Relevant Reasoners in a Classical World

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Omniscience and Epistemic clutter

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($\Box$M)	$\dfrac{\varphi \to \psi}{\Box\varphi \to \Box\psi}$	($\Box$CON1)	$\Box(\varphi \wedge \neg\varphi) \to \Box\psi$
($\Box$N)	$\dfrac{\varphi}{\Box\varphi}$	($\Box$CON2)	$\Box\varphi \wedge \Box\neg\varphi \to \Box\psi$
($\Box$C)	$\Box\varphi \wedge \Box\psi \to \Box(\varphi \wedge \psi)$		
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- Relevant epistemic logic: avoids epistemic clutter by restricting the underlying propositional logic, so that e.g. $\varphi \wedge \neg\varphi \to \psi$ is invalid.
- However, invalid principles of relevant logic are fine when seen as being about *truth* in a *world* (vs *information* received by an *agent*)

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- For each relevant logic L:

$$\frac{\vdash_{\mathsf{L}} \varphi_1 \wedge \ldots \wedge \varphi_n \to \psi}{\vdash_{\mathsf{CL}} \Box\varphi_1 \wedge \ldots \wedge \Box\varphi_n \to \Box\psi} \text{ (RR)}$$

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  (**Slogan:** Formulas in the scope of $\Box$ behave relevantly in CL)
- Epistemic clutter is avoided
- Logical omniscience is almost completely avoided:

$$(\Box\mathrm{C})\ \vdash_{\mathsf{CL}} \Box\varphi \wedge \Box\psi \to \Box(\varphi \wedge \psi) \qquad \frac{\vdash_{\mathsf{L}} \varphi \to \psi}{\vdash_{\mathsf{CL}} \Box\varphi \to \Box\psi}$$

Outline

1. From L to CL
2. Relevant modal logics
3. Classical epistemic logics for relevant reasoners
4. The axiomatization result
5. Conclusion
6. Bonus

1. From L to CL

The basic relevant system

Drawing from (Fuh90), we define the axiom system BM.C:

(a1)	$\varphi \to \varphi$	(a7)	$\psi \to (\varphi \vee \psi)$
(a2)	$\neg(\varphi \wedge \psi) \to (\neg\varphi \vee \neg\psi)$	(a8)	$((\varphi \to \psi) \wedge (\varphi \to \chi)) \to (\varphi \to (\psi \wedge \chi))$
(a3)	$(\neg\varphi \wedge \neg\psi) \to \neg(\varphi \vee \psi)$	(a9)	$((\varphi \to \chi) \wedge (\psi \to \chi)) \to ((\varphi \vee \psi) \to \chi)$
(a4)	$(\varphi \wedge \psi) \to \varphi$	(a10)	$(\varphi \wedge (\psi \vee \chi)) \to ((\varphi \wedge \psi) \vee (\varphi \wedge \chi)$
(a5)	$(\varphi \wedge \psi) \to \psi$	($\Box$C)	$\Box(\varphi \wedge \psi) \to (\Box\varphi \wedge \Box\psi)$
(a6)	$\varphi \to (\varphi \vee \psi)$	($\Box_L$C)	$\Box_L(\varphi \wedge \psi) \to (\Box_L\varphi \wedge \Box_L\psi)$

(MP) $\dfrac{\varphi \qquad \varphi \to \psi}{\psi}$ (Adj) $\dfrac{\varphi \qquad \psi}{\varphi \wedge \psi}$ (Aff) $\dfrac{\varphi' \to \varphi \qquad \psi \to \psi'}{(\varphi \to \psi) \to (\varphi' \to \psi')}$

(Con) $\dfrac{\varphi \to \psi}{\neg\psi \to \neg\varphi}$ ($\Box$-M) $\dfrac{\varphi \to \psi}{\Box\varphi \to \Box\psi}$ ($\Box_L$-M) $\dfrac{\varphi \to \psi}{\Box_L\varphi \to \Box_L\psi}$

Propositional extensions

Let L be BM.C + a subset of the following axioms and rules.

Modal		**Propositional**	
(□M)	$\dfrac{\varphi \to \psi}{\Box\varphi \to \Box\psi}$	(DN)	$\varphi \leftrightarrow \neg\neg\varphi$
(□N)	$\dfrac{\varphi}{\Box\varphi}$	(Cp)	$(\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$
(□C)	$\Box\varphi \wedge \Box\psi \to \Box(\varphi \wedge \psi)$	(WB)	$((\varphi \to \psi) \wedge (\psi \to \chi)) \to (\varphi \to \chi)$
(□K)	$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$	(X)	$\varphi \vee \neg\varphi$
! (□T)	$\Box\varphi \to \varphi$	(Rd)	$(\varphi \to \neg\varphi) \to \neg\varphi$
(□D)	$\Box\neg\varphi \to \neg\Box\varphi$	(B)	$(\varphi \to \psi) \to ((\chi \to \varphi) \to (\chi \to \psi))$
(□4)	$\Box\varphi \to \Box\Box\varphi$	(CB)	$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$
(□5)	$\neg\Box\varphi \to \Box\neg\Box\varphi$	(W)	$(\varphi \to (\varphi \to \psi)) \to (\varphi \to \psi)$
		(C)	$(\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))$
		(M)	$\varphi \to (\varphi \to \varphi)$
		(ER)	$\dfrac{\varphi}{(\varphi \to \psi) \to \psi}$

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$\vdash_{\mathsf{L}} \varphi$ *iff* $\vdash_{\mathsf{CL}} \Box_L \varphi$.

CL

The relevant reasoning meta-rule is “admissible”.

$$\frac{\vdash_{\mathsf{L}} \varphi_1 \wedge \ldots \wedge \varphi_n \to \psi}{\vdash_{\mathsf{CL}} \Box\varphi_1 \wedge \ldots \wedge \Box\varphi_n \to \Box\psi}\text{(RR)}$$

$\vdash_{\mathsf{L}} \varphi_1 \wedge \ldots \wedge \varphi_n \to \psi$	
$\vdash_{\mathsf{L}} \Box\varphi_1 \wedge \ldots \wedge \Box\varphi_n \to \Box\psi$	$\Box$M and $\Box$C in L
$\vdash_{\mathsf{CL}} \Box_L \left(\Box\varphi_1 \wedge \ldots \wedge \Box\varphi_n \to \Box\psi\right)$	Lemma 1
$\vdash_{\mathsf{CL}} \Box\varphi_1 \wedge \ldots \wedge \Box\varphi_n \to \Box\psi$	(BR)

2. Relevant modal logics

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Definition (Model)

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- $Q, Q_L \in S(\downarrow\uparrow)$
- $V : At \to S(\uparrow)$

Definition (L-Model)

$\boldsymbol{M} = (M, L)$ such that $L \in S(\uparrow)$ and

$$\forall s \exists x (x \in L \ \& \ Rxss) \tag{1}$$

$$s \in L \ \& \ Rstu \implies t \leq u \tag{2}$$

Relevant modal logics

Valuation extended to the full language as follows:

$$
\begin{aligned}
&\dots \\
\neg[\![\varphi]\!]_M &= \{s \mid s^* \notin [\![\varphi]\!]_M\} \\
[\![\varphi]\!]_M \to [\![\varphi]\!]_M &= \{s \mid \forall t, u : Rstu \ \ \& \ \ t \in [\![\varphi]\!]_M \implies u \in [\![\psi]\!]_M\} \\
\Box[\![\varphi]\!]_M &= \{s \mid \forall t : Qst \implies t \in [\![\varphi]\!]_M\} \\
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For all φ, $[\![\varphi]\!]_{\boldsymbol{M}} \in S(\uparrow)$.

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$M \models \varphi$ iff $L \subseteq [\![\varphi]\!]_M$ for all M.

Lemma 2 (Heredity)

For all φ, $[\![\varphi]\!]_M \in S(\uparrow)$.

Lemma 3 (Verification)

For all $\varphi, \psi \models \varphi \to \psi$ *iff* $[\![\varphi]\!]_M \subseteq [\![\psi]\!]_M$.

3. Classical epistemic logics for relevant reasoners

W-model

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- A set of possible worlds in a bounded model $\boldsymbol{M}$ is $W \subseteq S$ such that $w \in W$ iff

$$w^* = w \tag{3}$$

$$Rwww \tag{4}$$

$$Rwst \implies (s = 0 \text{ or } w \leq t) \tag{5}$$

$$Rwst \implies (t = 1 \text{ or } s \leq w^*) \tag{6}$$

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- $Q_L(W)$ simulates the behaviour of L in L-models, i.e.

$$\forall s \exists u (u \in Q_L(W) \;\&\; Russ)$$
$$u \in Q_L(W) \;\&\; Rust \implies s \leq t$$

Putting things together

Definition

A W-model is $\boldsymbol{M} = (M, W)$ where M is a bounded model and W is a set of possible worlds in F such that

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Lemma 4

For all φ, ψ*:* $\models \Box_L(\varphi \rightarrow \psi)$ *iff* $[\![\varphi]\!]_{\boldsymbol{M}} \subseteq [\![\psi]\!]_{\boldsymbol{M}}$

Facts about W-models

Proposition 1

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If w *is a possible world in a bounded model* $\boldsymbol{M}$*, then:*

- $\boldsymbol{M}, w \models \neg\varphi$ *iff* $\boldsymbol{M}, w \not\models \varphi$*;*
- $\boldsymbol{M}, w \models \varphi \to \psi$ *iff* $\boldsymbol{M}, w \not\models \varphi$ *or* $\boldsymbol{M}, w \models \psi$*.*

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$\varphi \in$ **CL** iff $\varphi \in$ **CPC** for φ propositional

Invalidities in W-frames

(□E)	$\dfrac{\varphi \leftrightarrow \psi}{\Box\varphi \leftrightarrow \Box\psi}$	X
(□M)	$\dfrac{\varphi \rightarrow \psi}{\Box\varphi \rightarrow \Box\psi}$	X
(□N)	$\dfrac{\varphi}{\Box\varphi}$	X
(□C)	$\Box\varphi \wedge \Box\psi \rightarrow \Box(\varphi \wedge \psi)$	✓
(□K)	$\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$	X
(□W)	$\Box\varphi \rightarrow \Box(\psi \rightarrow \psi)$	X
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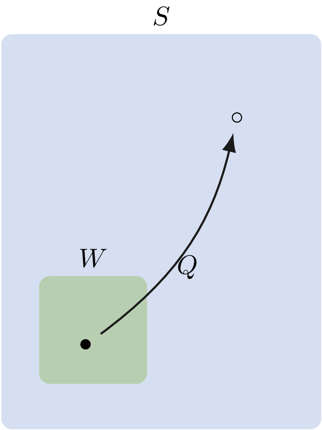

Similar idea in (Lev84)

4. The axiomatization result

The fundamental bridge lemma

Recall the axiomatisation of CL:

1. Axiomatisation of CPC;
2. for each axiom φ of L, an axiom $\Box_L \varphi$;
3. for each rule $\dfrac{\varphi_1 \dots \varphi_n}{\psi}$ of L, a rule $\dfrac{\Box_L \varphi_1 \dots \Box_L \varphi_n}{\Box_L \psi}$;
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Proof. One direction by easy induction on the length of proofs, the other by a model transformation (see the paper (SV22)), and soundness (see next slide). □

Soundness and completeness

Theorem 1

*For all φ and L: $\vdash_{\mathsf{CL}} \varphi$ iff $\varphi \in$ **CL**.*

Proof. Soundness by induction using W-frames conditions and $W \subseteq [\![\Box_L(\varphi \to \psi)]\!]_M$ iff $[\![\varphi]\!]_M \subseteq [\![\psi]\!]_M$. Completeness by canonical model construction.

M^{CL} is defined as follows:

- S^{CL} is the set of all prime L-theories ordered by set inclusion
- W^{CL} is the set of all non-empty proper prime CL-theories
- $R^{\mathsf{CL}}stu$ iff $\varphi \to \psi \in s$ and $\varphi \in t$ imply $\psi \in u$
- $s^{*^{\mathsf{CL}}} = \{\varphi \mid \neg\varphi \notin s\}$
- $Q^{\mathsf{CL}}_{(L)}st$ iff $\Box_{(L)}\varphi \in s$ implies $\varphi \in t$
- $V^{\mathsf{CL}}(p) = \{s \mid p \in s\}$

The canonical model

Lemma 5

For all L, M^{CL} is a model for ***CL****.*

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Proof. ... The fact that $Q_L^{\mathsf{CL}}(W^{\mathsf{CL}})$ "behaves like L" uses Lemma 1 ($\vdash_{\mathsf{L}} \varphi \iff \vdash_{\mathsf{CL}} \Box_L \varphi$).
Note: Canonicity of frame conditions for logics stronger than L is standard, e.g. as in (RPMB82). $\square$

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Lemma 6

$\varphi \in s$ *iff* $(\boldsymbol{M}^{\mathsf{CL}}, s) \models \varphi$.

Proof. Standard. □

5. Conclusion

Conclusion and future work

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- We provide a framework for moderately idealised agents that avoids logical omniscience and epistemic clutter
- We obtain a general completeness theorem

Conclusion and future work

- Relevant and classical modal logic can “live together”
- We provide a framework for moderately idealised agents that avoids logical omniscience and epistemic clutter
- We obtain a general completeness theorem
- Some topics of ongoing or future work:
  - Neighbourhood semantics (we’re on it)
  - First-order versions studied by Nick Ferenz
  - Distributed and common knowledge (...build on relevant PDL (TB22))
  - Epistemic dynamics (...build on relevant public update logic (ST21))
  - Algebraic formulation

6. Bonus

Bonus: Neighborhood version

- $\vdash_{\mathsf{CL}} \Box\varphi \wedge \Box\psi \to \Box(\varphi \wedge \psi)$ and $\dfrac{\vdash_{\mathsf{L}} \varphi \to \psi}{\vdash_{\mathsf{CL}} \Box\varphi \to \Box\psi}$ since (□C) and (□M) are in L

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- $\vdash_{CL} \Box\varphi \wedge \Box\psi \rightarrow \Box(\varphi \wedge \psi)$ and $\dfrac{\vdash_{L} \varphi \rightarrow \psi}{\vdash_{CL} \Box\varphi \rightarrow \Box\psi}$ since ($\Box$C) and ($\Box$M) are in L
- Use neighborhood semantics to model a non-normal modality $\Box$! We define BM.E:

(a1)	$\varphi \rightarrow \varphi$	(a7)	$\psi \rightarrow (\varphi \vee \psi)$
(a2)	$\neg(\varphi \wedge \psi) \rightarrow (\neg\varphi \vee \neg\psi)$	(a8)	$((\varphi \rightarrow \psi) \wedge (\varphi \rightarrow \chi)) \rightarrow (\varphi \rightarrow (\psi \wedge \chi))$
(a3)	$(\neg\varphi \wedge \neg\psi) \rightarrow \neg(\varphi \vee \psi)$	(a9)	$((\varphi \rightarrow \chi) \wedge (\psi \rightarrow \chi)) \rightarrow ((\varphi \vee \psi) \rightarrow \chi)$
(a4)	$(\varphi \wedge \psi) \rightarrow \varphi$	(a10)	$(\varphi \wedge (\psi \vee \chi)) \rightarrow ((\varphi \wedge \psi) \vee (\varphi \wedge \chi))$
(a5)	$(\varphi \wedge \psi) \rightarrow \psi$	($\Box_L$C)	$\Box_L(\varphi \wedge \psi) \rightarrow (\Box_L\varphi \wedge \Box_L\psi)$
(a6)	$\varphi \rightarrow (\varphi \vee \psi)$		

(MP) $\dfrac{\varphi \qquad \varphi \rightarrow \psi}{\psi}$ (Adj) $\dfrac{\varphi \qquad \psi}{\varphi \wedge \psi}$ (Aff) $\dfrac{\varphi' \rightarrow \varphi \qquad \psi \rightarrow \psi'}{(\varphi \rightarrow \psi) \rightarrow (\varphi' \rightarrow \psi')}$

(Con) $\dfrac{\varphi \rightarrow \psi}{\neg\psi \rightarrow \neg\varphi}$ ($\Box$-E) $\dfrac{\varphi \leftrightarrow \psi}{\Box\varphi \leftrightarrow \Box\psi}$ ($\Box_L$-M) $\dfrac{\varphi \rightarrow \psi}{\Box_L\varphi \rightarrow \Box_L\psi}$

Axiomatisation of CN

- Take an extension of BM.E with a subset of modal and propositional axioms/rules as above;
- For any such N:
  1. Axiomatisation of CPC;
  2. for each axiom φ of N, an axiom $\Box_L \varphi$;
  3. for each rule $\frac{\varphi_1 \ldots \varphi_n}{\psi}$ of N, a rule $\frac{\Box_L \varphi_1 \ldots \Box_L \varphi_n}{\Box_L \psi}$;
  4. $\frac{\Box_L (\varphi \to \psi)}{\varphi \to \psi}$ (BR).

Note: We still have $\frac{\vdash_{\mathsf{L}} \varphi \leftrightarrow \psi}{\vdash_{\mathsf{CL}} \Box\varphi \leftrightarrow \Box\psi}$ since $\frac{\varphi \leftrightarrow \psi}{\Box\varphi \leftrightarrow \Box\psi}$ is a rule of L

Bonus: Neighborhood version

Definition (W-model)

$M = (S, W, \textit{Prop}, L, \leq, R, *, N, Q_L, V)$ where:

- $(S, \leq), W, R, *, Q_L$ as in relational W-models

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Note: General frames not only useful for a uniform completeness proof, but essential for the canonicity of the conditions on 0,1.

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- $N : S \rightarrow P(P(S))$ neighborhood function
- $V : At \rightarrow \textit{Prop}$
- $\Box[\![\varphi]\!]_M = \{s \mid Ns[\![\varphi]\!]_M\}$

Theorem 2

*For all φ and N: $\vdash_{\mathsf{CN}} \varphi$ iff $\varphi \in$ **CN**.*

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