Many-valued coalgebraic modal logic with a semi-primal algebra of truth-degrees

Wolfgang Poiger

University of Luxembourg

Joint work (in progress) with Alexander Kurz and Bruno Teheux

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Modal logic, coalgebraically

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$$\mathsf{Krip} \rightleftarrows \mathsf{MA}$$

Krip: Kripke frames with bounded morphisms
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Going many-valued:

$$\mathsf{T}' \circlearrowright \mathcal{X} \rightleftarrows \mathcal{V}\mathbf{L} \circlearrowleft \mathsf{M}'$$

Replace the two-element Boolean algebra **2** by another finite algebra **L**.

On the choice of algebra

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Main Assumption

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A finite algebra $\mathbf{P}$ is *primal* if every $f\colon P^k \to P$ is term-definable in $\mathbf{P}$.

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A finite algebra **L** is *semi-primal* if every $f\colon L^k \to L$ which preserves subalgebras [i.e. $f(S^k) \subseteq S$ for all $\mathbf{S} \in \mathbb{S}(\mathbf{L})$] is term-definable in **L**.

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- They are special instances of finite discriminator algebras (also called quasi-primal algebras)

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Let **L** be a finite bounded-lattice expansion. Then **L** is semi-primal if and only if for every $a \in L$ the following $T_a : L \to L$ is term-definable in **L**:

$$T_a(x) = \begin{cases} 1 & \text{if } x = a \\ 0 & \text{if } x \neq a. \end{cases}$$

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- Therefore, there exists a semi-primal algebra based on *any* bounded lattice.
- Adding the terms T_a is *not the same* as adding constants a.
- Given a finite bounded distributive lattice **D**, there is an axiomatization of modal logic over $(\mathbf{D}, \to, (T_d)_{d \in D})$ with Heyting implication interpreted on (crisp) Kripke frames. (Maruyama 2009)

Examples (2)

Proposition (Niederkorn 2001)

Every finite Łukasiewicz chain $\mathbf{Ł}_n = (\{0, \frac{1}{n}, \dots, \frac{n-1}{n}, 1\}, \wedge, \vee, 0, 1, \oplus, \odot, \neg)$ where $\neg x = 1 - x, \quad x \oplus y = min(x + y, 1)$ and $x \odot y = max(x + y - 1, 0)$ is semi-primal.

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Therefore:
Every finite Moisil chain $\mathbf{ŁM}_n = (\{0, \frac{1}{n}, \dots, \frac{n-1}{n}, 1\}, \wedge, \vee, 0, 1, \neg, (\tau_{\frac{i}{n}})_{i=1}^{n})$ is semi-primal.

Examples (3)

Definition

A *FL_{ew}-algebra* is an algebra $\mathbf{R} = (R, \wedge, \vee, 0, 1, \odot, \rightarrow)$ such that $(R, \odot, 1)$ is a commutative monoid and which satisfies the residuation condition:

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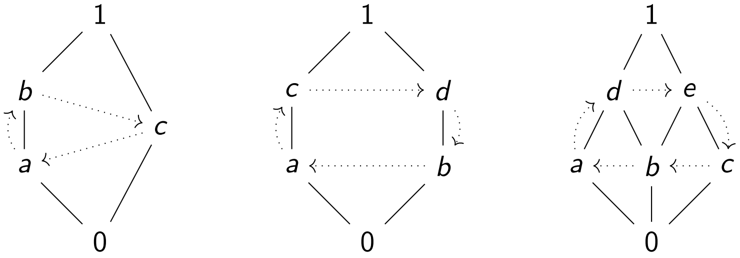
- There are also examples of semi-primal commutative bounded residuated lattices (i.e. where the monoid has neutral element $e \neq 1$).

Examples (4)

Call an algebra $\mathbf{L} = (L, \wedge, \vee, 0, 1, ')$ *pseudo-logic* if $0' = 1$ and $1' = 0$.

Examples of semi-primal pseudo-logics: (Davey, Schumann, Werner 1991)



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Morphisms $(X, \mathbf{r}) \to (X', \mathbf{r}')$ are continuous maps $f\colon X \to X'$ with

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Theorem (Keimel, Werner 1974 & Clark, Davey 1998)

For semi-primal algebras without trivial subalgebras there is a dual equivalence (with schizophrenic object L)

$$\mathsf{Stone}_{\mathbf{L}} \rightleftarrows \mathcal{V}\mathbf{L}$$

Forgetting topology

To obtain our coalgebraic base category $\mathcal{X}$, we *forget topology*:

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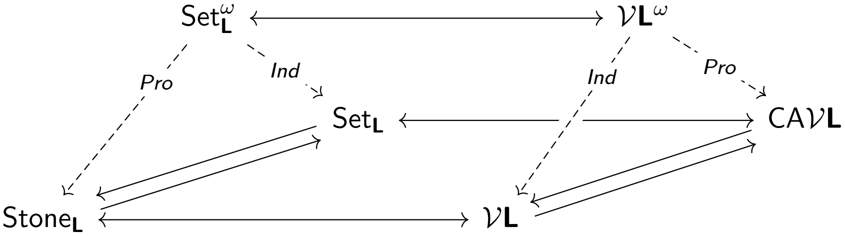
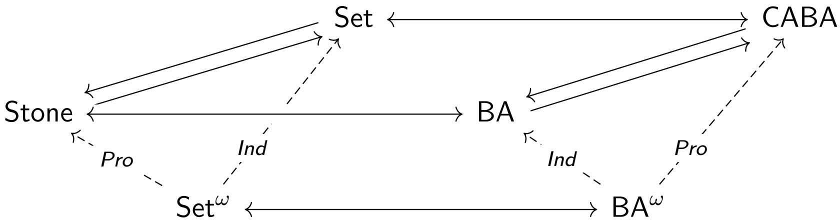
Pro- and Ind- completions





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Proposition

To every Set-endofunctor T there is natural way to associate a $\mathsf{Set}_{\mathbf{L}}$-endofunctor T' with $\mathsf{T}'\mathsf{U} = \mathsf{T}$ (here U is the forgetful functor).

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Example 1: $T = \mathcal{P}$, the (covariant) powerset functor. Then $\mathrm{Coalg}(\mathcal{P}')$ corresponds to the following:

Definition

A *crisp* $\mathbf{L}$-*frame* is a triple $(W, R, \mathbf{r})$ such that

- (W, R) is a Kripke-frame.
- $(W, \mathbf{r}) \in \mathrm{Set}_{\mathbf{L}}$ (i.e. $\mathbf{r} : W \to \mathbb{S}(\mathbf{L})$).
- Compatibility: $wRw' \Rightarrow \mathbf{r}(w') \subseteq \mathbf{r}(w)$.

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Example 2: $T = \mathcal{L}$, given on objects by $\mathcal{L}(X) = L^X$. Then $\mathrm{Coalg}(\mathcal{L}')$ corresponds to the following:

Definition

A **L**-*frame* is a triple $(W, R, \mathbf{r})$ such that

- (W, R) is a **L**-labeled Kripke-frame, i.e. $R\colon W \times W \to L$.
- $(W, \mathbf{r}) \in \mathsf{Set}_{\mathbf{L}}$ (i.e. $\mathbf{r} : W \to \mathbb{S}(\mathbf{L})$).
- Compatibility: $R(w, w') \neq 0 \Rightarrow \mathbf{r}(w') \subseteq \mathbf{r}(w)$.

Abstract coalgebraic modal logic

$$\mathsf{T}' \circlearrowright \mathsf{Set}_{\mathbf{L}} \underset{\Sigma}{\overset{\Pi}{\rightleftarrows}} \mathcal{V}\mathbf{L} \circlearrowleft \mathsf{M}'$$

Ingredients:

- A (representation of the) functor M' for the 'syntax'.
- A natural transformation $\delta\colon \mathsf{M}'\Pi \Rightarrow \Pi\mathsf{T}'$ for the 'semantics'.

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$$W \xrightarrow{\ \xi\ } \mathsf{T}'W$$

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$$\Pi\mathsf{T}'W \xrightarrow{\ \Pi\xi\ } \Pi W$$

Abstract coalgebraic modal logic

$$\mathsf{T}' \circlearrowright \mathsf{Set}_{\mathbf{L}} \underset{\Sigma}{\overset{\Pi}{\rightleftarrows}} \mathcal{V}\mathbf{L} \circlearrowleft \mathsf{M}'$$

Ingredients:

- A (representation of the) functor M' for the 'syntax'.
- A natural transformation $\delta \colon \mathsf{M}'\Pi \Rightarrow \Pi\mathsf{T}'$ for the 'semantics'.

From coalgebras to algebras:

$$W \xrightarrow{\ \xi\ } \mathsf{T}'W$$

$$\mathsf{M}'\Pi W \xrightarrow{\ \delta_W\ } \Pi\mathsf{T}'W \xrightarrow{\ \Pi\xi\ } \Pi W$$

Example

Definition

A *crisp* $\mathbf{L}$-*frame* is $(W, R, \mathbf{r})$ such that

- (W, R) is a Kripke-frame.
- $(W, \mathbf{r}) \in \mathrm{Set}_{\mathbf{L}}$ (i.e. $\mathbf{r} : W \to \mathbb{S}(\mathbf{L})$).
- Compatibility: $wRw' \Rightarrow \mathbf{r}(w') \subseteq \mathbf{r}(w)$.

A *crisp* $\mathbf{L}$-*model* is $(W, R, \mathbf{r}, Val)$ such that, in addition

$$Val\colon W \times \mathrm{Prop} \to L$$

always satisfies $Val(w, p) \in \mathbf{r}(w)$. We extend Val to all modal formulas using the rule

$$Val(w, \Box\varphi) = \bigwedge\{Val(w', \varphi) \mid wRw'\}$$

Example

$$\mathcal{P}' \circlearrowleft \mathsf{Set}_{\mathbf{L}} \underset{\Sigma}{\overset{\Pi}{\rightleftarrows}} \mathcal{V}\mathbf{L} \circlearrowright \mathsf{M}'$$

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- M' takes an algebra $\mathbf{A} \in \mathcal{V}\mathbf{L}$ to the free algebra generated by $\{\Box a \mid a \in A\}$ quotiented by the equations

$$\Box 1 \approx 1, \quad \Box(a \wedge b) \approx \Box a \wedge \Box b, \quad \Box \tau_\ell(a) \approx \tau_\ell(\Box a)$$

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- $\delta_X \colon \mathsf{M}'\Pi X \to \Pi\mathcal{P}'X$ is determined by

$$\Box f \mapsto (Y \mapsto \bigwedge_{y \in Y} f(y))$$

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- Completeness amounts to injectivity of δ.

On the choice of base category

Usually in MV-modal logic, people seem more interested in the correspondence

$$\mathsf{T} \circlearrowright \mathsf{Set} \rightleftarrows \mathcal{V}\mathbf{L} \circlearrowleft \mathsf{M}'$$

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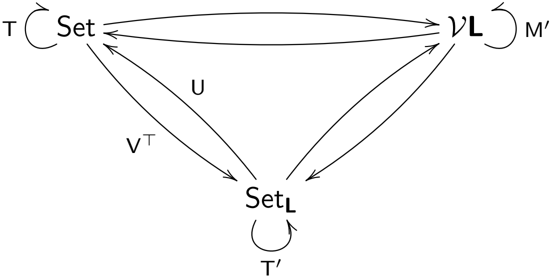

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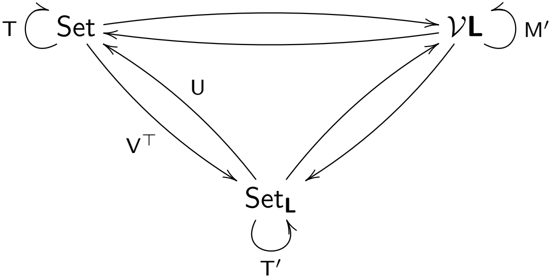

$V^{\top} \dashv U$

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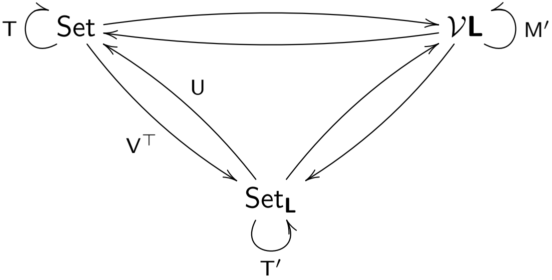

$\mathsf{V}^\top \dashv \mathsf{U}$

$\mathsf{V}^\top(X) = (X, \mathbf{r}^\top)$ assigns $\mathbf{r}^\top(x) = \mathbf{L}$ for all $x \in X$.

The '**L**-versions' of the Goldblatt-Thomason theorem

Theorem

Let F be an elementary class of crisp **L**-frames. Then F is modally definable if and only if F is closed under disjoint unions, generated sub-**L**-frames and bounded morphic images, and it reflects canonical extensions.

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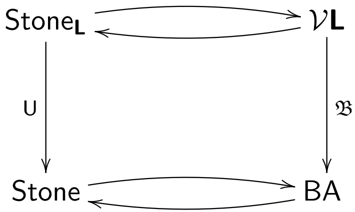
Corollary

Let F be a class of Kripke-frames closed under ultrapowers. Then F is **L**-definable if and only if F is closed under disjoint unions, generated subalgebras and bounded morphic images, and it reflects ultrafilter extensions.

The Boolean skeleton



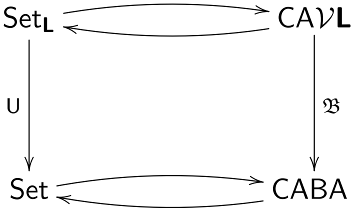
Let U be the forgetful functor. Its dual $\mathfrak{B}$ is given by the following.

Definition (Maruyama 2011)

Given any $\boldsymbol{A} \in \mathcal{V}\mathbf{L}$ we define $\mathfrak{B}(A) = \{a \in A \mid T_1(a) = a\}$. The *Boolean skeleton* of $\boldsymbol{A}$ is the Boolean algebra

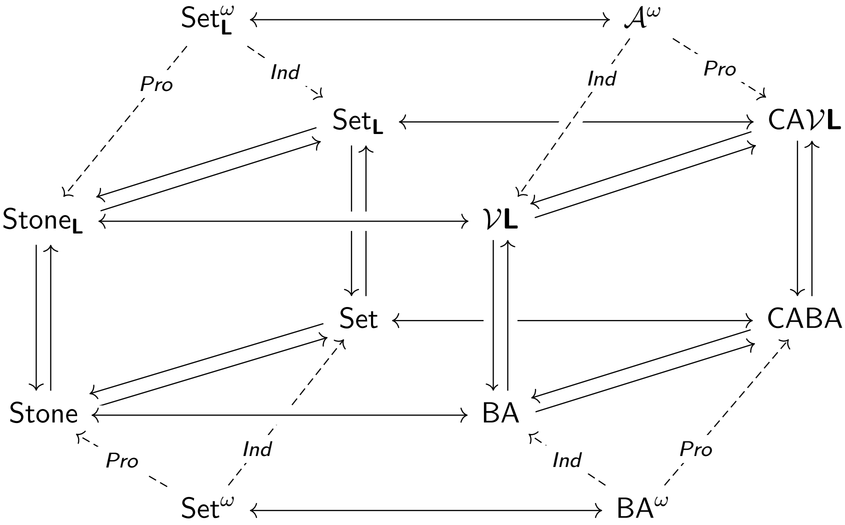
$$\mathfrak{B}(\boldsymbol{A}) = (\mathfrak{B}(A), \wedge, \vee, T_0, 0, 1).$$

Canonical extensions



Theorem

An algebra $\mathbf{A}$ *is in* CA$\mathcal{V}\mathbf{L}$ *iff its lattice-reduct is complete and* $\mathfrak{B}(\mathbf{A})$ *is in CABA.*


$\mathsf{Set}^{\omega}_{\mathbf{L}}$
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Thanks for your attention!