{b : \Gamma \longrightarrow !^a A}$$

Cut-elimination FAILS!!

Theorem

There is no function $f : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ such that the rule

$$\frac{a : !\Gamma, \Delta_1 \longrightarrow A \quad b : !\Gamma, \Delta_2, A \longrightarrow C}{f(a,b) : !\Gamma, \Delta_1, \Delta_2 \longrightarrow C}\ cut$$

is admissible in $\mathcal{C}^\ell(\mathbb{R}^+)$.

What if we add exponentials to succedents?

$$\frac{b : \Gamma^{\preceq a} \longrightarrow A}{b : \Gamma \longrightarrow !^a A}$$

Cut-elimination FAILS!!

Theorem

There is no function $f : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ *such that the rule*

$$\frac{a : !\Gamma, \Delta_1 \longrightarrow A \quad b : !\Gamma, \Delta_2, A \longrightarrow C}{f(a, b) : !\Gamma, \Delta_1, \Delta_2 \longrightarrow C} \; cut$$

is admissible in $\mathcal{C}^\ell(\mathbb{R}^+)$.

Proof: Take

$a : !^{1/k} p \longrightarrow !^{1/k} p^{\otimes(k \cdot a)}$

$b : !^{1/k} p^{\otimes(k \cdot a)} \longrightarrow p^{\otimes(k \cdot k \cdot a \cdot b)}$

$k.a.b : !^{1/k} p \longrightarrow p^{\otimes(k \cdot k \cdot a \cdot b)}$

Restrict the cut-formula!

Theorem (Lang'21)

If A is bang-free and $c \neq 0$, then the following cut rule is admissible in $\mathcal{C}^{\ell}(\mathbb{R}^{+})$:

$$\frac{a : !\Gamma, \Delta_1 \longrightarrow !^{c}A \quad b : !\Gamma, \Delta_2, !^{c}A \longrightarrow C}{f(a, b, c) : !\Gamma, \Delta_1, \Delta_2 \longrightarrow C}\ cut_{\ell}$$

where $f(a, b, c) = b + \lfloor b/c \rfloor .a$

Restrict the cut-formula!

Theorem (Lang'21)

If A is bang-free and $c \neq 0$, then the following cut rule is admissible in $\mathcal{C}^{\ell}(\mathbb{R}^{+})$:

$$\frac{a : !\Gamma, \Delta_1 \longrightarrow !^{c}A \quad b : !\Gamma, \Delta_2, !^{c}A \longrightarrow C}{f(a,b,c) : !\Gamma, \Delta_1, \Delta_2 \longrightarrow C}\ cut_{\ell}$$

where $f(a,b,c) = b + \lfloor b/c \rfloor .a$

- Last case = particular case with no bangs.
- General case where A is not bang-free: open problem.

Enhance the notion of cut rule!

Definition

Let $\mathcal{E} = \{a_b \mid a, b \in \mathbb{R}^+\}$ be such that

1. $a_b \geq_{\mathcal{E}} a_c$ (i.e., the ordering $\geq_{\mathcal{E}}$ ignores the subindices).
2. $a_b >_{\mathcal{E}} c_d$ iff $a > c$.

For any formula $A \in \mathcal{C}^{\ell}(\mathbb{R}^+)$, we define $[A]_c$ as the formula that substitutes any modality $!^{a_b}$ with $!^{a_{b+c}}$.

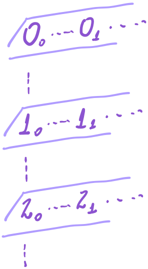
Enhance the notion of cut rule!

Definition

Let $\mathcal{E} = \{a_b \mid a, b \in \mathbb{R}^+\}$ be such that

1. $a_b \geq_{\mathcal{E}} a_c$ (i.e., the ordering $\geq_{\mathcal{E}}$ ignores the subindices).
2. $a_b >_{\mathcal{E}} c_d$ iff $a > c$.

For any formula $A \in \mathcal{C}^{\ell}(\mathbb{R}^+)$, we define $[A]_c$ as the formula that substitutes any modality $!^{a_b}$ with $!^{a_{b+c}}$.



Enhance the notion of cut rule!

Definition

Let $\mathcal{E} = \{a_b \mid a, b \in \mathbb{R}^+\}$ be such that

1. $a_b \geq_{\mathcal{E}} a_c$ (i.e., the ordering $\geq_{\mathcal{E}}$ ignores the subindices).
2. $a_b >_{\mathcal{E}} c_d$ iff $a > c$.

For any formula $A \in \mathcal{C}^\ell(\mathbb{R}^+)$, we define $[A]_c$ as the formula that substitutes any modality $!^{a_b}$ with $!^{a_{b+c}}$.

- Sequent labels belong to $\mathbb{R}^+$, modal labels belong to $\mathcal{E}$.
- Promotion of $!^{a_0}$ has the same effect/constraints that the promotion of $!^{a_b}$.
- Dereliction of the latter requires a greater budget ($a + b$ instead of a).
- $!^{a_b}A \equiv !^{a_c}A$, each direction requiring a different budget.
- $\mathcal{E}_0 = \{a_0 \mid a \in \mathbb{R}^+\} \simeq \mathbb{R}^+$, that is, each element $a \in \mathbb{R}^+$ can be seen as the equivalence class of a_0 in $\mathbb{R}^+ \times \mathbb{R}^+$ modulo $\mathbb{R}^+$.

The new $\mathcal{C}^\ell(\mathbb{R}^+)$

Definition

A is $\multimap$-linear if for all subformulas of the form $B \multimap C$, B doesn't have occurrences of $!^a$.

Theorem ($\multimap$-linear cut)

If A is a $\multimap$-linear formula, then the following rule is admissible

$$\frac{a : !\Gamma, \Delta_1 \longrightarrow A \quad b : !\Gamma, \Delta_2, [A]_a \longrightarrow C}{a + b : !\Gamma, \Delta_1, \Delta_2 \longrightarrow C} \; cut_{LL}$$

Moreover, if $a : \Gamma \longrightarrow C$ is provable using cut_{LL}, then there is a cut-free proof of $a' : \Gamma \longrightarrow C$ for some $a \geq a'$.

The source of evil tamed!

$$\dfrac{\dfrac{\overset{\pi_1}{c : !\Gamma, \Delta_1, A \longrightarrow B}}{c : !\Gamma, \Delta_1 \longrightarrow A \multimap B} \quad \dfrac{\overset{\pi_2}{b_1 : !\Gamma, \Delta_2' \longrightarrow [A]_c} \quad \overset{\pi_3}{b_2 : !\Gamma, \Delta_2'', [B]_c \longrightarrow C}}{b_1 + b_2 : !\Gamma, \Delta_2, [A \multimap B]_c \longrightarrow C}}{c + b_1 + b_2 : !\Gamma, \Delta_1, \Delta_2 \longrightarrow C}$$

reduces to

$$\dfrac{\overset{\pi_2}{b_1 : !\Gamma, \Delta_2' \longrightarrow A} \quad \dfrac{\overset{\pi_1}{c : !\Gamma, \Delta_1, [A]_{b_1} \longrightarrow B} \quad \overset{\pi_3}{b_2 : !\Gamma, \Delta_2'', [B]_c \longrightarrow C}}{c + b_2 : !\Gamma, \Delta_1, \Delta_2'', [A]_{b_1} \longrightarrow C}}{c + b_1 + b_2 : !\Gamma, \Delta_1, \Delta_2 \longrightarrow C}$$

Note: $[A]_c = [A]_{b_1} = A$.

Outline

What next?

- Classical LL;
- Non-affine LL;
- Bounds of computation;
- Complexity of cut-elimination.

Thanks!!!

☺