Intuitionistic modal algebras and twist representations

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(joint work with Sergio Celani)

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- In the case of Nelson's constructive logic with strong negation (the starting point of this research), **A** is an involutive commutative integral residuated lattice that satisfies the so-called Nelson identity (i.e. a Nelson residuated lattice) and **H** is a Heyting algebra.

Twist representations and Nelson logics

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- A quasi-Nelson algebra **A** is represented as a twist-algebra over a Heyting algebra **H** enriched with a modal operator (a modal intuitionistic algebra).
- This suggests the possibility of extending the studies of Vakarelov, Sendlewski etc. by relating the theory of quasi-Nelson logic/algebras to the theory of modal intuitionistic logics/algebras.

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- Despite the above works, IMAs have not been studied in depth from a logical and (universal) algebraic point of view.

Modal operators on intuitionistic algebras

Definition 1

Given any algebra having a (bounded) meet semilattice reduct $\langle L; \leq, \wedge, 0 \rangle$, a modal operator on L (also known as a nucleus) is a unary operation $\Box$ satisfying:

(NS1) $x \leq \Box x = \Box\Box x$

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Obs.: both definitions coincide on Heyting algebras and on (bounded) implicative meet semilattices.

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Some examples

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- In particular, the identity map and the double negation map are dense modal operators on every Heyting algebra
(indeed, each dense operator satisfies $x \leq \Box x \leq \neg\neg x$).
- Given a topological space S and a subset $A \subseteq S$, the map $O \mapsto \text{Int}(A \cup O)$ is a modal operator on the Heyting algebra of open sets of S.

IMAs as twist-algebra factors

- Every quasi-Nelson algebra $\mathbf{A} = \langle A; \wedge, \vee, *, \to, \Rightarrow, 0, 1 \rangle$ can be represented as a twist-algebra $\mathbf{A} \leq \mathbf{H}^{\bowtie}$ where $\mathbf{H}$ is a Heyting algebra enriched with a dense modal operator.

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  - $\{\to, \sim\}$-subreducts as twist-algebras over nuclear Hilbert semigroups $\langle S; \odot, \to, \Box, 0, 1 \rangle$

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  - $\{*, \wedge, \to, \sim\}$-subreducts as twist-algebras over $\oplus$-implicative semilattices $\langle S; \wedge, \oplus, \to, \Box, 0, 1 \rangle$.
  - $\{\wedge, \vee, \sim, \neg\}$-subreducts (where $\neg x := x \to 0$) as twist-algebras over modal pseudo-complemented lattices $\langle L; \wedge, \vee, \neg, \Box, 0, 1 \rangle$.

The twist-algebra construction

Let $\mathbf{H} = \langle H; \wedge, \vee, \rightarrow, \Box, 0, 1\rangle$ be (e.g.) a Heyting algebra with a dense modal operator. The full twist-algebra over **H** is the algebra

$$\mathbf{H}^{\bowtie} = \langle H \times \Box H; \wedge, \vee, *, \rightarrow, \Rightarrow, 0, 1\rangle$$

where $\Box H := \{\Box a : a \in H\}$, with operations given by:

$$\begin{aligned}
\langle a_1, a_2\rangle \wedge \langle b_1, b_2\rangle &:= \langle a_1 \wedge b_1, \Box(a_2 \vee b_2)\rangle \\
\langle a_1, a_2\rangle \vee \langle b_1, b_2\rangle &:= \langle a_1 \vee b_1, a_2 \wedge b_2\rangle \\
\langle a_1, a_2\rangle * \langle b_1, b_2\rangle &= \langle a_1 \wedge b_1, (a_1 \rightarrow b_2) \wedge (b_1 \rightarrow a_2)\rangle \\
\langle a_1, a_2\rangle \rightarrow \langle b_1, b_2\rangle &= \langle a_1 \rightarrow b_1, \Box(a_1 \wedge b_2)\rangle \\
\langle a_1, a_2\rangle \Rightarrow \langle b_1, b_2\rangle &:= \langle (a_1 \rightarrow b_1) \wedge (b_2 \rightarrow a_2), \Box(a_1 \wedge b_2)\rangle \\
1 &:= \langle 1, 0\rangle \\
0 &:= \langle 0, 1\rangle.
\end{aligned}$$

A twist-algebra over **H** is any subalgebra $\mathbf{A} \leq \mathbf{H}^{\bowtie}$ satisfying $\pi_1[A] = H$ and $a_1 \wedge a_2 = 0$ for all $\langle a_1, a_2\rangle \in A$.

Twist factors of subreducts

The preceding construction can be adapted to obtain a representation for the above-mentioned subreducts of quasi-Nelson algebras:

- for $\{*, \sim\}$-subreducts, we only need the operations $\wedge$ and $x \rightharpoonup y := x \to \Box y$, the modal operator being defined as $\Box x := 1 \rightharpoonup x$ (weak implicative semilattices)

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- for $\{*, \wedge, \to, \sim\}$-subreducts, we need $\wedge$, $\to$ and the semigroup operation $x \oplus y := \Box(x \vee y)$, defining $\Box x := x \oplus x$ ($\oplus$-implicative semilattices).

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- for $\{\wedge, \vee, \sim, \neg\}$-subreducts, we need $\wedge$, $\vee$, $\neg$ (the pseudo-complement) and $\Box$ (modal pseudo-complemented lattices).

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- The above classes of algebras (weak implicative semilattices, nuclear Hilbert semigroups etc.) are obtained by extending the corresponding algebraic operations from the algebra of fixpoints $\mathbf{A}_{\Box}$ to the whole algebra $\mathbf{A}$.
- Also recall that each class generalizes non-modal intuitionistic algebras, for both the identity and the double negation map are (dense) modal operators.

Two examples: nuclear Hilbert semigroups

A nuclear Hilbert semigroup (NHS) is an algebra $\mathbf{S} = \langle S; \odot, \to, 0, 1 \rangle$ such that:

1. $\langle S; \to, 0, 1 \rangle$ is a bounded Hilbert algebra
2. $\langle S; \odot \rangle$ is a commutative semigroup
3. The operation $\Box x := x \odot x$ is a dense modal operator on $\langle S; \to, 0, 1 \rangle$
4. $x \odot y = x \odot (x \to y)$
5. $\Box x \to (\Box y \to z) = (x \odot y) \to z$
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- Any bounded dense nuclear implicative semilattice $\langle A; \wedge, \to, \Box, 0, 1 \rangle$ gives rise to an NHS upon defining $x \odot y := \Box(x \wedge y)$.

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A weak implicative semilattice (WIS) is an algebra $\mathbf{S} = \langle S; \wedge, \rightharpoonup, 0, 1 \rangle$ such that (upon defining $\Box x := 1 \rightharpoonup x$):

1. $\langle S; \wedge, \Box, 0, 1 \rangle$ is a bounded semilattice with a dense modal operator
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3. $x \rightharpoonup (y \wedge z) = (x \rightharpoonup y) \wedge (x \rightharpoonup z)$
4. $x \wedge \Box y = x \wedge (x \rightharpoonup y)$
5. $x \leq y \rightharpoonup z$ if and only if $x \wedge y \leq \Box z$
6. $x \rightharpoonup y = \Box x \rightharpoonup \Box y$.

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Examples:

- The $\vee$-free reduct of any Visser's basic algebra $\langle A; \wedge, \vee, \rightarrow, 0, 1 \rangle$ is a WIS.

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1. $\langle S; \wedge, \Box, 0, 1 \rangle$ is a bounded semilattice with a dense modal operator
2. $x \rightharpoonup (y \rightharpoonup z) = (x \wedge y) \rightharpoonup z$
3. $x \rightharpoonup (y \wedge z) = (x \rightharpoonup y) \wedge (x \rightharpoonup z)$
4. $x \wedge \Box y = x \wedge (x \rightharpoonup y)$
5. $x \leq y \rightharpoonup z$ if and only if $x \wedge y \leq \Box z$
6. $x \rightharpoonup y = \Box x \rightharpoonup \Box y$.

Examples:

- The $\vee$-free reduct of any Visser's basic algebra $\langle A; \wedge, \vee, \to, 0, 1 \rangle$ is a WIS.
- Any pseudo-complemented semilattice $\langle P; \wedge, \neg, 0, 1 \rangle$ gives rise to a WIS upon defining $x \rightharpoonup y := \neg(x \wedge \neg y)$.

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- Any bounded dense nuclear implicative semilattice $\langle A; \wedge, \rightarrow, \Box, 0, 1 \rangle$ gives rise to a WIS upon defining $x \rightharpoonup y := x \rightarrow \Box y$.

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- We can thus obtain a duality for NHSs by imposing the preceding properties.
- Spatial morphisms are serial relations, corresponding to algebraic homomorphisms that preserve the modal operator and the minimum. (The operation $\odot$ is automatically preserved.)

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- The implication $\rightharpoonup$ can be represented via the specialization order $\leq_{\tau_{\mathcal{K}}}$ (which determines an intuitionistic implication $\rightarrow$), defining $x \rightharpoonup y := x \rightarrow \Box y$ (with further suitable requirements, e.g. $R \subseteq \leq_{\tau_{\mathcal{K}}}$).

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- By imposing suitable restrictions on meet relations, we obtain a spatial category dual to WISs.

Results and plans

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- Dualities for IMAs might give us "two-sorted dualities" (in the sense of Jung & Rivieccio 2021) for subreducts of quasi-Nelson algebras (namely, for the $\{*, \to, \sim\}$-, the $\{*, \wedge, \to, \sim\}$- and the $\{\wedge, \vee, \sim, \neg\}$-subreducts).

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- Dualities for IMAs could themselves be rephrased as two-sorted, where each IMA is represented as a pair $\langle \mathbf{A}, \mathbf{A}_\square \rangle$ connected by back-and-forth maps.

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