Quantified Relevant logic **RQ** with Constant Domains!? A Perspective from Quantified Modal Logics

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Is the first-order relevant logic **RQ** sound and complete w.r.t. constant domain semantics?

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- No

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- No
- Yes

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- No
- Yes
- Maybe

Outline

- History of the problem
  - **RQ**
  - Fine's incompleteness proof for constant domains
  - Fine's semantics
- Mares-Goldblatt Semantics for **RQ**
- Constant domains *and Tarskian interpretations!?*
- Analogous results in quantified modal classical logic
- What would such a model look like for **RQ**?

History of the problem

The Logic **RQ**

(Id) $\mathcal{A} \to \mathcal{A}$

(B) $(\mathcal{A} \to \mathcal{B}) \to ((\mathcal{C} \to \mathcal{A}) \to (\mathcal{C} \to \mathcal{B}))$

(C) $(\mathcal{A} \to (\mathcal{B} \to \mathcal{C})) \to (\mathcal{B} \to (\mathcal{A} \to \mathcal{C}))$

(W) $(\mathcal{A} \to (\mathcal{A} \to \mathcal{B})) \to (\mathcal{A} \to \mathcal{B})$

($\wedge$E) $\mathcal{A} \wedge \mathcal{B} \to \mathcal{A}, \mathcal{A} \wedge \mathcal{B} \to \mathcal{B}$

($\wedge$I) $(\mathcal{A} \to \mathcal{B}) \wedge (\mathcal{A} \to \mathcal{C}) \to (\mathcal{A} \to \mathcal{B} \wedge \mathcal{C})$

($\vee$I) $\mathcal{A} \to \mathcal{A} \vee \mathcal{B}, \mathcal{B} \to \mathcal{A} \vee \mathcal{B}$

($\vee$E) $(\mathcal{A} \to \mathcal{C}) \wedge (\mathcal{B} \to \mathcal{C}) \to (\mathcal{A} \vee \mathcal{B} \to \mathcal{C})$

(Dist) $\mathcal{A} \wedge (\mathcal{B} \vee \mathcal{C}) \to (\mathcal{A} \wedge \mathcal{B}) \vee \mathcal{C}$

(DNE) $\mathcal{A} \leftrightarrow \neg\neg\mathcal{A}$

(Cont) $(\mathcal{A} \to \neg\mathcal{B}) \to (\mathcal{B} \to \neg\mathcal{A})$

(RCM) $(\mathcal{A} \to \neg\mathcal{A}) \to \neg\mathcal{A}$

(t) $\mathbf{t}$

(rt) $\mathcal{A} \leftrightarrow \mathbf{t} \to \mathcal{A}$

(R$\circ$) $\mathcal{A} \to (\mathcal{B} \to \mathcal{C}) \Longleftrightarrow (\mathcal{A} \circ \mathcal{B}) \to \mathcal{C}$

($\forall$E) $\forall x\mathcal{A} \to \mathcal{A}[\tau/x]$, with τ free for x in $\mathcal{A}$

(EC) $\forall x(\mathcal{A} \vee \mathcal{B}^x) \to \forall x\mathcal{A} \vee \mathcal{B}^x$

(rMP) $\mathcal{A} \to \mathcal{B}, \mathcal{A} \Rrightarrow \mathcal{B}$

(rAdj) $\mathcal{A}, \mathcal{B} \Rrightarrow \mathcal{A} \wedge \mathcal{B}$

(r$\forall$I) $\mathcal{A}^x \to \mathcal{B} \Rrightarrow \mathcal{A}^x \to \forall x\mathcal{B}$

Ternary Relation Semantics for **R**

An **R**-frame is a tuple $F = \langle K, N, R, * \rangle$, where $\varnothing \neq N \subseteq K$, $R \subseteq K^3$, $* : K \longrightarrow K$, and where (with lower-case Greek letters range over K), we have the following:

(DF1) $\alpha \leq \beta =_{df} \exists \gamma \in N(R\gamma\alpha\beta)$

(DF2) $\wp(K)^{\uparrow} =_{df} \{X \subseteq K \mid \forall \alpha, \beta \in K((\alpha \leq \beta \;\&\; \alpha \in X) \Rightarrow \beta \in X)\}$

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(C1) $\langle K, \leq \rangle$ is a pre-ordered set (≤ is reflexive, transitive)

(C2) $N \in \wp(K)^{\uparrow}$

(C3) $R\alpha\beta\gamma \,\&\, \alpha' \leq \alpha \,\&\, \beta' \leq \beta \,\&\, \gamma \leq \gamma' \Rightarrow R\alpha'\beta'\gamma'$ (for short, $R \downarrow\downarrow\uparrow$)

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(C4) $R\alpha\beta\gamma \Rightarrow R\beta\alpha\gamma$

(C5) $\exists x(R\alpha\beta x \;\&\; Rx\gamma\delta) \Rightarrow \exists x(R\alpha\gamma x \;\&\; Rx\beta\delta)$

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(C6) $R\alpha\alpha\alpha$

(C7) $R\alpha\beta\gamma \Rightarrow R\alpha\gamma^*\beta^*$

(C8) $\alpha^{**} = \alpha$

Ternary Relation Semantics - the Basic Framework

Given a frame F, and $X, Y \subseteq W$, define $X \cap Y, X \cup Y$ as usual, and:

$$\begin{aligned} \neg X &= \{\alpha \mid \alpha^* \notin X\} \\ X \to Y &= \{\alpha \mid \forall \beta, \gamma((R\alpha\beta\gamma \,\&\, \beta \in X) \Rightarrow \gamma \in Y)\} \\ X \circ Y &= \{\alpha \mid \exists \beta, \gamma(R\beta\gamma\alpha \,\&\, \beta \in X \,\&\, \gamma \in Y)\} \end{aligned}$$

Fact (Heredity)

If $X, Y \in \wp(K)^{\uparrow}$, then $\neg X, X \cap Y, X \cup Y, X \to Y, X \circ Y \in \wp(K)^{\uparrow}$.

Ternary Relation Semantics - the Basic Framework

Given a frame F, a model M *on* F is given by a function $M : \mathbb{P} \longrightarrow \wp(K)^{\uparrow}$, extended to a homomorphism $||\cdot||^M : \mathcal{L} \longrightarrow \wp(K)^{\uparrow}$ – i.e., the following clauses obtain:

$$||p||^M = M(p) \qquad ||t||^M = N$$

$$||\neg A||^M = \neg(||A||^M) \qquad ||A \wedge B||^M = ||A||^M \cap ||B||^M$$

$$||A \to B||^M = ||A||^M \to ||B||^M \qquad ||A \vee B||^M = ||A||^M \cup ||B||^M$$

$$||A \circ B||^M = ||A||^M \circ ||B||^M$$

A is true in M ($\vDash_M A$) iff $N \subseteq ||A||^M$.
A is true in F ($\vDash_F A$) iff $\vDash_M A$ for all M on F.
A is valid in $\mathbf{F}$ ($\vDash_{\mathbf{F}} A$) iff $\vDash_F A$ for all $F \in \mathbf{F}$.

RQ's Incompleteness

Standard one universal constant domain semantics build on an R-frame: add a constant (universal) domain, set

$$||\forall x_n A||^M f = \bigcap_{f' \sim_{x_n} f} ||A||^M f'$$

$$||\exists x_n A||^M f = \bigcup_{f' \sim_{x_n} f} ||A||^M f'$$

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Fine [14] demonstrated that the standard constant domain semantics, build on the typical ternary relational **R** semantics, validates formulas not provable in **RQ**

A_0 $[(P \to \exists xEx) \wedge \forall x((P \to Fx) \vee (Gx \to Hx))] \to$
$\to \{[\forall x((Ex \wedge Fx) \to Q) \vee \forall x((Ex \to Q) \vee Gx)] \to [\exists xHx \vee (P \to Q)]\}$

Completeness

1988: Fine [13] gives a genius but complicated and not very intuitive semantics for which **RQ**:

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2006: Mares and Goldblatt [23] give a simpler, more natural semantics for **RQ**:

- Employs general frames (a trick often used to obtain completeness results)
- Employs a non-Tarskian/Kripkean approach to the quantifiers. That is:
  - Non-Kripkean: $\bigcap_{f' \sim_{x_n} f} ||A||^M f'$ is not always admissible
  - Non-Tarskian: **not always that** $a, f \models \forall x_n \mathcal{A}$ iff $a, f' \models \mathcal{A}$, for every $f' \sim_{x_n} f$

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Also 1988: Ross Brady's Content Semantics for some quantified relevant logics

Mares-Goldblatt Semantics

Mares Goldblatt Semantics for **RQ**

An MG frame is a tuple $\langle K, N, R, *, Prop, U, PropFun\rangle$, where $\langle K, N, R, *\rangle$ is an **R**-frame complete with defined $\leq$ and $\wp(K)^{\uparrow}$, U is a non-empty set of individuals, $N \in Prop \subseteq \wp(K)^{\uparrow}$, and $PropFun \subseteq \{\varphi \mid \varphi : U^{\omega} \longrightarrow Prop\}$.

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(c1.0) There is a $\varphi_N \in PropFun$ s.t. for any $f \in D^{\omega}$, $\varphi_N f = N$.

(c1.1) Given $\varphi \in PropFun$, there is a $\neg\varphi \in PropFun$ s.t. $(\neg\varphi)f = \neg(\varphi f)$

(c1.2) For $\varphi, \psi \in PropFun$, there is a $\varphi \otimes \psi \in PropFun$ s.t. $(\varphi \otimes \psi)f = \varphi f \otimes \psi f$ for $\otimes \in \{\cap, \cup, \rightarrow, \circ\}$

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(c1.3) For any $\varphi \in PropFun$ and $n \in \omega$, there is a $\forall_n\varphi \in PropFun$ s.t.

$$(\forall_n\varphi)f = \bigsqcap_{f' \sim_{x_n} f} \varphi f' = \bigcup\{X \in Prop \mid X \subseteq \bigcap_{f \sim_{x_n} f} \varphi f'\}$$

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(c1.4) For any $\varphi \in PropFun$ and $n \in \omega$, there is a $\exists_n\varphi \in PropFun$ s.t.

$$(\exists_n\varphi)f = \bigsqcup_{f' \sim_{x_n} f} \varphi f' = \bigcap\{X \in Prop \mid \bigcup_{f' \sim_{x_n} f} \varphi f' \subseteq X\}$$

Mares Goldblatt Semantics for **RQ**

A model on an MG frame F is a multi-type function M that assigns:

- an individual $M(c) \in U$ to each constant symbol c;
- a function $M(P^n) : U^n \longrightarrow Prop$ to each n-ary predicate symbol P^n;
- $M_f(x_n) = fn$; $M_f(c) = M(c)$.

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Extend this to a valuation $|| \cdot ||^M : \mathcal{L} \longrightarrow PropFun$ as follows:

- $||P(\tau_1, \ldots, \tau_n)||_f^M = M(P)(M_f(\tau_1), \ldots, M_f(\tau_n))$ $\quad ||\mathbf{t}||_f^M = \varphi_N f = N$
- $||\neg\mathcal{A}||_f^M = \neg(||\mathcal{A}||_f^M)$
- $||\mathcal{A} \wedge \mathcal{B}||_f^M = ||\mathcal{A}||_f^M \cap ||\mathcal{B}||_f^M$ $\quad ||\mathcal{A} \vee \mathcal{B}||_f^M = ||\mathcal{A}||_f^M \cup ||\mathcal{B}||_f^M$
- $||\mathcal{A} \rightarrow \mathcal{B}||_f^M = ||\mathcal{A}||_f^M \rightarrow ||\mathcal{B}||_f^M$ $\quad ||\mathcal{A} \circ \mathcal{B}||_f^M = ||\mathcal{A}||_f^M \circ ||\mathcal{B}||_f^M$
- $||\forall x_n \mathcal{A}||_f^M = (\forall_n ||\mathcal{A}||^M)f$ $\quad ||\exists x_n \mathcal{A}||_f^M = (\exists_n ||\mathcal{A}||^M)f$

$\mathcal{A}$ is satisfied on M just in case for every $f \in D^\omega$, $N \subseteq ||\mathcal{A}||_f^M$.
$\mathcal{A}$ is satisfied on F just if its satisfied on every M on F.
$\mathcal{A}$ is valid on a set $\mathbf{F}$ of frames just in case it's satisfied on every $F \in \mathbf{F}$.

Constant domains and Tarskian Interpretations!?

Can we Regain the Tarskian Truth Condition?

- For "weaker" relevant logics, this should be no problem
  - As Fine notes, the problem appears to be with (B) and (B').
  - Completeness for the standard constant (universal) domain should work.

But can we obtain completeness for **RQ** using only general frames (and retain the Tarskian truth condition)?

Let's look at a similar case in quantified modal classical logic.

Analogous Results in Quantified Modal Classical Logic

Completeness and Incompleteness of Quantified **S4.2B**

Goldblatt and Mares:

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- Similar results hold for **QS4M** $+$ CQ and related systems for variable domain models.

Regain the Tarskian Truth Condition?

- Note that the problematic axioms, (B) and (B'), also have existential consequents in their corresponding semantic condition(s).

$$Rabx \;\&\; Rxcd \;\Rightarrow\; \exists y(Racy \;\&\; Rybd)$$

- Question: Can we give a general frame, Tarskian constant domain semantics for **RQ**, where the underlying frame is no longer an **R**-frame?

Regain the Tarskian Truth Condition?

Idea:

- Build a canonical model
  - Use only ω-complete theories
  - For a closed $\mathcal{A}$, $||\mathcal{A}|| = \{\alpha \in K : \mathcal{A} \in \alpha\}$
  - $Prop = \{X \subseteq K : X = ||\mathcal{A}||$ for a closed $\mathcal{A}\}$
- Show: $\mathcal{A} \to \mathcal{B} \in \alpha$ iff $R\alpha\beta\gamma$ & $\mathcal{A} \in \beta \;\Rightarrow\; \mathcal{B} \in \gamma$
- Show: Truth Lemma
- Soundness: All theorems of **RQ** are in each regular theory, closure of *Prop* under the right operations $\Rightarrow$ truth sets of theorems always include N.
- Completeness: The usual route with the truth lemma.

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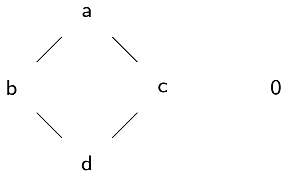
- Show: $\mathcal{A} \rightarrow \mathcal{B} \in \alpha$ iff $R\alpha\beta\gamma$ & $\mathcal{A} \in \beta \ \Rightarrow \ \mathcal{B} \in \gamma$
- Show: Truth Lemma
- Soundness: All theorems of **RQ** are in each regular theory, closure of *Prop* under the right operations $\Rightarrow$ truth sets of theorems always include N.
- Completeness: The usual route with the truth lemma.

So What Would Such a Model Look Like?

Toy Example



1. $K = \{a, b, c, d, 0\}$;
2. $N = \{a, b\}$;
3. $R = \{(d, a, d), (b, y, z), (y, b, z) | x \leq y\}$
   $\cup\{(d, x, y) | x \leq y \vee (x = a \ \& \ y = 0)\}$
   $\cup\{(x, a, a), (x, x, x) | x \in \{a, b, c, d\}\}$;
4. $a^* = d, \ b^* = c, \ c^* = b, \ d^* = a$;
5. *Prop* is the set of upsets on $\{a, b, c, d\}$
6. U is some set; *PropFun* is full
7. $||\cdot||$ assigns every formula such that $[d)$ is in the truth set of **no** formula(-variable assignment pair).

Thanks!

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