Strong standard completeness for $S5$-modal Łukasiewicz logics

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Outline

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Introduction

Petr Hájek. *Metamathematics of fuzzy logic*. Vol. 4. Trends in Logic—Studia Logica Library. Kluwer Academic Publishers, Dordrecht

- (Strong completeness) Lukasiewicz logic ($\mathcal{L}$) $\iff$ MV-algebras
- Defines an S5 modal extension of $\mathcal{L}$ (S$5(\mathcal{L})$).

(Finitary completeness) Lukasiewicz logic ($\mathcal{L}$) $\iff$ $[0,1]_L$

Agnieszka Kułacka. "Strong standard completeness for continuous t-norms". In: *Fuzzy Sets and Systems* 345

- (Strong Completeness) Infinitary Lukasiewicz logic ($\mathcal{L}$) $\iff$ $[0, 1]_L$

Diego Castaño, Cecilia Cimadamore, José Patricio Díaz Varela, and Laura Rueda. "Completeness for monadic fuzzy logics via functional algebras". In: *Fuzzy Sets and Systems* 407

- (Strong completeness) S5-modal extension of $\mathcal{L}$ (S5($\mathcal{L}$)) $\iff$ structures based on MV-chains (S5($\mathbb{MV}_{to}$))

Diego Castaño, José Patricio Díaz Varela, and Gabriel Savoy. "Strong standard completeness theorems for S5-modal Lukasiewicz logics". In: *gabriel_savoy@hotmail.com*

(Finitary completeness) S5-modal extension of $\mathcal{L}$ (S$5(\mathcal{L})$) $\iff$ structures based on $[0,1]_L$ ($S5([0,1]_L)$)

Strong completeness

- Lukasiewicz logic ($\mathcal{L}$) $\iff$ MV-algebras
- S5($\mathcal{L}$) $\iff$ S5($\mathbb{MV}_{to}$)

Weak completeness

- Lukasiewicz logic ($\mathcal{L}$) $\iff$ $[0,1]_L$
- S5($\mathcal{L}$) $\iff$ $S5([0,1]_L)$

Strong completeness for infinitary extensions

- Infinitary Lukasiewicz logic ($\mathcal{L}$) $\iff$ $[0,1]_L$
- Infinitary S5($\mathcal{L}$) $\iff$ S5($[0,1]_L$) ?

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Lukasiewicz t-norm

Is the algebra $[0,1]_L = ([0,1], \cdot, \to, 0, 1)$ where:

- $a \cdot b = \max(0, 1 - (a + b))$
- $a \to b = \min((1 - a) + b, 1)$

MV-algebras

It is known that the variety generated by the Lukasiewicz t-norm is the class of all MV-algebras.

Formulas

The formulas we are going to consider, are constructed from a propositional language constituted by:

- propositional letters $p, q, r, \ldots$. The set of all propositional letters is Fm,
- logical operations $\rightarrow, \&$, and $\Box, \Diamond$,
- logical constants $0, 1$.

For example $p \rightarrow (q \& q)$ and $\Diamond(r \rightarrow 0) \rightarrow 0$ are formulas.

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Structures based on MV-algebras

Given a set X, an MV-chain A, and a function $e : X \times Prop \to A$ we say a triple (X, e, A) is a structure based on A.
Given a structure **K**, the *truth value* $\|\phi\|_{\mathbf{K},x}$ of a formula ϕ at a point x is defined inductively:

- if $\phi \in Prop$ then $\|\phi\|_{\mathbf{K},x} = e(x, \phi)$,
- if $\phi = \alpha \to \beta$ then $\|\phi\|_{\mathbf{K},x} = \|\alpha\|_{\mathbf{K},x} \to \|\beta\|_{\mathbf{K},x}$,
- if $\phi = \alpha \& \beta$ then $\|\phi\|_{\mathbf{K},x} = \|\alpha\|_{\mathbf{K},x} \cdot \|\beta\|_{\mathbf{K},x}$,
- if $\phi = \Box\alpha$ then $\|\phi\|_{\mathbf{K},x} = \inf_{x \in X} \|\alpha\|_{\mathbf{K},x}$,
- if $\phi = \Diamond\alpha$ then $\|\phi\|_{\mathbf{K},x} = \sup_{x \in X} \|\alpha\|_{\mathbf{K},x}$.

Observe that the infima and suprema appearing in the definition not necessarily exists. In the case they exist for all formulas we say that the structure is *safe*.

Semantic Logic associated to a class of models

Given a set of formulas Γ, we say that a safe model $\mathbf{K} = (X, e, A)$ is a *model* of Γ if $\|\phi\|_{\mathbf{K},x} = 1$ for every $x \in X$ and $\phi \in \Gamma$.
Consider a class $\mathbb{C}$ of MV-chains. For a set of formulas $\Gamma \cup \{\phi\}$ we write $\Gamma \models_{S5(\mathbb{C})} \phi$ if every model, based on A, of Γ is also a model of ϕ, for all $A \in \mathbb{C}$.

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A candidate for syntactic logic

We denote by $\mathcal{L}$ the Lukasiewicz logic. Consider now the logic $\vdash$ defined by the following axiomatic system:

- Axioms:
  - Instantiations of axiom-schemata of $\mathcal{L}$
  - $\Box\phi \to \phi$
  - $\phi \to \Diamond\phi$
  - $\Box(\nu \to \phi) \to (\nu \to \Box\phi)$
  - $\Box(\phi \to \nu) \to (\Diamond\phi \to \nu)$
  - $\Box(\phi \vee \nu) \to (\Box\phi \vee \nu)$
  - $\Diamond(\phi \& \phi) \equiv (\Diamond\phi)\&(\Diamond\phi)$
- Rules of Inference:
  - Modus Ponens: $\frac{\phi, \phi\to\psi}{\psi}$
  - Necessitation: $\frac{\phi}{\Box\phi}$
  - Infinitary Rule: $\frac{\Box\chi\vee(\Box\phi\to\Box\psi^n) : n\in\mathbb{N}}{\Box\chi\vee(\Box\phi\to(\Box\phi\&\Box\psi))}$

  where ϕ, ψ, χ are formulas, ν is any propositional combination of formulas beginning with $\Box$ or $\Diamond$, and $\alpha \equiv \beta$ abbreviates $(\alpha \to \beta) \wedge (\beta \to \alpha)$.

Proofs

If $\Gamma \cup \{\phi\}$ is a set of formulas, the notation $\Gamma \vdash \phi$ means that there is a *proof* from Γ to ϕ. This proofs are defined in the usual way, but they can be of infinite length.

Syntactic Lemmas

Lemma

Let $\Gamma \cup \{\alpha, \beta, \psi\}$ *be a set of formulas. Then,*

$$\Gamma, \Box\beta \text{ implies } \Gamma, \Box\alpha \vee \Box\beta \vdash \Box\alpha \vee \psi.$$

Lemma

Let $\Gamma \cup \{\alpha, \beta, \phi, \psi\}$ *be a set of formulas. Then,*

$$\Gamma, \Box\alpha \vdash \phi \text{ and } \Gamma, \Box\beta \vdash \psi \text{ imply } \Gamma, \Box\alpha \vee \Box\beta \vdash \phi \vee \psi.$$

Corollary

Let $\Gamma \cup \{\alpha, \beta, \phi\}$ *be a set of formulas. Then,*

$$\Gamma, \Box\alpha \to \Box\beta \vdash \phi \text{ and } \Gamma, \Box\beta \to \Box\alpha \vdash \phi \text{ imply } \Gamma \vdash \phi.$$

Corollary

Let $\Gamma \cup \{\phi, \psi\}$ *be a set of formulas. Then,*

$$\Gamma \vdash \phi \vee \Box\psi \text{ and } \Gamma, \Box\psi \vdash \phi \text{ imply } \Gamma \vdash \phi.$$

Theorem

Let $\Gamma \cup \{\phi\}$ *be a set of formulas such that* $\Gamma \nvdash \phi$. *Then, there is a* $\Box$*-prelinear theory* Γ^* *containing* Γ *such that* $\Gamma^* \nvdash \phi$.

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Monadic MV-algebras

With Γ^* we can define an equivalence relation on Fm. This gives us an algebra $\mathbf{L} = (Fm/\equiv, \wedge, \vee, *, \rightarrow, 0, 1, \exists, \forall)$. This algebra will be a *simple monadic MV-algebra*.

Definition

An algebra $(A, \vee, \wedge, \rightarrow, \cdot, \forall, \exists, 0, 1)$ is a monadic MV-algebra if $(A, \vee, \wedge, \rightarrow, \cdot, 0, 1)$ is a MV-algebra and

- $\forall x \rightarrow x = 1$,
- $\forall(x \rightarrow \forall y) = \exists x \rightarrow \forall y$,
- $\forall(\forall x \rightarrow y) = \forall x \rightarrow \forall y$,
- $\forall(\exists x \vee y) = \exists x \vee \forall y$,

for each $x, y \in A$.

We write $(\mathbf{A}, \forall, \exists)$ for $(A, \vee, \wedge, \rightarrow, \cdot, \forall, \exists, 0, 1)$.

$[0,1]_L^X$

If we consider in the MV-algebra $[0,1]_L^X$ these two new operations

- $\forall_\wedge f = \inf_{x\in[0,1]} f(x)$,
- $\exists_\vee f = \sup_{x\in[0,1]} f(x)$,

then $([0,1]_L, \forall_\wedge, \exists_\vee)$ is a monadic MV-algebra.
Observe that if h is a function from $Prop$ to $[0,1]_L^X$, then one can easily obtain a structure $\mathbf{K} = (X, e, [0,1]_L)$, where $e(x,p) = (h(p))(x)$.

Embedding in $[0,1]_L$

Theorem

Let $(\mathbf{A}, \exists, \forall)$ be a simple monadic MV-algebra. Then, there is a set I and an embedding of $(\mathbf{A}, \exists, \forall)$ into $([0,1]_L^I, \exists_\vee, \forall_\wedge)$. Moreover, the set I can be taken to be the set of maximal filters of $\mathbf{A}$.

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Strong standard completeness

Theorem

Let $\Gamma \cup \{\phi\}$ *be a set of formulas, then*

$$\Gamma \vdash \phi \textit{ if and only if } \Gamma \models_{S5([0,1]_L)} \phi.$$

Fix k. In the same work we also proved that an extension $\vdash_k$ of this infinitary logic by the axiom

$$\bigwedge_{1\leq i<j\leq k+1} \Box(\phi_i \vee \phi_j) \to \bigvee_{i=1}^{k+1} \Box\phi_i$$

is strongly complete with respect to models based on $[0,1]_L$ whose universe is bounded with respect to k ($5_k([0,1]_L)$).

Theorem

Let $\Gamma \cup \{\phi\}$ *be formulas, then*

$$\Gamma \vdash_k \phi \textit{ if and only if } \Gamma \models_{S5_k([0,1]_L)} \phi$$

Thank you!

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