

Some properties of residuated lattices using two parameter derivations

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Outline

- 1 Introduction
- 2 Preliminaries
- 3 Two parameter derivations in residuated lattices
- 4 Characterization of Boolean center by using (f, g) - multiplicative derivations

Overview of the derivation

- The notion of derivation, which comes from mathematical analysis, is useful for studying some structural properties of various kinds of algebra,
- Indeed, it has been applied to the theory of algebras with two binary operations; particularly to the theory of commutative rings in 1957 by Posner as follows: For a ring $\mathcal{R} := (R; +, \cdot)$, a map $d : R \rightarrow R$ is called a derivation if it satisfies the condition : For all $x, y \in R$,

$$\begin{aligned}d(x + y) &= d(x) + d(y) \\d(x \cdot y) &= d(x) \cdot y + x \cdot d(y).\end{aligned}$$

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Overview of the derivation

- The notion of derivation was extended to the theory of lattice by Szász in 1975, where the binary ring operations were interpreted as lattice operations and some fundamental properties were established in 2008 by Xin et al.
- Based on Xin, the concept of derivation was extended to f -derivation in lattices by Çeven and Öztürk, the authors characterized distributive and modular lattices by using f -derivation.
- The concept was further explored in the form of (f, g) -derivations in lattices by Asci in 2013, later by Alsatayhi on BL-algebras in 2017.

Main Objective

- The main purpose of this presentation, is to introduce the notion of (f, g) - multiplicative derivations in residuated lattice and investigate some related properties.
- We characterize the Boolean center of L in case where $f = g$.

Preliminaries

The concept of residuated lattices will be used extensively in this presentation. Therefore, we recall its definitions and summarize its main properties.

Definition 1

Ward-Dilworth, (1939)

A bounded commutative residuated lattice is an algebra $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ such that ,

(L_1) $(L, \wedge, \vee, 0, 1)$ is a bounded lattice,

(L_2) $(L, \odot, 1)$ is a commutative monoid,

(L_3) For all $x, y, z \in L$, $x \odot y \leq z$ if and only if $y \leq x \rightarrow z$.

We will denote by the letter L the 7-tuple $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$.

Preliminaries

Example 2

Given a commutative unitary ring L ; the set of ideals of L will be denoted by $Id(L)$, form a complete lattice with set intersection acting as the meet operation and "ideal sum" acting as the join operation. The monoid operation is given by "ideal multiplication", and the element L of $Id(L)$ acts as the identity for this operation. That is, given two ideals I and J in $Id(L)$, we have:

$$X \wedge Y := I \cap J, X \vee Y := I + J = [I \cup J] = \{x + y : x \in I \text{ and } y \in J\}$$

$$I \odot J = \left\{ \sum_{k=1}^n x_k y_k, n \in \mathbb{N}, x_k \in I, y_k \in J \right\} \text{ and the residuals are given by}$$

$$I \rightarrow J := \{a \in L : aJ \subseteq I\}.$$

Therefore, by taking $L := Id(L)$, $0 := \{0\}$ and $1 := L$,

$(Id(L), \wedge, \vee, \odot, \rightarrow, 0, 1)$ is a residuated lattice.

Preliminaries

Definition 3

Hajek, (1996)

A bounded commutative residuated lattice L is called **BL-algebra**, if the following two identities hold in L :

- (1) $x \odot (x \rightarrow y) = x \wedge y$, (divisibility),
- (2) $(x \rightarrow y) \vee (y \rightarrow x) = 1$ (prelinearity).

- Let L be a bounded commutative residuated lattice, for any $x \in L$, we define $x^* := x \rightarrow 0$.
- We adopt the notations:
 $x \oslash y := x^* \rightarrow y$ for every $x, y \in L$, $(x^*)^* = x^{**}$, $x^0 = 1$ and $x^n = x^{n-1} \odot x$ for natural number $n \geq 1$.

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Definition 4

A bounded commutative residuated lattice L is **regular** if for all $x \in L$, $x^{**} = x$.

Preliminaries

Definition 5

Yi Lui et al. (2014)

An **ideal** of a bounded commutative residuated lattice L is a non empty subset I of L satisfying the following conditions: for any $x, y \in L$,

- (1) $x, y \in I$ implies $x \odot y \in I$,
- (2) if $x \leq y$ and $y \in I$, then $x \in I$.

Theorem 6

Yi Lui et al.(2014)

Let L be a bounded commutative residuated lattice. I is an ideal of L if and only if I satisfies the following conditions:

- (1) $0 \in I$,
- (2) for any $x, y \in L$, if $x^* \odot y \in I$ and $x \in I$, then $y \in I$.

Preliminaries

We now, give some properties of the bounded commutative residuated lattices necessary in the proofs of some results.

Proposition 2.1

Ward-Dilworth(1939), Hajek(1996), Raluca(2006)

Let L be a bounded commutative residuated lattice, for any $x, y, z \in L$, we have the following properties:

- (L₄) $1 \rightarrow x = x, \quad x \rightarrow x = 1, \quad 1^* = 0, 0^* = 1, \quad x \rightarrow 1 = 1, 0 \rightarrow x = 1,$*
- (L₅) $x \odot y \leq x, y$, therefore $x \odot y \leq x \wedge y$ and $x \odot 0 = 0,$*
- (L₆) $x \odot y \leq x \rightarrow y, \quad y \leq x \rightarrow y,$*
- (L₇) $x \leq y$ if and only if $x \rightarrow y = 1,$*

Preliminaries

(L₈) $x \rightarrow y = y \rightarrow x = 1$ if and only if $x = y$,

(L₉) $x \odot (x \rightarrow y) \leq x$,

(L₁₀) $x \leq y$ implies $x \odot z \leq y \odot z$,

(L₁₁) $x \odot (y \vee z) = (x \odot y) \vee (x \odot z)$ and $x \odot (y \wedge z) \leq (x \odot y) \wedge (x \odot z)$,

(L₁₂) $x \odot x^* = 0$ and $x \odot y = 0$ if and only if $x \leq y^*$,

(L₁₃) $x \leq x^{**}$, $x^{***} = x^*$,

(L₁₄) $x \leq y$ implies $y^* \leq x^*$,

Preliminaries

Let L be a bounded commutative residuated lattice.

- An element $x \in L$ is called a Boolean element if $x \vee x^* = 1$. We will denote by $B(L) := \{x \in L : x \vee x^* = 1\}$ the set of all Boolean elements of L .
- The set $B(L)$ is the universe of a Boolean subalgebra of a residuated lattice L (called the boolean center of L).

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Preliminaries

Let L be a bounded commutative residuated lattice, for $x \in L$, we consider the following statements:

- (1) $x \in B(L)$,
- (2) $x \odot x = x$ and $x^{**} = x$,
- (3) $x^* \rightarrow x = x$,
- (4) $(x \rightarrow e) \rightarrow x = x$, for any $e \in L$,
- (5) $x^* \wedge x = 0$.

We have the following logical dependency:

- $(1) \Rightarrow (2), (3), (4)$ and $(1) \Rightarrow (5)$; moreover, $(3) \Leftrightarrow (4)$;
- If L is a BL-Algebra, then the statements (1) – (5) are equivalent;
- If $e \in B(L)$, then $e \odot x = e \wedge x$, for all $x \in L$.

Preliminaries

Definition 7

We must note that a homomorphism between two bounded commutative residuated lattices L and L' is mappings $f : L \rightarrow L'$ that satisfied:

- i) $f(x \wedge y) = f(x) \wedge f(y)$ and $f(x \vee y) = f(x) \vee f(y)$,
- ii) $f(x \odot y) = f(x) \odot f(y)$,
- iii) $f(x \rightarrow y) = f(x) \rightarrow f(y)$,
- iv) $f(0) = 0$ and $f(1) = 1$.

Some properties of two parameter derivations in residuated lattices

2. Two parameters derivations in residuated lattices.

- In this section, we extend the notions of a multiplicative derivation by introducing the notion of (f, g) -multiplicative derivation and discuss some related properties. The condition for (f, g) -multiplicative derivation to be monotone are provided.
- Moreover, the set of fixed points is defined, also the conditions for the set of fixed points to be a down closed set and an ideal of L are given.

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- Moreover, the set of fixed points is defined, also the conditions for the set of fixed points to be a down closed set and an ideal of L are given.

Definition of (f, g) -derivation

Definition 8

Let L be a bounded commutative residuated lattice. A map $d : L \rightarrow L$ is called (f, g) -multiplicative derivation on L for the given two homomorphisms $f, g : L \rightarrow L$ if $d(x \odot y) = (d(x) \odot f(y)) \vee (g(x) \odot d(y))$ for all $x, y \in L$.

Lemma 3.1

Let L be a residuated lattice. Let $d : L \rightarrow L$ be a map, and $f, g : L \rightarrow L$ two homomorphisms.

The following assertions are equivalent.

$$\mathbf{A}_1 : \forall x, y \in L, d(x \odot y) = (d(x) \odot f(y)) \vee (g(x) \odot d(y))$$

$$\mathbf{A}_2 : \forall x, y \in L, d(x \odot y) = (d(x) \odot g(y)) \vee (f(x) \odot d(y))$$

Proof.

If d satisfies \mathbf{A}_1 then for all $x, y \in L$ and by using the commutativity of the law \odot , we have $d(x \odot y) = d(y \odot x) = (d(y) \odot f(x)) \vee (g(y) \odot d(x))$; and with the commutativity of \vee , it follows that d satisfies \mathbf{A}_2 . The converse is similar. □

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Remark 3.1

- *It is obvious that (f, f) -derivation coincide with f -multiplicative derivation,*
- *If $f = g = id_M$, then we call it multiplicative derivation.*

Example of (f, g) -derivation

Example 9

Let $L := \{0, a, b, c, 1\}$ such that the order on L is the chain $0 \leq a \leq b \leq c \leq 1$. The operations \odot and \rightarrow on L are defined by the following tables:

\odot	0	a	b	c	1	\rightarrow	0	a	b	c	1
0	0	0	0	0	0	0	1	1	1	1	1
a	0	a	a	a	a	a	0	1	1	1	1
b	0	a	a	a	b	b	0	c	1	1	1
c	0	a	a	c	c	c	0	b	b	1	1
1	0	a	b	c	1	1	0	a	b	c	1

Then $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is a residuated lattice which is not a BL-algebra because $b = b \wedge c \neq c \odot (c \rightarrow b) = c \odot b = a$.

Now, we define the functions f, g and d on L by:

$$\begin{aligned} f(0) &= 0, & f(a) &= a, & f(b) &= a, & f(c) &= 1, & f(1) &= 1, \\ g(0) &= 0, & g(a) &= c, & g(b) &= c, & g(c) &= 1, & g(1) &= 1 \end{aligned} \quad \text{and}$$

$$d(x) = \begin{cases} a & \text{if } x \in \{a, b\} \\ 0 & \text{otherwise} \end{cases}.$$

It is easy to verify that d is an (f, g) -multiplicative derivation on L .

Proposition 3.1

Let d be an (f, g) -multiplicative derivation on L . Then the following conditions hold: for all $x, y \in L$,

- (1) $d(0) = 0$,
- (2) $d(x) \leq f(x)^{**}$ and $d(x) \leq g(x)^{**}$. If $x \leq y$ then $d(x) \leq f(y)^{**}$ and $d(x) \leq g(y)^{**}$, and moreover $x \in B(L)$ implies $d(x) \leq f(x)$ and $d(x) \leq g(x)$,
- (3) $d(x) = d(x) \vee (d(1) \odot f(x)) = d(x) \vee (g(x) \odot d(1))$ and also $f(x) \odot d(1) \odot d(x)^* = g(x) \odot d(1) \odot d(x)^* = 0$,
- (4) $d(x^*) \leq d(x)^*$,
- (5) $d(x) = 1$ implies $f(x)^{**} = 1 = g(x)^{**}$ for all $x \in L$,
- (6) $d(x^*) \leq f(x^*)$ and $d(x^*) \leq g(x^*)$.

Proof.

(1)

$$\begin{aligned}
 d(0) &= d(0 \odot 0) = (d(0) \odot f(0)) \vee (g(0) \odot d(0)) \\
 &= (d(0) \odot 0) \vee (0 \odot d(0)) \\
 &= 0.
 \end{aligned}$$

(2) Let's prove that: $d(x) \leq f(x)^{**}$ and $d(x) \leq g(x)^{**}$.

$x \odot x^* = 0$, so $d(0) = d(x \odot x^*) = (d(x) \odot f(x)^*) \vee (g(x) \odot d(x^*))$,
 therefore $d(x) \odot f(x)^* = 0$, hence $d(x) \leq f(x)^{**}$ by Proposition 2.1.

By the similar way, we get

$0 = d(x^* \odot x) = (d(x^*) \odot f(x)) \vee (g(x)^* \odot d(x))$, then
 $g(x)^* \odot d(x) = 0$, hence $d(x) \leq g(x)^{**}$.

Always by the similar way, if $x \leq y$, then $x \odot y^* = 0$, and we get
 $d(x) \leq f(y)^{**}$ and $d(x) \leq g(y)^{**}$. If moreover, we have $x \in B(L)$,
 then by using $x^{**} = x$, we conclude that $d(x) \leq f(x)$ and
 $d(x) \leq g(x)$.

Proof.

(3) Let's prove that: $d(x) = d(x) \vee (d(1) \odot f(x))$.

We have,

$$\begin{aligned} d(x) &= d(x \odot 1) \\ &= (d(x) \odot f(1)) \vee (g(x) \odot d(1)) \\ &= d(x) \vee (g(x) \odot d(1)) \end{aligned}$$

so, $g(x) \odot d(1) \leq d(x)$, hence $g(x) \odot d(1) \odot d(x)^* = 0$ by Proposition 2.1. We also get $d(x) = d(x) \vee (d(1) \odot f(x))$, thus $f(x) \odot d(1) \leq d(x)$ by Theorem 2.1, hence $f(x) \odot d(1) \odot d(x)^* = 0$.

(4) Let's prove that: $d(x^*) \leq d(x)^*$.

We have $d(x) \leq f(x)^{**}$ by part (2), so $d(x^*) \leq f(x^*)^{**}$ that is $d(x^*) \leq f(x)^{***} = f(x)^*$.

Furthermore,

$0 = d(0) = d(x \odot x^*) = (d(x) \odot f(x)^*) \vee (g(x) \odot d(x^*))$; thus, $d(x) \odot f(x)^* = f(x)^* \odot d(x)$ that is $f(x)^* \leq d(x)^*$, hence

Proof.

(5) $d(x) = 1$ implies $f(x)^{**} = 1 = g(x)^{**}$ for all $x \in L$.

By using part (2) and the fact that 1 is the greatest element of L .

(6) $d(x^*) \leq f(x^*)$ and $d(x^*) \leq g(x^*)$.

Since $x^{***} = x^*$ and $d(x) \leq f(x)^{**}$, $d(x) \leq g(x)^{**}$, we have $d(x^*) \leq f(x^*)^{**} = f(x^*)$ and $d(x^*) \leq g(x^*)^{**} = g(x^*)$, hence $d(x^*) \leq f(x^*)$ and $d(x^*) \leq g(x^*)$.



Proposition 3.2

Let d be an (f, g) -multiplicative derivation on L such that $d(1) = 1$. Then the following conditions hold:

- (1) $f(x) \leq d(x)$ and $g(x) \leq d(x)$ for all $x \in L$,
- (2) For all $x \in B(L)$, we have $d(x) = f(x) = g(x)$.

Proposition 3.3

Let d be an (f, g) -multiplicative derivation on L , then we have:

- (1) $d(x \odot y) \leq d(x) \vee d(y)$,
- (2) $\text{Ker}_d = \{x \in L : d(x) = 0\}$ is closed under \odot .

Characterization of Boolean center using (f, g) -multiplicative derivations

3. Characterization of Boolean center using (f, g) - multiplicative derivations

Characterization of Boolean center using f -multiplicative derivations

From now on, we will use the following notations for the sets of (f, g) -fixed points, f -fixed points and g -fixed points:

$$\text{Fix}_d^{f,g}(L) := \{x \in L : d(x) = f(x) = g(x)\};$$

$$\text{Fix}_d^{f,-}(L) := \{x \in L : d(x) = f(x)\}; \quad \text{Fix}_d^{-,g}(L) := \{x \in L : d(x) = g(x)\}$$

where d is an (f, g) -derivation on L .

Characterization of Boolean center using f -multiplicative derivations

Example 10

In Example 9 $\text{Fix}_d^{f,-}(L) = \{0, a, b\}$ and $\text{Fix}_d^{-,g}(L) = \{0\}$. Furthermore, by using part (2) of Proposition 3.2, we have:

$$B(L) \subseteq \text{Fix}_d^{f,g}(L) = \text{Fix}_d^{f,-}(L) \cap \text{Fix}_d^{-,g}(L).$$

Theorem 11

Let d be an (f, g) -multiplicative derivation on the Boolean center $B(L)$. If d is a homomorphism, then $\text{Fix}_d^{f,-}(L)$ is an ideal of L .

Definition of (f, g) -contractive and isotone (f, g) -multiplicative derivations

Definition 12

Let L be a residuated lattice and d be a (f, g) -derivation on L .

- 1) d is a **f -contractive** (f, g) -derivation if $d(x) \leq f(x)$ for all $x \in L$ and **g -contractive** (f, g) -derivation if $d(x) \leq g(x)$ for all $x \in L$. d is called **(f, g) -contractive** (f, g) -multiplicative derivation if d is both f -contractive and g -contractive.
- 2) d is an **ideal** (f, g) -multiplicative derivation if d is both monotone and (f, g) -contractive.

In particular, if d is both monotone and f -contractive, we call d an **f -ideal** (f, g) -multiplicative derivation.

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In what follows, we assume that $f = g$ and obtain the following results.

Proposition 4.1

Let $a \in L$, we define a map $d : L \rightarrow L$ by $d_{(a,f)}(x) = a \odot f(x)$ for all $x \in L$, where f is a homomorphism. Then $d_{(a,f)}$ is an f -multiplicative derivation on L .

It is easy to see that $d_{(a,f)}$ is monotone and f -contractive on L .

Definition 13

An ideal f -multiplicative derivation d is said to be good if $d(1) \in B(L)$.

Proposition 4.2

Let $d : L \rightarrow L$ be an ideal f -multiplicative derivation on L such that $d(1) \in B(L)$, then $d(x) = d(1) \odot f(x)$.

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The next proposition shows that good ideal f -multiplicative derivations on L are one-to-one correspond with a boolean center of L .

Proposition 4.3

An f -multiplicative derivation d on L is a good ideal f -multiplicative derivation if and only if there exists a unique $a \in B(L)$ such that $d = d_{(a,f)}$.

Let $\mathcal{GID}_f(L)$ denotes the set of all good ideal f -multiplicative derivation on L . By the preceding proposition, $\mathcal{GID}_f(L) := \{d_{(x,f)} : x \in B(L)\}$. In addition, we define $\preceq, \otimes, \rightarrow$ on $\mathcal{GID}_f(L)$ by: $d_{(x,f)} \preceq d_{(x',f)}$, if $x \leq x'$; $d_{(x,f)} \otimes d_{(x',f)} = d_{(x \odot x',f)}$, $d_{(x,f)} \rightarrow d_{(x',f)} = d_{(x \rightarrow x',f)}$. Then it is straightforward to see that $(\mathcal{GID}_f(L); \preceq, \otimes, \rightarrow, d_{(\perp,f)}, d_{(\top,f)})$ is a Boolean center, that is naturally isomorphic to the Boolean algebra $B(L)$.

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Let $\mathcal{GID}_f(L)$ denotes the set of all good ideal f -multiplicative derivation on L . By the preceding proposition, $\mathcal{GID}_f(L) := \{d_{(x,f)} : x \in B(L)\}$.

In addition, we define $\preceq, \otimes, \rightarrow$ on $\mathcal{GID}_f(L)$ by: $d_{(x,f)} \preceq d_{(x',f)}$, if $x \leq x'$; $d_{(x,f)} \otimes d_{(x',f)} = d_{(x \odot x',f)}$, $d_{(x,f)} \rightarrow d_{(x',f)} = d_{(x \rightarrow x',f)}$. Then it is straightforward to see that $(\mathcal{GID}_f(L); \preceq, \otimes, \rightarrow, d_{(\perp,f)}, d_{(\top,f)})$ is a Boolean center, that is naturally isomorphic to the Boolean algebra $B(L)$.

The next proposition shows that good ideal f -multiplicative derivations on L are one-to-one correspond with a boolean center of L .

Proposition 4.3

An f -multiplicative derivation d on L is a good ideal f -multiplicative derivation if and only if there exists a unique $a \in B(L)$ such that $d = d_{(a,f)}$.

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THANK YOU