Logical approximations of Qualitative Probability

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Qualitative probability

An approach to (foundations of) probability, taking *comparisons* of events as a primitive notion.

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- Arguably more realistic basis for representing uncertainty.

De Finetti axioms (De Finetti, 1931)

Definition (qualitative probability)

$(\mathcal{A}, \trianglelefteq)$ is a *qualitative probability* if $\mathcal{A}$ is a boolean algebra and

1. $\trianglelefteq$ is a total preorder over $\mathcal{A}$;
2. $\bot \triangleleft \top$;
3. if $\alpha \sqsubseteq \beta$ then $\alpha \trianglelefteq \beta$ and
4. if $\alpha \wedge \gamma = \bot$ and $\beta \wedge \gamma = \bot$ then

$$\alpha \trianglelefteq \beta \text{ if and only if } \alpha \vee \gamma \trianglelefteq \beta \vee \gamma.$$

From qualitative to quantitative probability

Definition (Representability)

A qualitative probability $(\mathcal{A}, \trianglelefteq)$ is said to be :

- *almost representable* if there exists a finitely additive probability P such that $\alpha \trianglelefteq \beta$ implies $P(\alpha) \leq P(\beta)$.
- *representable* if there exists a finitely additive probability P such that $\alpha \trianglelefteq \beta$ iff $P(\alpha) \leq P(\beta)$;

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Not all qualitative probabilities are representable (Kraft C., Pratt J., and Seidenberg A., 1959).

Two problems

- Qualitative probabilities are based on classical logic/Boolean algebras, and they inherit its computational untractability.
- To obtain representability one need to add further, less intuitive axioms. In particular (Savage, 1972) requires that a qualitative probability allows for arbitrarily fine-grained comparisons.

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- Qualitative probabilities are based on classical logic/Boolean algebras, and they inherit its computational untractability.
- To obtain representability one need to add further, less intuitive axioms. In particular (Savage, 1972) requires that a qualitative probability allows for arbitrarily fine-grained comparisons. More precisely:

Definition

A qualitative probability $(\mathcal{A}, \trianglelefteq)$ is said to be *fine* if, for any $\alpha \in \mathcal{A}$ such that $\bot \lhd \alpha$, there exists a partition $\beta_1, \ldots, \beta_n$ of $\mathcal{A}$ such that $\beta_i \lhd \alpha$ for each $i = 1, \ldots, n$.

Depth-bounded Boolean logics

We base our construction on the hierarchy $\{\vdash_k\}_{k\in\mathbb{N}}$ of Depth-bounded Boolean logics (D'Agostino, Finger, and Gabbay, 2013).

- k is a parameter capturing the maximum nested use of *hypothetical information* allowed, $\vdash_0$ does not allow for any use of *hypothetical information*.

0-depth: the logic of actual information

$$\frac{\varphi \quad \psi}{\varphi \wedge \psi}\ (\wedge\mathcal{I}) \qquad \frac{\neg\varphi}{\neg(\varphi \wedge \psi)}\ (\neg\wedge\mathcal{I}1) \qquad \frac{\neg\psi}{\neg(\varphi \wedge \psi)}\ (\neg\wedge\mathcal{I}2)$$

$$\frac{\neg\varphi \quad \neg\psi}{\neg(\varphi \vee \psi)}\ (\neg\vee\mathcal{I}) \qquad \frac{\varphi}{\varphi \vee \psi}\ (\vee\mathcal{I}1) \qquad \frac{\psi}{\varphi \vee \psi}\ (\vee\mathcal{I}2)$$

$$\frac{\varphi \quad \neg\varphi}{\bot}\ (\bot\mathcal{I}) \qquad \frac{\varphi}{\neg\neg\varphi}\ (\neg\neg\mathcal{I})$$

$$\frac{\varphi \vee \psi \quad \neg\varphi}{\psi}\ (\vee\mathcal{E}1) \qquad \frac{\varphi \vee \psi \quad \neg\psi}{\varphi}\ (\vee\mathcal{E}2) \qquad \frac{\neg(\varphi \vee \psi)}{\neg\varphi}\ (\neg\vee\mathcal{E}1)$$

$$\frac{\neg(\varphi \vee \psi)}{\neg\psi}\ (\neg\vee\mathcal{E}2) \qquad \frac{\varphi \wedge \psi}{\varphi}\ (\wedge\mathcal{E}1) \qquad \frac{\varphi \wedge \psi}{\psi}\ (\wedge\mathcal{E}2)$$

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$$\frac{\neg\neg\varphi}{\varphi}\ (\neg\neg\mathcal{E}) \qquad \frac{\bot}{\varphi}\ (\bot\mathcal{E})$$

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- $\vdash_0$ is tractable, decidable in polynomial time.

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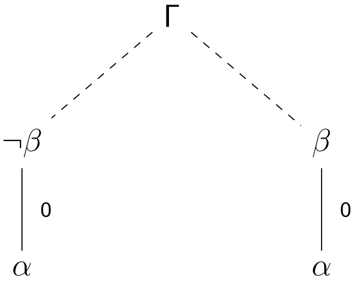
$$p \vdash_0 p \vee \neg p \qquad \neg p \vdash_0 p \vee \neg p \qquad \not\vdash_0 p \vee \neg p$$

- $\vdash_0$ is tractable, decidable in polynomial time.
- Has also a non-deterministic information-based three-valued semantics. ($v(\varphi) = 1$ means: *I am informed that φ is the case*)

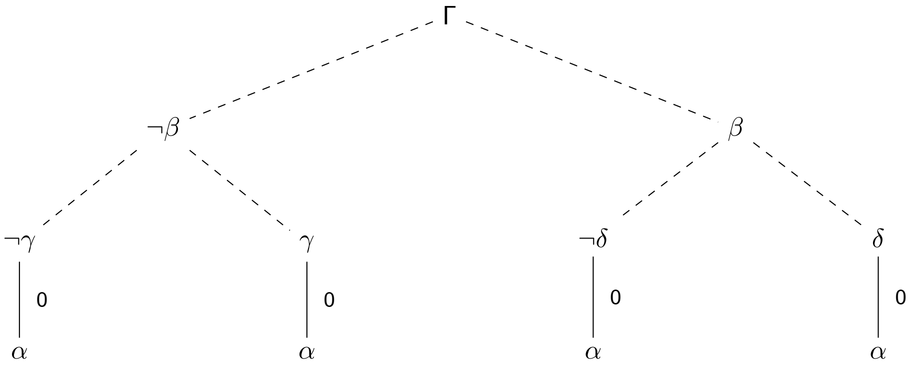
k-depth logics: using hypothetical information

$\Gamma \vdash_1 \alpha$:



k-depth logics: using hypothetical information

$\Gamma \vdash_2 \alpha$:



The hierarchy of DBBLs

- For each k, we have $\vdash_k \subset \vdash_{k+1}$
- The hierarchy approximate classical logic : $\lim_{k\to\infty} \vdash_k \; = \; \vdash$
- Each $\vdash_k$ is feasible, i.e. decidable in PTIME.

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We define sequences of qualitative probabilities

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- Based on the ideas of Depth-Bounded logics. Key distinction between:
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- *Qualitative*, and *bounded* counterparts of Belief functions.

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- *Qualitative*, and *bounded* counterparts of Belief functions.

Two refinements processes at once

- From qualitative probabilities based on depth-bounded logics to qualitative probabilities based on classical logic.
- From qualitative to quantitative: refine qualitative probabilities, so to obtain (almost) representability.

Refining comparisons: from qualitative to quantitative

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- (More refined): R is more likely than obtaining three tails when flipping three times a coin, less likely than obtaining two tails, when flipping it two times ($1/8 \leq P(R) \leq 1/4$)

Requires manipulation of hypothetical information

An example

Let R denote “It will rain tomorrow”, T_n “n-th flip of a coin gives tails”

- Initial information

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- Expand the tree, up to depth 3

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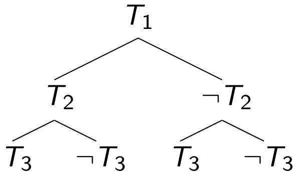

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  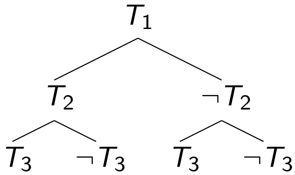

  Define a qualitative probability $\trianglelefteq_3$ on subsets of the leaves. For instance, we might have:

$$\{T_1 \wedge T_2 \wedge T_3\} \trianglelefteq_3 \{R\} \trianglelefteq_3 \{T_1 \wedge T_2 \wedge T_3, T_1 \wedge T_2 \wedge \neg T_3\}$$

Our framework: actual and hypothetical information

- The agent starts with a set $Supp \subseteq Fm$ which represents the formulas upon which explicit information is provided.
- $Supp$ is at each iteration enriched by case-distinctions (RB). Not due to external information, but to hypothetical reasoning performed by the agent.
- We thus build a sequence of forests, representing an exploration of hypothetical information, starting from $Supp$. We denote by $Supp_k$ all the leaves of a forest at depth k.

Definition (Qualitative sequence)

We say that $\mathcal{F} = (\mathcal{F}_k)_{k\in\mathbb{N}}$ is a *depth-bounded qualitative sequence* (qualitative sequence, for short), if:

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- (Stable) For every $k \in \mathbb{N}$, and every $\Delta, \Gamma \subseteq Supp_k$, we have that $\Delta \trianglelefteq_k \Gamma$ implies $(\Delta)_{k'} \trianglelefteq_{k'} (\Gamma)_{k'}$ for every $k' \geq k$.

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- Each $\mathcal{F}_k$ plays the role of a (qualitative,depth-bounded) mass function in Dempster-Shafer theory.
- The support is only over the formulas in $Supp_k$.

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$\varphi \trianglelefteq_k^{pl} \psi$ iff

$$\{\alpha \in Supp_k \mid \alpha \nvdash_0 \neg\varphi\} \ \trianglelefteq_k \ \{\alpha \in Supp_k \mid \alpha \nvdash_0 \neg\psi\}.$$

Properties of $\trianglelefteq_k^b$ and DBBLs

Let $\mathcal{F} = (\mathcal{F}_k)_{k\in\mathbb{N}}$ be a qualitative sequence. The relation $\trianglelefteq_k^b$ satisfies the following:

1. $\trianglelefteq_k^b$ is a total preorder.
2. $\bot \triangleleft_k^b \top$.
3. For any $\varphi, \psi \in Fm$, if $\varphi \vdash_k \psi$ then there is a $n \geq k$ such that $\varphi \trianglelefteq_n^b \psi$.
4. Let $\varphi, \chi, \psi \in Fm$, with $\varphi, \chi \vdash_0 \bot$ and $\psi, \chi \vdash_0 \bot$. There is a $\overline{k}$ such that then $\varphi \trianglelefteq_k^b \psi$ iff $\varphi \vee \chi \trianglelefteq_k^b \psi \vee \chi$, for any $k \geq \overline{k}$.
5. Let $\varphi, \psi, \gamma \in Fm$, with $\varphi \vdash \psi$ and $\psi, \chi \vdash_0 \bot$. Then there is a k such that $\varphi \vee \chi \triangleleft_k^b \psi \vee \chi$

The limit of a sequence

Definition (Limit)

Consider the sequence of structures $(Supp_k, \trianglelefteq_k)_{k \in \mathbb{N}}$.
The *limit* of the sequence is the structure $(Fm, \trianglelefteq)$ defined by:

$$\varphi \trianglelefteq \psi \text{ iff there is a } \overline{k} \text{ such that } \varphi \trianglelefteq_k^b \psi \text{ for every } k \geq \overline{k}$$

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Theorem

Let $\mathcal{A}$ be the Lindenbaum-Tarski algebra over a countable language. If $(\mathcal{A}, \unlhd)$ is an almost representable qualitative probability, then there exists a qualitative sequence $\mathcal{F}$ such that $(\mathcal{A}, \unlhd)$ is the limit of $\mathcal{F}$.

The main results: from qualitative to quantitative

A qualitative sequence $(\mathcal{P}(Supp_k), \trianglelefteq_k)_{k\in\mathbb{N}}$ is:

- *Refinable* if whenever $\alpha \trianglelefteq_k \beta$ for some $\alpha, \beta \in Supp_k$ and $k \in \mathbb{N}$, there is a $k' \geq k$ such that

$$\text{for every } \gamma \in (\beta)_{k'} \text{ we have } \gamma \triangleleft_{k'} (\alpha)_{k'}.$$

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- *Coverable* if whenever $\alpha \triangleleft_k \beta$ for some $\alpha, \beta \in Supp_k$ and $k \in \mathbb{N}$, there is a $k' \geq k$ and $C \subseteq Supp_{k'}$ such that $C \cap \alpha_{k'} = \emptyset$ and

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$$(\alpha)_{k'} \cup C \bowtie_{k'} (\beta)_{k'}$$

Theorem

(Baldi and Hosni, 2021)

- *If $\mathcal{F}$ is refinable, then the corresponding limit structure is almost representable.*
- *If $\mathcal{F}$ is defined over a finite language and is coverable, then the corresponding limit structure is representable.*

Conclusions and Future Work

- Two approximations processes at once:
  - Approximate classical logic by DBBLs
  - Approximate standard probabilities by increasingly more refined qualitative probabilities

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- Two approximations processes at once:
  - Approximate classical logic by DBBLs
  - Approximate standard probabilities by increasingly more refined qualitative probabilities
- Computational complexity: we have results (Baldi and Hosni, 2022) to the effect that a probabilistic satisfiability problem (PSAT) dealing with approximations of (quantitative) probability at a given depth k, is feasible. The same should hold in the qualitative setting.
- Modal logics with qualitative probabilities in the language.
- Apply these ideas in broader decision-theoretic framework.

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