A fresh view of linear logic as a logical framework

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Joint work with Carlos Olarte & Bruno Xavier
LATD&MOSAIC

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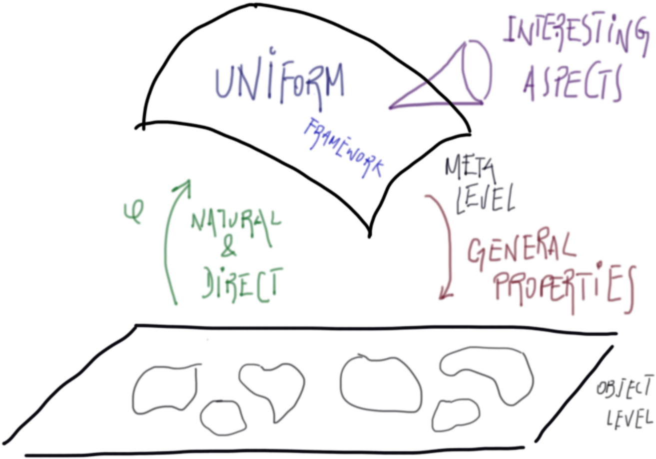
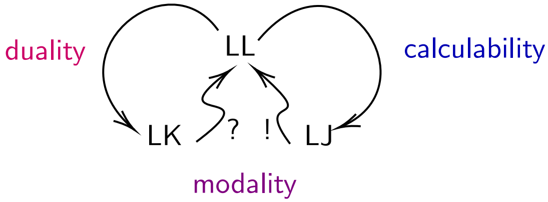
In classical ballet...

Uniformity of movement, different individuals.

… and in logical frameworks!



Linear logic LL [Girard]



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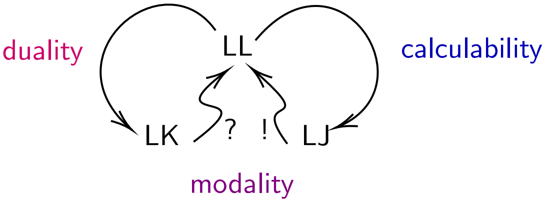

- … emphasizes the role of formulas as resources, while classical logic: emphasizes truth and intuitionistic logic: emphasizes proof.

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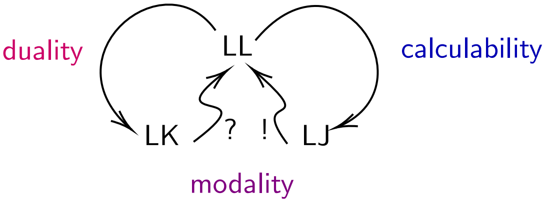

- ... emphasizes the role of formulas as resources, while classical logic: emphasizes truth and intuitionistic logic: emphasizes proof.
- ... has a way of representing both the needs of intuitionism and the elegance of classical logic: negation is involutive, sequents are symmetric, connectives are inter-definable, models are simple.

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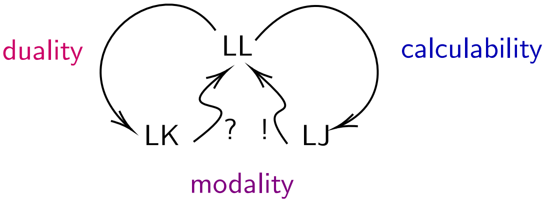

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- ... is resource conscious: specification of states.

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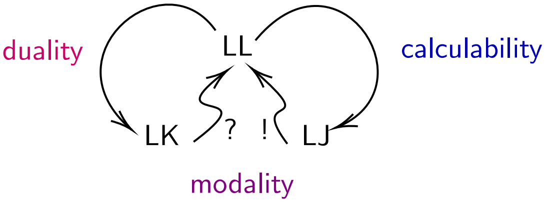

- ... emphasizes the role of formulas as resources, while classical logic: emphasizes truth and intuitionistic logic: emphasizes proof.
- ... has a way of representing both the needs of intuitionism and the elegance of classical logic: negation is involutive, sequents are symmetric, connectives are inter-definable, models are simple.
- ... is resource conscious: specification of states.
- ... captures several notions of dualities.

Outline

Dualities

Multi-modalities

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Dualities

Multi-modalities

Main idea [Dale & me]

$$\underbrace{B_1, \ldots, B_n \vdash C_1, \ldots, C_m}_{\text{Object-logic}} \quad \rightsquigarrow \quad \underbrace{\vdash \lfloor B_1 \rfloor, \ldots, \lfloor B_n \rfloor, \lceil C_1 \rceil, \ldots, \lceil C_m \rceil}_{\text{Meta-logic (LL)}}$$

Main idea [Dale & me]

$$\underbrace{B_1, \ldots, B_n \vdash C_1, \ldots, C_m}_{\text{Object-logic}} \quad \leadsto \quad \underbrace{\vdash \lfloor B_1 \rfloor, \ldots, \lfloor B_n \rfloor, \lceil C_1 \rceil, \ldots, \lceil C_m \rceil}_{\text{Meta-logic (LL)}}$$

$$\frac{\Gamma \vdash F, \Delta}{\Gamma \vdash F \vee G, \Delta} \vee_{R1} \quad \frac{\Gamma \vdash G, \Delta}{\Gamma \vdash F \vee G, \Delta} \vee_{R2} \quad \frac{\Gamma, F \vdash \Delta \quad \Gamma, G \vdash \Delta}{\Gamma, F \vee G \vdash \Delta} \vee_L$$

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Encoding:

$$\vee_R : \lceil F \vee G \rceil^{\perp} \otimes (\lceil F \rceil \oplus \lceil G \rceil) \quad \vee_L : \lfloor F \vee G \rfloor^{\perp} \otimes (\lfloor F \rfloor \,\&\, \lfloor G \rfloor)$$

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Adequacy:

$$\dfrac{\dfrac{\dfrac{}{\vdash \Theta, \mathcal{T}; \Gamma_1 \Downarrow \lceil F \wedge G \rceil^\perp}\ \mathrm{I} \quad \dfrac{\dfrac{\vdash \Theta, \mathcal{T}; \Gamma_2, \lceil F \rceil \Uparrow \cdot}{\vdash \Theta, \mathcal{T}; \Gamma_2 \Downarrow \lceil F \rceil}\ \mathrm{R_n, store}}{\vdash \Theta, \mathcal{T}; \Gamma_2 \Downarrow \lceil F \rceil \oplus \lceil G \rceil}\ \oplus_1}{\vdash \Theta, \mathcal{T}; \Gamma_1, \Gamma_2 \Downarrow \exists F, G. \lceil F \vee G \rceil^\perp \otimes (\lceil F \rceil \oplus \lceil G \rceil)}\ \exists, \exists, \otimes}{\vdash \Theta, \mathcal{T}; \Gamma_1, \Gamma_2 \Uparrow \cdot}\ \mathrm{D_c}$$

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Properties: cut-elimination

ML clause for OL cut:

$$\frac{\Gamma_1 \vdash \Delta_1, F \quad \Gamma_2, F \vdash \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\ \text{cut} \qquad \rightsquigarrow \qquad \text{cut} : \lfloor F \rfloor \otimes \lceil F \rceil$$

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ML left/right clauses for OL connective $\vee$:

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ML left/right clauses for OL connective $\star$:

$$\lfloor \star(\overline{F}) \rfloor^{\perp} \otimes \mathtt{B}\llbracket \star \rrbracket \qquad \text{and} \qquad \lceil \star(\overline{F}) \rceil^{\perp} \otimes \mathtt{B}\llbracket \star \rrbracket$$

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Cut-coherence:

$$\vdash \text{cut}; \cdot \Uparrow \mathtt{B}\llbracket \star \rrbracket^{\perp} \parr \mathtt{B}\llbracket \star \rrbracket^{\perp}$$

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Theorem: ML cut-coherence implies OL cut-elimination.

Dualities!

$$\lfloor \star(\overline{F}) \rfloor \otimes \lceil \star(\overline{F}) \rceil$$

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$$\lfloor \star(\overline{F}) \rfloor \otimes \lceil \star(\overline{F}) \rceil$$

$$\Downarrow$$

$$\mathrm{B}\llbracket \star \rrbracket \otimes \mathrm{B}\llbracket \star \rrbracket$$

Dualities!

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$$\Downarrow$$

$$\mathtt{B}\llbracket \star \rrbracket \otimes \mathtt{B}\llbracket \star \rrbracket, \mathtt{B}\llbracket \star \rrbracket^{\perp} \parr \mathtt{B}\llbracket \star \rrbracket^{\perp}$$

Dualities!

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Mechanized in Coq [Carlos & Bruno & Amy]

Properties: identity expansion

ML clause for OL initial axiom:

$$\frac{}{\Gamma, F \vdash F, \Delta}\ \mathsf{init} \qquad \rightsquigarrow \qquad \mathsf{init} : \ \lfloor F \rfloor^{\perp} \parr \lceil F \rceil^{\perp}$$

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Init-coherence:

$$\vdash \text{init}; \cdot \Uparrow \mathtt{B}\llbracket \star \rrbracket \parr \mathtt{B}\llbracket \star \rrbracket$$

Properties: identity expansion

ML clause for OL initial axiom:

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ML left/right clauses for OL connective $\star$:

$$\lfloor \star(\overline{F}) \rfloor^{\perp} \otimes \mathtt{B}[\![\star]\!] \qquad \text{and} \qquad \lceil \star(\overline{F}) \rceil^{\perp} \otimes \mathtt{B}\lceil\!\lceil \star \rceil\!\rceil$$

Init-coherence:

$$\vdash \text{init}; \cdot \Uparrow \mathtt{B}[\![\star]\!] \parr \mathtt{B}\lceil\!\lceil \star \rceil\!\rceil$$

Theorem: ML init-coherence implies OL identity expansion.

Properties: harmony [Dickhoff, Dowek]

Insufficient connectives: no cut-elimination.

$$\frac{\Gamma \vdash \Delta, F, G}{\Gamma \vdash \Delta, F \star G} \star R \qquad\qquad \frac{\Gamma, F, G \vdash \Delta}{\Gamma, F \star G \vdash \Delta} \star L$$

Properties: harmony [Dickhoff, Dowek]

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$$\frac{\Gamma \vdash \Delta, F, G}{\Gamma \vdash \Delta, F \star G}\ \star R \qquad\qquad \frac{\Gamma, F, G \vdash \Delta}{\Gamma, F \star G \vdash \Delta}\ \star L$$

For any A and B:

$$\frac{\dfrac{\dfrac{}{B \vdash A, B}\ I}{B \vdash A \star B}\ \star R \qquad \dfrac{\dfrac{}{A, B \vdash A}\ I}{A \star B \vdash A}\ \star L}{B \vdash A}\ cut$$

Properties: harmony [Dickhoff, Dowek]

Insufficient connectives: no cut-elimination.

$$\frac{\Gamma \vdash \Delta, F, G}{\Gamma \vdash \Delta, F \star G} \star R \qquad \frac{\Gamma, F, G \vdash \Delta}{\Gamma, F \star G \vdash \Delta} \star L$$

Excessive connectives: no identity expansion.

$$\frac{\Gamma \vdash \Delta, F \quad \Gamma \vdash \Delta, G}{\Gamma \vdash \Delta, F \bullet G} \bullet R \quad \frac{\Gamma, F \vdash \Delta}{\Gamma, F \bullet G \vdash \Delta} \bullet L$$

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$$\frac{\dfrac{\dfrac{}{A \vdash A}\ \text{init}}{A \bullet B \vdash A}\ \bullet L \qquad \dfrac{\overset{\frown}{A \vdash B}}{A \bullet B \vdash B}\ \bullet L}{A \bullet B \vdash A \bullet B}\ \bullet R$$

Properties: harmony [Dickhoff, Dowek]

Insufficient connectives: no cut-elimination.

$$\frac{\Gamma \vdash \Delta, F, G}{\Gamma \vdash \Delta, F \star G} \star R \qquad \frac{\Gamma, F, G \vdash \Delta}{\Gamma, F \star G \vdash \Delta} \star L$$

Excessive connectives: no identity expansion.

$$\frac{\Gamma \vdash \Delta, F \quad \Gamma \vdash \Delta, G}{\Gamma \vdash \Delta, F \bullet G} \bullet R \quad \frac{\Gamma, F \vdash \Delta}{\Gamma, F \bullet G \vdash \Delta} \bullet L$$

Harmonious connectives: cut-elimination+identity expansion

Properties: harmony [Dickhoff, Dowek]

Insufficient connectives: no cut-elimination.

$$\star R : \lceil F \star G \rceil^{\perp} \otimes \lceil F \rceil \parr \lceil G \rceil \qquad \star L : \lfloor F \star G \rfloor^{\perp} \otimes \lfloor F \rfloor \parr \lfloor G \rfloor$$

Excessive connectives: no identity expansion.

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Harmonious connectives: cut-elimination+identity expansion

$$\|$$

(cut+init) coherence

Open problems

What about

- formalization of insufficient and excessive connectives;
- application to Quantum computing;
- what happens when cut-elimination does not hold?
- analytic cuts.

Outline

Dualities

Multi-modalities

Sequent systems and modalities

$$4: \ \Box A \to \Box\Box A \qquad\qquad T: \ \Box A \to A$$

$$\frac{F, \Diamond\Gamma}{\Box F, \Diamond\Gamma, \Delta}\ 4 \qquad\qquad \frac{F, \Gamma}{\Diamond F, \Gamma}\ T$$

Sequent systems and modalities

$$4: \; !A \multimap !!A \qquad\qquad T: \; !A \multimap A$$

$$\frac{F, ?\Gamma}{!F, ?\Gamma}\ \text{prom} \qquad\qquad \frac{F, \Gamma}{?F, \Gamma}\ \text{der}$$

Sequent systems and modalities

$$4 : \ !A \multimap !!A \qquad \mathsf{T} : \ !A \multimap A$$

$$\frac{F, ?\Gamma}{!F, ?\Gamma}\ \text{prom} \qquad \frac{F, \Gamma}{?F, \Gamma}\ \text{der}$$

How about

$$\mathsf{K} : \ !(A \multimap B) \multimap (!A \multimap !B) \quad \text{and} \quad \mathsf{D} : \ !A \multimap ?A$$

Sequent systems and modalities

$$\mathsf{K}: \ !(A \multimap B) \multimap (!\, A \multimap !\, B)$$

$$\frac{\mathcal{G} \mathbin{/\!/} \Gamma \mathbin{/\!/} F}{\mathcal{G} \mathbin{/\!/} \Gamma, !\, F}\ ! \qquad \frac{\mathcal{G} \mathbin{/\!/} \Gamma \mathbin{/\!/} F, \Delta}{\mathcal{G} \mathbin{/\!/} \Gamma, ?F \mathbin{/\!/} \Delta}\ ?$$

How about

$$4: \ !\, A \multimap !\, !\, A, \quad \mathsf{T}: \ !\, A \multimap A \quad \text{and} \quad \mathsf{D}: \ !\, A \multimap ?A$$

Sequent systems and modalities

$$\mathsf{K}: \; !(A \multimap B) \multimap (!A \multimap !B)$$

$$\frac{\mathcal{G} /\!/ \Gamma /\!/ F}{\mathcal{G} /\!/ \Gamma, !F} \; ! \qquad \frac{\mathcal{G} /\!/ \Gamma /\!/ F, \Delta}{\mathcal{G} /\!/ \Gamma, ?F /\!/ \Delta} \; ?$$

How about

$$\mathsf{4}: \; !A \multimap !!A, \quad \mathsf{T}: \; !A \multimap A \quad \text{and} \quad \mathsf{D}: \; !A \multimap ?A$$

$$\frac{\mathcal{G} /\!/ \Gamma /\!/ ?F, \Delta}{\mathcal{G} /\!/ \Gamma, ?F /\!/ \Delta} \; ?_4 \qquad \frac{\mathcal{G} /\!/ \Gamma, F}{\mathcal{G} /\!/ \Gamma, ?F} \; ?_t \qquad \frac{\mathcal{G} /\!/ \Gamma /\!/ F}{\mathcal{G} /\!/ \Gamma, ?F} \; ?_d$$

Sequent systems and modalities

$$\mathsf{K}: \; !^i(A \multimap B) \multimap (!^i A \multimap !^i B)$$

$$\frac{\mathcal{G} \mathbin{/\!/} \Gamma \mathbin{/\!/}^i F}{\mathcal{G} \mathbin{/\!/} \Gamma, !^i F} \; !^i \qquad \frac{\mathcal{G} \mathbin{/\!/} \Gamma \mathbin{/\!/}^i F, \Delta}{\mathcal{G} \mathbin{/\!/} \Gamma, ?^j F \mathbin{/\!/}^i \Delta} \; ?^i \; (i \preceq j)$$

How about

$$4: \; !^i A \multimap !^i !^i A, \quad \mathsf{T}: \; !^i A \multimap A \quad \text{and} \quad \mathsf{D}: \; !^i A \multimap ?^i A$$

$$\frac{\mathcal{G} \mathbin{/\!/} \Gamma \mathbin{/\!/}^i ?^j F, \Delta}{\mathcal{G} \mathbin{/\!/} \Gamma, ?^j F \mathbin{/\!/}^i \Delta} \; ?^i{}_4 \; (i \preceq j) \quad \frac{\mathcal{G} \mathbin{/\!/}^k \Gamma, F}{\mathcal{G} \mathbin{/\!/}^k \Gamma, ?^i F} \; ?^i{}_{\mathsf{t}} \quad \frac{\mathcal{G} \mathbin{/\!/} \Gamma \mathbin{/\!/}^i F}{\mathcal{G} \mathbin{/\!/} \Gamma, ?^i F} \; ?^i{}_{\mathsf{d}}$$

Sequent systems and modalities

$$\mathsf{K}: \; !^i(A \multimap B) \multimap (!^i A \multimap !^i B)$$

$$\frac{\mathcal{G} \mathbin{/\!/} \Gamma \mathbin{/\!/}^i F}{\mathcal{G} \mathbin{/\!/} \Gamma, !^i F} \; !^i \qquad \frac{\mathcal{G} \mathbin{/\!/} \Gamma \mathbin{/\!/}^i F, \Delta}{\mathcal{G} \mathbin{/\!/} \Gamma, ?^j F \mathbin{/\!/}^i \Delta} \; ?^i \; (i \preceq j)$$

How about

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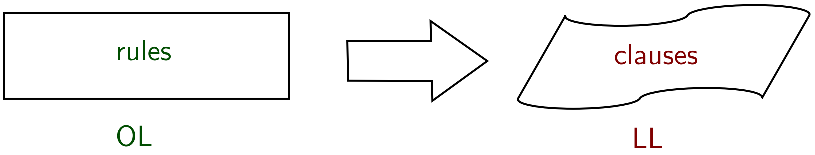
Plus

$$\mathsf{C}: \; !^i B \multimap !^i B \otimes !^i B \quad \text{and} \quad \mathsf{W}: \; !^i B \multimap 1$$

Uniformity [Björn & Carlos & me]

1. LL: $I = \{i\}$ and $\{4, \mathsf{T}, \mathsf{C}, \mathsf{W}\} \in i$.
2. LL with bounded exponentials (LL_b): $I = \{i\}$ and $\{4, \mathsf{T}\} \in i$.
3. Affine LL (LL_a): $I = \{i\}$ and $\{4, \mathsf{T}, \mathsf{W}\} \in i$ and every formula is question-marked.
4. Elementary LL: $I = \{i\}$ and $\{\mathsf{D}, \mathsf{C}, \mathsf{W}\} \in i$ [Guerrini et al].
5. easy to handle LLL;
6. SELL with signature $\langle I, \preccurlyeq, U\rangle$: $\{\mathsf{T}, 4, \mathsf{C}, \mathsf{W}\} \in i$ if $i \in U$ and $\{\mathsf{T}, 4\} \in i$ otherwise.

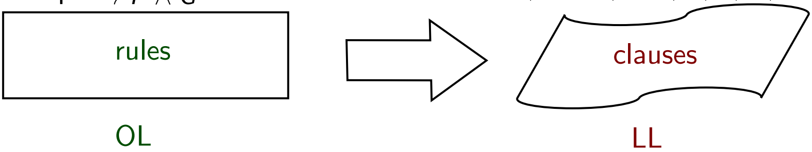
Harmony [Dale & me]



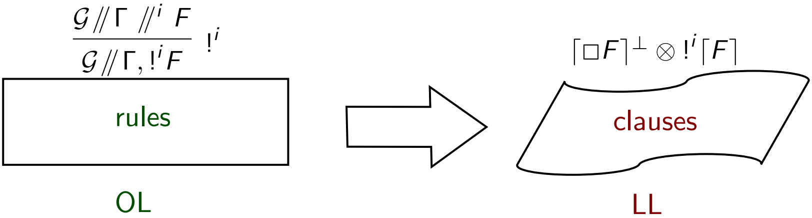
Harmony [Dale & me]

$$\frac{\Gamma \longrightarrow F \quad \Gamma \longrightarrow G}{\Gamma \longrightarrow F \wedge G} \wedge R$$

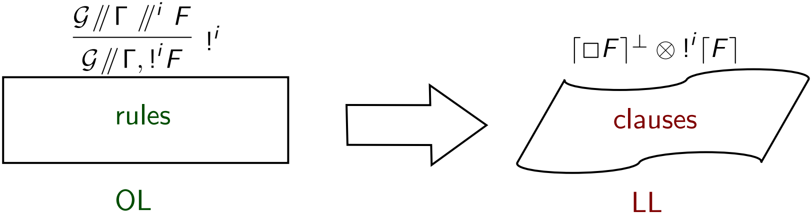
$$\lceil F \wedge G \rceil^{\perp} \otimes \lceil F \rceil \& \lceil G \rceil$$



Harmony [Carlos & Bruno & me]



Harmony [Carlos & Bruno & me]

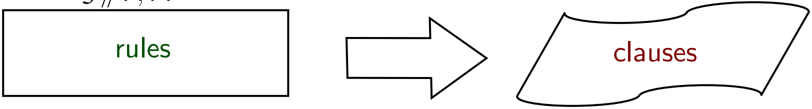


cut-coherence:

$$\vdash \text{cut}; \cdot \Uparrow \mathrm{B}\llbracket \star \rrbracket^{\perp} \parr \mathrm{B}\lceil\!\lceil \star \rceil\!\rceil^{\perp}$$

Harmony [Carlos & Bruno & me]

$$\frac{\mathcal{G} /\!/ \Gamma \ /\!/^i \ F}{\mathcal{G} /\!/ \Gamma, !^i F} \ !^i \qquad\qquad \lceil \Box F \rceil^\perp \otimes !^i \lceil F \rceil$$



OL LL

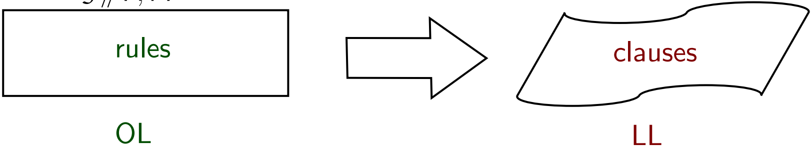
cut-coherence:

$$\vdash \text{cut}; \cdot \Uparrow \mathrm{B}\lfloor \star \rfloor^\perp \parr \mathrm{B}\lceil \star \rceil^\perp$$

Theorem: ML cut-coherence implies OL cut-elimination.

Harmony [Carlos & Bruno & me]

$$\frac{\mathcal{G} \,/\!/\, \Gamma \;\; /\!/^{i} \;\; F}{\mathcal{G} \,/\!/\, \Gamma, !^{i} F} \; !^{i} \qquad\qquad \lceil \Box F \rceil^{\perp} \otimes !^{i} \lceil F \rceil$$

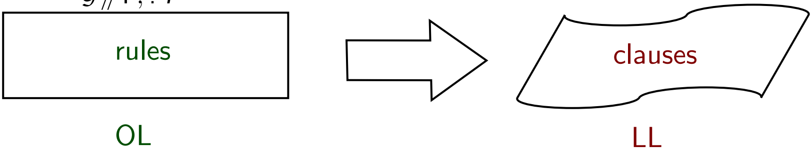


init-coherence:

$$\vdash \text{init}; \cdot \Uparrow \mathrm{B}\llbracket \star \rrbracket \parr \mathrm{B}\llbracket \star \rrbracket$$

Harmony [Carlos & Bruno & me]

$$\frac{\mathcal{G} /\!/ \Gamma \ /\!/^i \ F}{\mathcal{G} /\!/ \Gamma, !^i F} \ !^i \qquad \qquad \lceil \Box F \rceil^\perp \otimes !^i \lceil F \rceil$$



init-coherence:

$$\vdash \text{init}; \cdot \Uparrow \mathrm{B}\lfloor\!\lfloor \star \rfloor\!\rfloor \parr \mathrm{B}\lceil\!\lceil \star \rceil\!\rceil$$

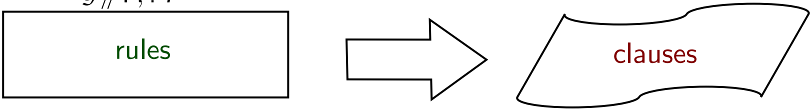
Theorem:

ML init-coherence implies OL identity expansion.

Harmony [Carlos & Bruno & me]

$$\frac{\mathcal{G} \,/\!\!/\, \Gamma \;/\!\!/^{i}\; F}{\mathcal{G} \,/\!\!/\, \Gamma, !^{i} F}\; !^{i}$$

$$\lceil \Box F \rceil^{\perp} \otimes !^{i} \lceil F \rceil$$



OL

LL

init-coherence:

$$\vdash \text{init}; \cdot \Uparrow \mathrm{B}\llbracket \star \rrbracket \parr \mathrm{B}\llbracket \star \rrbracket$$

Theorem:

ML init-coherence implies OL identity expansion.

init + cut = harmony $\leadsto$ mechanized in Coq [Carlos & Bruno]

Thanks!!!

☺