Modal Algebraic Models For Counterfactual Conditionals

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Counterfactuals are subjunctive conditional statements of the form “If [antecedent] were the case, then [consequent] would be the case”.

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How?

We introduce an algebraic setting for counterfactuals reasoning based on a modal extensions of Boolean Algebras of Conditionals and their associated logic.

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3. consider all Boolean combinations of conditional events, $\mathbf{Free}(A \mid A') = (Free(A \mid A'), \sqcap, \sqcup, \sim, \bot^*, \top^*)$; space of conditional events
4. impose some "rules of behavior" to those conditionals (laws of probability):

(C1) $(b \mid b) \equiv_{\mathfrak{C}} \top^*$, for all $b \in A'$;
(C2) $(a_1 \mid b) \sqcap (a_2 \mid b) \equiv_{\mathfrak{C}} (a_1 \wedge a_2 \mid b)$, for all $a_1, a_2 \in A$, $b \in A'$;
(C3) $\sim(a \mid b) \equiv_{\mathfrak{C}} (\neg a \mid b)$, for all $a \in A$, $b \in A'$;
(C4) $(a \wedge b \mid b) \equiv_{\mathfrak{C}} (a \mid b)$, for all $a \in A$, $b \in A'$;
(C5) $(a \mid b) \sqcap (b \mid c) \equiv_{\mathfrak{C}} (a \mid c)$, for all $a \in A$, $b, c \in A'$ such that $a \leq b \leq c$.

Definition: Boolean Algebra of Conditionals (BAC)

The *Boolean Algebra of Conditionals* (BAC) of $\mathbf{A}$, $\mathcal{C}(\mathbf{A})$, is the quotient structure:

$$\mathcal{C}(\mathbf{A}) = \mathbf{Free}(A \mid A')/_{\equiv_{\mathfrak{C}}} = (\mathcal{C}(A), \sqcap, \sqcup, \sim, \bot_{\mathfrak{C}}, \top_{\mathfrak{C}})$$

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- let $\omega_1, \ldots, \omega_m$ be the atoms of $\mathcal{C}(\mathbf{A})$, $\mathrm{at}(\mathcal{C}(\mathbf{A})) = \{\omega_1, \ldots, \omega_m\}$; conditionals
- let $\mathrm{Seq}(\mathbf{A})$ be the permutations of $\mathrm{at}(\mathbf{A})$, $\mathrm{Seq}(\mathbf{A}) = \{\langle \alpha^1, \ldots, \alpha^n \rangle : \alpha^i \in \mathrm{at}(\mathbf{A})\}$;

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$$\omega = (\alpha^1 \mid \top_{\mathfrak{C}}) \sqcap (\alpha^2 \mid \neg\alpha^1) \sqcap \cdots \sqcap (\alpha^n \mid \neg\alpha^1 \wedge \cdots \wedge \alpha^{n-1})$$

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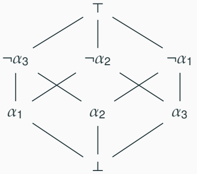
3. by 1 and 2, each atom $\omega_i \in \mathrm{at}(\mathcal{C}(\mathbf{A}))$ can be univocally identified with the permutation $\langle \alpha^1, \ldots, \alpha^n \rangle$ that induces ω_i (and viceversa);

Notation: $\omega[i]$ denotes the *i*-th element in the permutation identified with/that induces ω



- $\text{at}(\mathbf{A}_3) = \{\alpha_1, \alpha_2, \alpha_3\}$;
- $\text{Seq}(\mathbf{A}_3) = \{$
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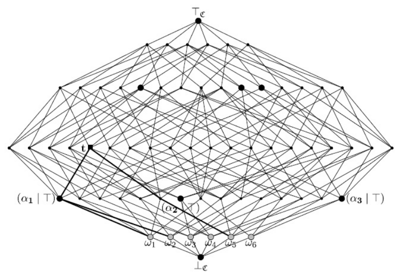

$\langle\alpha_1, \alpha_2, \alpha_3\rangle \mapsto \omega_1 = (\alpha_1 \mid \top) \sqcap (\alpha_2 \mid \neg\alpha_1)$
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Boolean Algebras of Conditionals - Syntax & Semantics

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$\mathcal{L}^{LBC}$ is a language obtained by expanding $\mathcal{L}$ with the conditional connective $|$:

- if φ, ψ are formulas of $\mathcal{L}$ and $\nvdash_{CPL} \neg\varphi$, then $(\psi \mid \varphi)$ is a formula of $\mathcal{L}^{LBC}$;
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- An interpretation of $\mathcal{L}^{LBC}$ is any $\omega_{\mathbf{L}} \in \text{at}(\mathcal{C}(\mathbf{L}))$. Observe: any $\omega_{\mathbf{L}} \in \text{at}(\mathcal{C}(\mathbf{L}))$ is a permutation of classical valuations of $\mathcal{L}$.

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Proposition

Consider **A** and its BAC $\mathcal{C}(\mathbf{A})$; for every conditional $(a \mid b)$ in $\mathcal{C}(\mathbf{A})$ and $\omega \in \mathrm{at}(\mathcal{C}(\mathbf{A}))$,

$$\omega \sqsubseteq (a \mid b) \Leftrightarrow \exists j : \omega[j] \leq a \wedge b \text{ and } \forall i < j, \omega[i] \nleq b$$

Boolean Algebras of Conditionals - Syntax & Semantics

Hence we get:

$\omega_{\mathbf{L}} \Vdash (\varphi \mid \psi)$	$\Leftrightarrow$	The first (from the left) valuation in $\omega_{\mathbf{L}}$ that makes ψ true also makes φ true
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Example

Assume $v_2(\psi) = v_2(\varphi) = 1$, $v_1(\varphi) = v_1(\psi) = 0$, $v_3(\varphi) = 0$, $v_3(\psi) = 1$.

- $\omega_{\mathbf{L}} = \langle v_1, v_2, v_3, \ldots \rangle$, $\omega_{\mathbf{L}} \Vdash (\varphi \mid \psi)$
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Moreover, for $\Gamma \cup \{\Phi\} \subseteq \mathcal{L}^{LBC}$ we define logical consequence as:

$$\Gamma \models_{LBC} \Phi \Leftrightarrow \text{for all interpretation } \omega_{\mathbf{L}}, \text{ if } \omega_{\mathbf{L}} \Vdash \bigwedge \Gamma, \text{ then } \omega_{\mathbf{L}} \Vdash \Phi$$

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'If kangaroos had no tails, they would topple over' seems to mean something like this: in ***any possible state of affairs*** *in which kangaroos have no tails, and* ***which resembles our actual state of affairs*** *as much as kangaroos having no tails permits it to, the kangaroos would topple over. I shall give a general analysis of counterfactual conditionals along these lines. (Lewis 1973b)*

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Consider a classical language $\mathcal{L}$ with finitely many $p, q, r \ldots$ propositional variables. Let $\vdash_{CPL}$ denotes derivability in Classical Logic.

$\mathcal{L}^{\Box\!\!\rightarrow}$ is a language obtained from $\mathcal{L}$ by extending it with the counterfactual connective $\Box\!\!\rightarrow$, where $\varphi \Box\!\!\rightarrow \psi$ can be read as if φ were the case, then ψ would be the case. Formulas in $\mathcal{L}^{\Box\!\!\rightarrow}$ are defined as:

- if φ, ψ are formulas of $\mathcal{L}$ and $\nvdash_{CPL} \neg\varphi$, then $\varphi \Box\!\!\rightarrow \psi$, φ and ψ are formulas of $\mathcal{L}^{\Box\!\!\rightarrow}$;
- if A, B are formulas of $\mathcal{L}^{\Box\!\!\rightarrow}$, then $\neg A$ and $A \wedge B$ are formulas of $\mathcal{L}^{\Box\!\!\rightarrow}$;
- nothing else is a formula of $\mathcal{L}^{\Box\!\!\rightarrow}$.

Observe: we restrict to a fragment of the original Lewis' language of counterfactuals.

Lewis' Logic of Counterfactuals

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C1

Rules:

(MP) form φ and $\varphi \rightarrow \psi$ infer ψ

(DWC) if $\vdash (\varphi_1 \wedge \cdots \wedge \varphi_n) \rightarrow \psi$ then $\vdash ((\delta \Box\!\!\rightarrow \varphi_1) \wedge \cdots \wedge (\delta \Box\!\!\rightarrow \varphi_n)) \rightarrow (\delta \Box\!\!\rightarrow \psi)$

Axioms:

1. all (substitutions instances of) classical tautologies
2. $\varphi \Box\!\!\rightarrow \varphi$
3. $((\varphi \Box\!\!\rightarrow \psi) \wedge (\psi \Box\!\!\rightarrow \varphi)) \rightarrow ((\varphi \Box\!\!\rightarrow \delta) \leftrightarrow (\psi \Box\!\!\rightarrow \delta))$
4. $((\varphi \vee \psi) \Box\!\!\rightarrow \varphi) \vee ((\varphi \vee \psi) \Box\!\!\rightarrow \psi) \vee (((\varphi \vee \psi) \Box\!\!\rightarrow \delta) \leftrightarrow ((\varphi \Box\!\!\rightarrow \delta) \wedge (\psi \Box\!\!\rightarrow \delta))$
5. $(\varphi \Box\!\!\rightarrow \psi) \rightarrow (\varphi \rightarrow \psi)$
6. $(\varphi \wedge \psi) \rightarrow (\varphi \Box\!\!\rightarrow \psi)$

Lewis defines: $\varphi \Diamond\!\!\rightarrow \psi := \neg(\varphi \Box\!\!\rightarrow \neg\psi)$; consider the following extension of **C1**:

C1$^+$

All the rules and axioms of **C1** plus: $\vdash (\varphi \Box\!\!\rightarrow \psi) \rightarrow (\varphi \Diamond\!\!\rightarrow \psi)$

Lewis' Logic of Counterfactuals - Semantics

Definition: Sphere Model

A *sphere model* is a tuple $\Sigma = (I, \mathscr{S}, v)$ where:

- I is a non-empty set (of possible worlds);
- $\mathscr{S}$ is a function $\mathscr{S} : I \to \wp(\wp(I))$ such that for each $i \in I$, $\mathscr{S}(i)$ is:

  (S1) nested: for all $S, T \in \mathscr{S}(i)$, either $S \subseteq T$ or $T \subseteq S$;
  (S2) non-empty: for all $S \in \mathscr{S}(i)$, $i \in S$;
  (S3) centered: either $\bigcup \mathscr{S}(i) = \emptyset$, or $\{i\} \in \mathscr{S}(i)$.

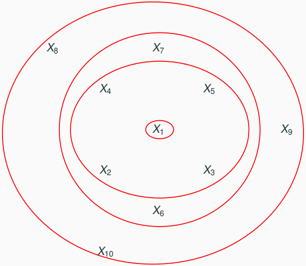
- v is a valuation function $v : \mathcal{P} \to \wp(I)$ that extends to compound formulas as follows:

  - $v(\neg\Phi) = I \setminus v(\Phi)$, $v(\Phi \wedge \Psi) = v(\Phi) \cap v(\Psi)$, $v(\Phi \vee \Psi) = v(\Phi) \cup v(\Psi)$
  - $v(\psi \mathrel{\Box\!\!\rightarrow} \varphi) = \{i \in I \mid v(\psi) \cap \bigcup \mathscr{S}(i) = \emptyset,$ or $\exists S \in \mathscr{S}(i)\, (\emptyset \neq (v(\psi) \cap S) \subseteq v(\varphi))\}$;

Informally, $\varphi \mathrel{\Box\!\!\rightarrow} \psi$ is true at a world i, $i \Vdash \varphi \mathrel{\Box\!\!\rightarrow} \psi$, iff ψ **is true at all the most similar worlds to** i **that make** φ **true**.

Sphere Model - Example

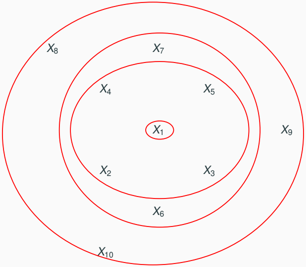
$\mathscr{S}(X_1) = \{$
$\{X_1\}$
$\{X_1, X_2, X_3, X_4, X_5\}$
$\{X_1, X_2, X_3, X_4, X_5, X_6, X_7\}$
$\{X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8, X_9, X_{10}\}$
$\}$



if $v(p) = \{X_5\}$ and $v(q) = \{X_5, X_6\}$, then $X_1 \Vdash p \,\Box\!\!\rightarrow q$ since $X_5 \Vdash p$ and $X_5 \Vdash q$

$\mathscr{S}_{X_1}$ is

- non-empty;
- centered in $\{X_1\}$;
- nested: all members of $\mathscr{S}(X_1)$ are totally ordered by set-inclusion.



if $v(p) = \{X_5\}$ and $v(q) = \{X_6\}$, then $X_1 \nVdash p \,\Box\!\!\rightarrow q$, since $X_5 \Vdash p$ but $X_5 \nVdash q$

Lewis' Logic of Counterfactuals

We define logical consequence as: for $\Gamma \cup \{A\} \subseteq \mathcal{L}^{\Box\!\!\rightarrow}$,

$\Gamma \models_{\mathbf{C1}} A \Leftrightarrow$ for all the sphere models Σ, for all the worlds i in Σ, if $i \Vdash \bigwedge \Gamma$ then $i \Vdash B$

Proposition - Soundness & Completeness C1

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Definition

A **total sphere model** is a sphere model $(I, \mathscr{S}, v)$ such that for all formulas $\varphi \in \mathcal{L}$, if $\nvdash_{CPL} \neg\varphi$ then

$$\text{for all } i \in I, \bigcup \mathscr{S}(i) \cap v(\varphi) \neq \emptyset$$

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Proposition - Soundness & Completeness C1$^+$

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Modal BACs

Lewis Algebras

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By Jónsson-Tarski duality, to each finite Lewis algebra $\mathcal{L}(\mathbf{A})$ we uniquely associate a dual frame $(\mathrm{at}(\mathcal{C}(\mathbf{A})), R)$ where $R \subseteq \mathrm{at}(\mathcal{C}(\mathbf{A})) \times \mathrm{at}(\mathcal{C}(\mathbf{A}))$ is defined as:

$$\forall \omega, \omega' \in \mathrm{at}(\mathcal{C}(\mathbf{A})), \omega \mathrm{R} \omega' \text{ iff } \forall \mathrm{t} \in \mathcal{C}(\mathbf{A}) \text{ if } \omega \sqsubseteq \Box \mathrm{t}, \text{ then } \omega' \sqsubseteq \mathrm{t}$$

Modal BACs - Syntax & Semantics

$\mathcal{L}^{LBC}_{\Box}$ is a language obtained by expanding $\mathcal{L}^{LBC}$ with $\Box$ and where formulas are:

- if $(\varphi \mid \psi) \in \mathcal{L}^{LBC}$, then $\Box(\varphi \mid \psi), (\varphi \mid \psi) \in \mathcal{L}^{LBC}_{\Box}$;
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For every $\mathcal{L}^{LBC}_{\Box}$ interpretation $\mathcal{F} = \langle \text{at}(\mathcal{C}(\mathbf{L})), R\rangle$, for $\Gamma \cup \{\Phi\} \subseteq \mathcal{L}^{LBC}_{\Box}$, we set:

$\mathcal{F} \models \Phi \quad \Leftrightarrow \quad$ for all $\omega_{\mathbf{L}} \in \text{at}(\mathcal{C}(\mathbf{L})), \omega_{\mathbf{L}} \Vdash \Phi$

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Duality

Lewis Frames - Characterization

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Observe: by Jónsson-Tarski duality, every $\mathcal{L}_{\Box}^{LBC}$ interpretation validates the following

$L1^*$ $\Box(\varphi \mid \top) \leftrightarrow (\varphi \mid \top)$ (dual of (L1))

$L2^*$ $\Box(\varphi \mid \varphi \vee \psi) \vee \Box(\psi \mid \varphi \vee \psi) \vee (\Box(\delta \mid \varphi \vee \psi) \leftrightarrow \Box((\delta \mid \varphi) \wedge (\delta \mid \psi)))$ (dual of (L2))

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Proposition

Let $\mathcal{F} = \langle \text{at}(\mathcal{C}(\mathbf{L})), R\rangle$ be a $\mathcal{L}_\Box^{LBC}$ interpretation:

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(Centering) : - all the accessible worlds/permutations begin with the same element
- dual of Centered system of spheres

Lewis Frames - Characterization

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In order to understand the meaning of *Sphericity*, a more intricate and a peculiar construction is needed. The intuitive idea is that *Sphericity* induces a certain structure of $R[\omega] = \{\omega' \mid \omega R \omega'\}$ that allows us to extrapolate sphere models.

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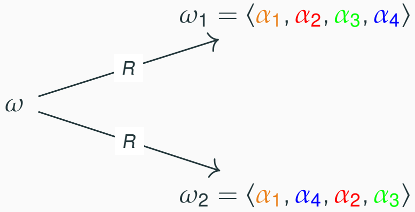
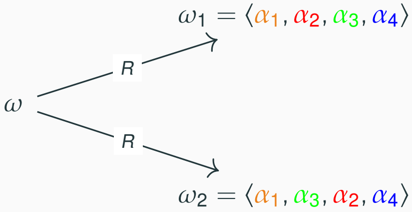


Non-Spheric



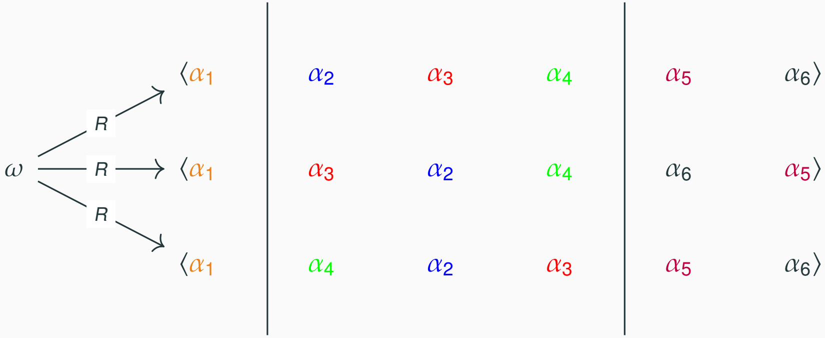
Spheric

Lewis Frame - Characterization

Consider a $\mathcal{L}_{\Box}^{LBC}$ interpretation $\langle \mathrm{at}(\mathcal{C}(\mathbf{L})), R \rangle$. Let $\mathbf{R}_{k,n}^{\omega}$ be the $k \times n$-matrix whose ith raw is a certain $\omega_{\mathbf{L}} \in R[\omega]$. We may refer to $\mathbf{R}_{k,n}^{\omega}$ as the matrix *generated* by $R[w]$.

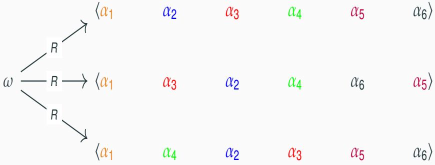
Proposition - Sphericity

A $\mathcal{L}_{\Box}^{LBC}$ interpretation $\langle \mathrm{at}(\mathcal{C}(\mathbf{L})), R \rangle$ has the Sphericity property iff for all $\omega \in \mathrm{at}(\mathcal{C}(\mathbf{L}))$ there exists a spheric partition of $\mathbf{R}_{k,n}^{\omega}$

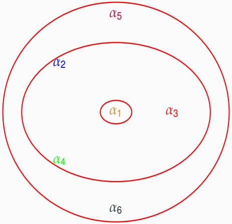


From Lewis Frames to Sphere Models... - Example

From every $\mathcal{L}_{\Box}^{LBC}$ interpretation we can extrapolate a sphere model preserving validity. We employ the sphericity and centering properties of Lewis frames.



$$\mathscr{S}(\alpha_1) =$$



$$\{\alpha_1\},$$
$$\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\},$$
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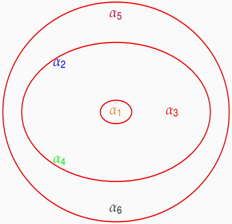
...and back - Example

Given a sphere model $(I, \mathscr{S}, v)$, for each $i \in I$, we define a binary relation $\prec_i \subseteq I \times I$ such that

$$j \prec_i k \text{ iff for all } S \in \mathscr{S}(i), \text{ if } k \in S, \text{ then } j \in S$$

From every total sphere model we can extrapolate a $\mathcal{L}_{\Box}^{LBC}$ interpretation preserving validity. We employ the totality condition and the definition of $\prec_i$.



$\alpha_1 \prec_1 \alpha_2 \prec_1 \alpha_3 \prec_1 \alpha_4 \prec_1 \alpha_5 \prec_1 \alpha_6$

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Counterfactuals as Modal Conditionals

Observe: two languages

1. $\mathcal{L}_{\Box}^{LBC}$: Modal Conditionals $\Box(\psi \mid \varphi)$ (and Boolean combinations of those);
2. $\mathcal{L}^{\Box\!\!\rightarrow}$: Classical formulas + counterfactuals $\varphi \Box\!\!\rightarrow \psi$ (and Boolean combinations of those)

Definition

We can translate $\mathcal{L}^{\rightarrow}$ into $\mathcal{L}_{\Box}^{LBC}$:

- if φ is a formula in $\mathcal{L}$, $\tau(\varphi) = \Box(\varphi \mid \top) = (\varphi \mid \top)$ by $(L1)$
- if φ is $\psi \Box\!\!\rightarrow \delta$, $\tau(\psi \Box\!\!\rightarrow \delta) = \Box(\delta \mid \psi)$
- if φ is a Boolean combination $\neg\psi$, $\psi \wedge \delta$, $\tau(\neg\psi) = \neg\tau(\psi)$, $\tau(\psi \wedge \delta) = \tau(\psi) \wedge \tau(\delta)$

Counterfactuals as Modal Conditionals

The fact that we can go from $\mathcal{L}_{\Box}^{LBC}$ interpretation to total sphere models and back preserving validity allows us to prove the following result

Theorem

For all $\varphi \in \mathcal{L}^{\Box\!\!\rightarrow}$,

$$\Gamma \models_{C1^+} \varphi \Leftrightarrow \tau[\Gamma] \models_{LBC^{\Box}} \tau(\varphi)$$

and in particular

$$\models_{C1^+} \varphi \Box\!\!\rightarrow \psi \Leftrightarrow \models_{LBC^{\Box}} \Box(\psi \mid \varphi)$$

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Counterfactuals can be interpreted as "necessary" conditionals

Observation

The counterfactual cannot be any strict conditional. *(Lewis 1973b)*

A strict conditional is $\Box(\varphi \rightarrow \psi)$ where $\rightarrow$ is classical implication:
$\varphi \Box\!\!\rightarrow \psi \not\equiv \Box(\varphi \rightarrow \psi)$

if we take the non-classical implication "|" obeying the laws of conditional probability, then a counterfactual can be interpreted as a conditional with a $\Box$ in front:
$\varphi \Box\!\!\rightarrow \psi \equiv \Box(\psi \mid \varphi)$

Probability

Probability of Conditionals

Proposition

Any positive probability $P : \mathbf{A} \to [0,1]$ naturally extends to a positive probability $\mu_P : \mathcal{C}(\mathbf{A}) \to [0,1]$ such that: for $\omega = \langle \alpha_1, \ldots, \alpha_n \rangle \in \text{at}(\mathcal{C}(\mathbf{A}))$,

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What happens to the probability of counterfactuals?

Belief Functions

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- Masses, m, are assigned to sets of possibilities $\{w_1, w_2, \ldots, w_n\}$ rather than single events $\{w_1\}, \{w_2\} \ldots$: their appeal rests on the fact they naturally encode evidence in favor of propositions. $Bel(A)$ is the degree to which the available evidence supports A.

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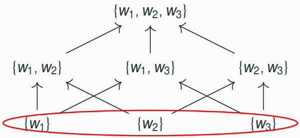
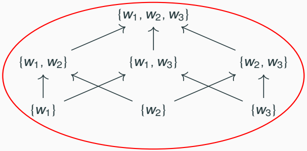


- $P(w_1) + P(\{w_2\}) + P(\{w_3\}) = 1$
- $P(\{w_1, w_2\}) = P(\{w_1\}) + P(w_2)$



- $\sum_{Y \subseteq \{w_1, w_2, w_3\}} m(Y) = 1$
- $Bel(\{w_1, w_2\}) = m(\{w_1\}) + m(\{w_2\}) + m(\{w_1, w_2\})$

Belief Functions from Modal Algebras

Belief Functions from Modal Algebras

A belief function *Bel* on a Boolean algebra $\mathbf{A}$ is a function $Bel : \mathbf{A} \to [0, 1]$ such that:

1. $Bel(\top) = 1$, $Bel(\bot) = 0$
2. $Bel(a_1 \vee \cdots \vee a_n) \geq \sum_{i=1}^{n} Bel(a_i) - \sum_{j<k} Bel(a_j \wedge a_k) + \cdots + (-1)^{n+1} Bel(a_1 \wedge \cdots \wedge a_n)$
   e.g. $Bel(A \cup B) \geq Bel(A) + Bel(B) - Bel(A \cap B)$.

A mass function m over a Boolean Algebra $\mathbf{A}$, $m : \mathbf{A} \to [0, 1]$, is such that:

1. $m(\bot) = 0$
2. $\sum_{a \in \mathbf{A}} m(a) = 1$

we can define a Belief function on $\mathbf{A}$, $Bel_m : \mathbf{A} \to [0, 1]$, as follows:

$$Bel_m(a) = \sum_{b \leq a} m(b)$$

By the results in (Harmanec, G. Klir, and Wang 1996), (Harmanec, G. J. Klir, and Resconi 1994), connecting belief functions and modal logic, we can show the following:

Belief Functions, Modal Algebras and Kripke Frames

Proposition

Consider a Kripke frame $\langle W, R\rangle$, its dual modal algebra $\langle \wp(W), \Box\rangle$ and a probability $P : \wp(W) \to [0,1]$. we have that:

$$m_p(X) = \sum_{R[w]=X} P(w) \text{ is a mass function on } \wp(W) \tag{1}$$

$$Bel_P(X) = \sum_{Y\subseteq X} m_P(Y) = \sum_{w\Vdash\Box X} P(w) = P(\Box X) \text{ is a Belief function on } \wp(W) \tag{2}$$

Proposition

Consider a Lewis algebra $\langle \mathcal{C}(\mathbf{A}), \Box\rangle$, its dual Lewis frame $\langle \mathrm{at}(\mathcal{C}(\mathbf{A})), R\rangle$, a probability $P : \mathbf{A} \to [0,1]$, and its extension to $\mathcal{C}(\mathbf{A})$ $\mu_P : \mathcal{C}(\mathbf{A}) \to [0,1]$. We get that:

$$\mu_P(\Box(a \mid b)) = \sum_{\omega\sqsubseteq\Box(a|b)} \mu_P(\omega) = Bel_{\mu_P}(a \mid b)$$

The probability of a counterfactual amounts to the belief of its corresponding conditional.

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Given a sphere model $\langle I, \mathscr{S}, v\rangle$, and a probability distribution $P : I \to [0, 1]$ on I, we can assign to each formula φ a probability:

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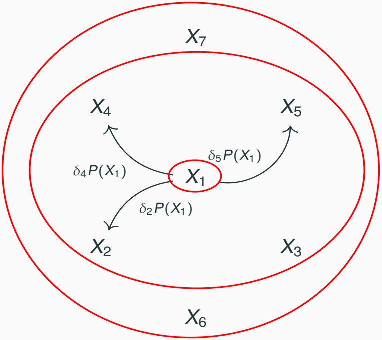
More attention has been dedicated to finding a method to calculate counterfactual probability, i.e. the probability that [consequent] would happen given that [antecedent] were the case.

E.g. Lewis' Imaging (Lewis 1973b), interventionist counterfactuals Pearl 2000).

Probability of Counterfactuals - Lewis' Imaging

Given a sphere model $\langle I, \mathscr{S}, v\rangle$, and a probability distribution $P : I \to [0,1]$ on I, one can define $P(\varphi \,\Box\!\!\to \psi) = P_\varphi(\psi)$ where P_φ is a new probability obtained from P by imaging on φ such that $P(\varphi) = 1$, i.e. φ were the case.



if $X_1 \nVdash \varphi$, then:

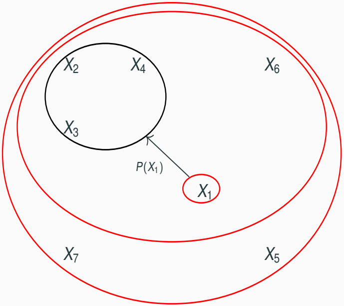
- $P_\varphi(X_1) = 0$
- X_1 transfers its mass to its closest worlds making φ true

$$P(\varphi \,\Box\!\!\to \psi) = P_\varphi(\psi) = \sum_{i \Vdash \psi} P_\varphi(i)$$

Probability of Counterfactuals

Consider $\varphi \,\Box\!\!\rightarrow\, \psi$; Dubois (1994) proposes that instead of transferring mass to single worlds, we can redistribute the lost mass to the entire set of closest worlds, so defining a mass function $m_\varphi : \wp(I) \rightarrow [0,1]$ such that $\sum_{Y \subseteq v(\varphi)} m_\varphi(Y) = 1$.

We formalize Dubois' intution



For $Y \in \wp(I)$, we can define:

$$m_\varphi(Y) = \sum_{Clos_\varphi(i)=Y} P(i)$$

$$Bel_\varphi(\psi) = \sum_{Y \subseteq v(\psi)} m_\varphi(Y) = \sum_{i \Vdash \varphi \Box\!\!\rightarrow \psi} P(i)$$

$$= P(\varphi \,\Box\!\!\rightarrow\, \psi)$$

We can interpret $P(\varphi \,\Box\!\!\rightarrow\, \psi)$ as the Belief of ψ given φ

Conclusions

Summing up

We introduced a novel framework to analyze conditional events, their logic and their probability.

1. we have expanded the framework of BACs to the modal case by introducing Lewis algebras and their dual Lewis frames;
2. we have analyzed the properties of this algebraic structures and characterized the class of Lewis frames;
3. we have proved soundness and completeness of $\mathbf{C1}^{+}$ with respect to Lewis Algebras/Lewis Frames;
4. we have provided an interpretation of counterfactual in terms of necessary conditionals;
5. we have analyzed the probability of counterfactuals in terms of Belief functions.

To do:

1. How to interpret the Belief $Bel_{\varphi}(\psi)$ (conditional belief?)
2. Study the logics arising from this framework (weaker/stronger than **C1**)
3. Philosophical interpretation of the modal operator $\Box(\varphi \mid \psi)$
4. Philosophical justification of this framework to analyze conditional events.

Conclusions

Thank You!

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